O Teorema de Gauss

O teor. de Gauss estabelece uma relação entre uma integral tripla, numa região $W \subseteq \mathbb{R}^3$, com uma integral de superf. na fronteira de W, ∂W.

Teorema de Gauss (ou Teorema da divergência)

Seja $W \subseteq \mathbb{R}^3$ um sólido com fronteira $\partial W = S$ uma superf. regular orientada por $\vec{n}$ exterior a W.
Seja $\vec{F} : U \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ um campo de classe C^1 definido num aberto U que contém W $(W \subseteq U)$
Então

$$\boxed{\iiint_W \operatorname{div} \vec{F} \, dv = \iint_{S=\partial W} \vec{F} \cdot \vec{n} \, dS}$$

Lembre que se $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$, então

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

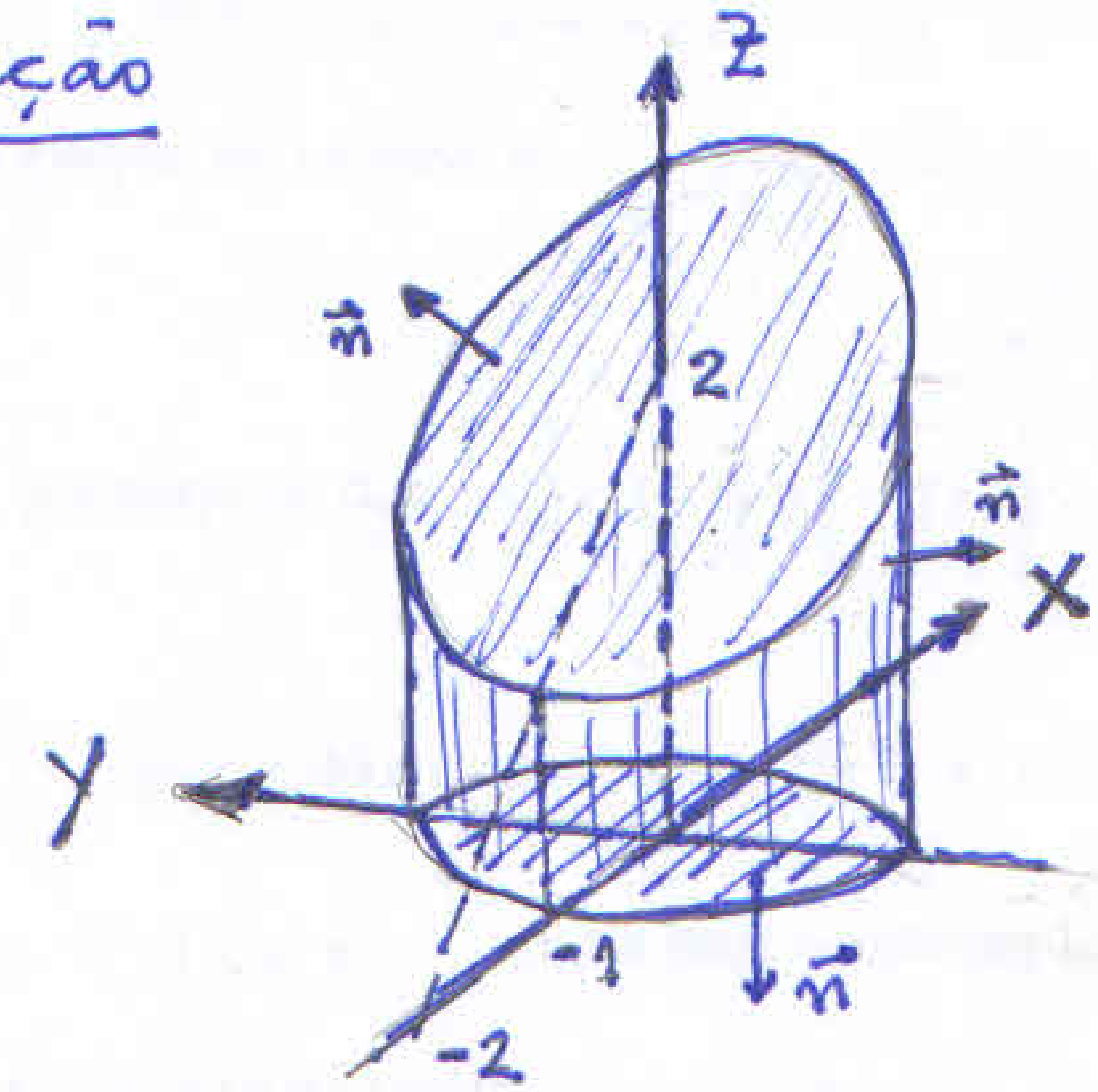
Exemplo 1:

Calcule $\iint_S \vec{F}\cdot\vec{n}\,dS$, onde

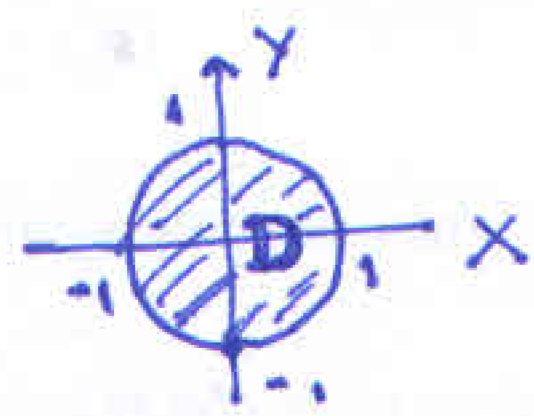
$$\vec{F}(x,y,z) = (x + ye^{z})\vec{i} + (y + ze^{x})\vec{j} + (z^2 + xe^{y})k$$ e

S é a fronteira do sólido interior ao cilindro $x^2+y^2=1$, entre os planos $z=0$ e $z=x+2$, $\vec{n}$ é a normal exterior a S.

Solução



W: = sólido limitado por S



$$\iint_S \vec{F}\cdot\vec{n}\,ds \overset{\text{Teor. Gauss}}{=} \iiint_W \operatorname{div}\vec{F}\cdot dV$$

$$\operatorname{div}\vec{F} = (1 + 1 + 2z) = 2(1+z)$$

$$\iiint_W \operatorname{div}\vec{F}\,dV = 2\iint_D \int_0^{x+2} (1+z)\,dz\,dx\,dy$$

$$= 2\iint_D \left(z + \frac{z^2}{2}\right)\Bigg|_0^{x+2} dx\,dy = 2\iint_D \left[x+2 + \frac{(x+2)^2}{2}\right]dx\,dy$$

$$\iiint_W \operatorname{div} \vec{F}\, dV = \iint_D (x^2+6x+8)\, dxdy$$

$$= 6\underbrace{\iint_D x\,dxdy}_{=0\ (*)} + 8\underbrace{\iint_D dxdy}_{\text{área}(D)=\pi} + \iint_D x^2\,dxdy$$

$$\left((*)\ \iint_D x\,dxdy = \int_0^{2\pi}\int_0^1 r\cos\theta \cdot r\,dr\,d\theta = \int_0^{2\pi} \frac{1}{3}\cos\theta\,d\theta = \frac{1}{3}\operatorname{sen}\theta \Big|_0^{2\pi} = 0 \right)$$

$$\iint_D x^2\,dxdy = \int_0^{2\pi}\int_0^1 r^2\cos^2\theta \cdot r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r^3\cos^2\theta\,dr\,d\theta$$

$$= \frac{1}{4}\int_0^{2\pi}\cos^2\theta\,d\theta = \frac{1}{4}\cdot\frac{1}{2}\left(\theta + \frac{\operatorname{sen}\theta}{2}\right)\Big|_0^{2\pi}$$

$$= \frac{1}{4}\cdot\frac{1}{2}\left(2\pi + 0\right) - 0$$

$$= \pi/4$$

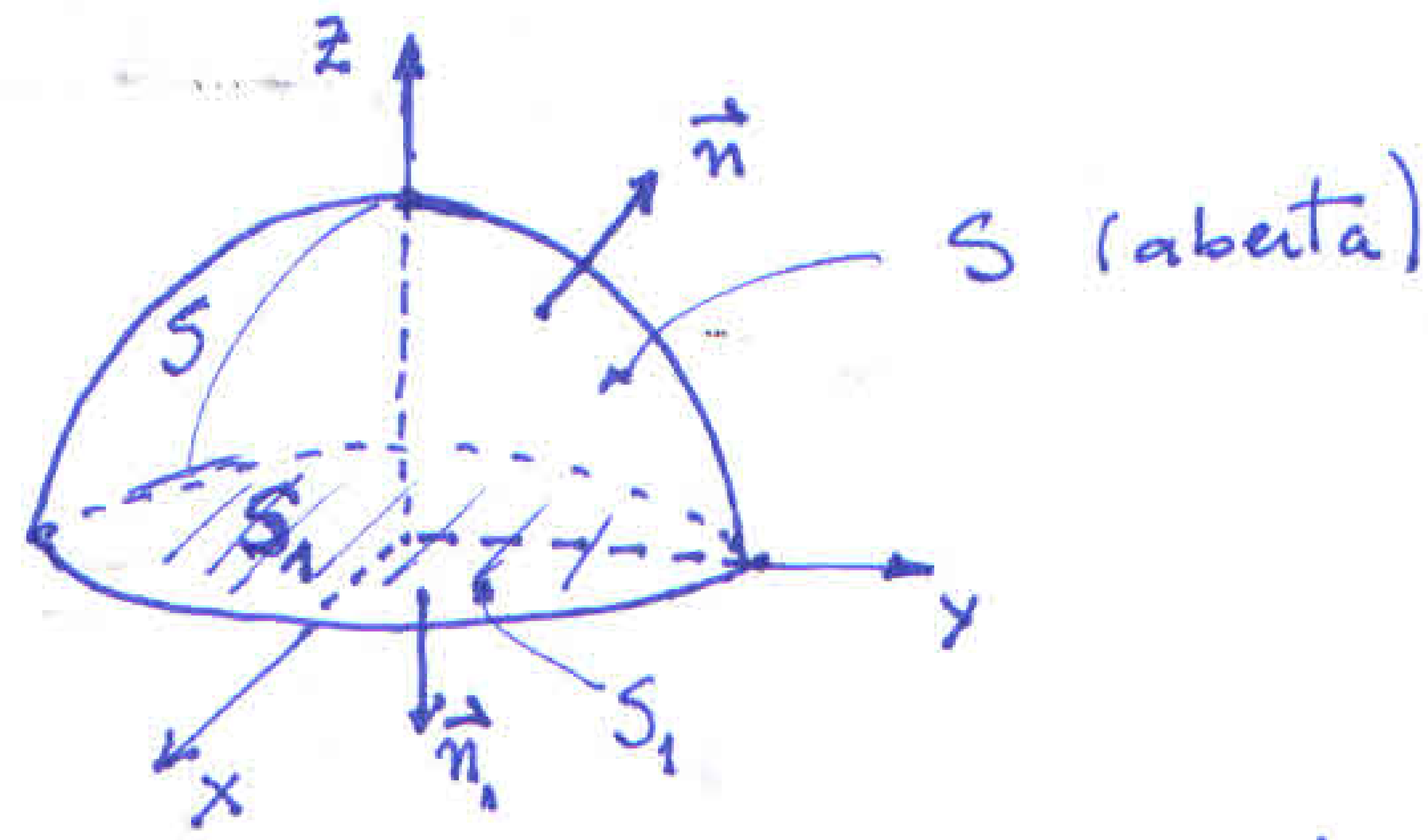
$$\therefore \iiint_W \operatorname{div} \vec{F}\, dv = 8\pi + \pi/4 = \frac{33\pi}{4}$$

Exemplo 2

Calcule $\iint_S \vec{F}\cdot\vec{n}\,dS$, onde $\vec{F}(x,y,z) = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$,

S é a superf. $x^2+y^2+z^2=1$ com $z \geq 0$ orientada por $\vec{n}$ a normal exterior.

Solução:



seja $\tilde{S} = S \cup S_1$, onde $S_1: \begin{cases} z=0 \\ x^2+y^2 \leq 1 \end{cases}$ com $\vec{n}_1 = -\vec{k}$

seja W o sólido limitado pela superf. fechada $\tilde{S}$. Então,

$$\iint_{\tilde{S}} \vec{F}\cdot\vec{n}\,dS = \iint_S \vec{F}\cdot\vec{n}\,dS + \iint_{S_1} \vec{F}\cdot\vec{n}\,dS$$

$$= \iiint_W \operatorname{div}\vec{F}\,dV$$

$$\operatorname{div}\vec{F} = 3x^2+3y^2+3z^2 = 3(x^2+y^2+z^2)$$

$$\iiint_W \operatorname{div} \vec{F}\, dV = 3 \iiint_W (x^2+y^2+z^2)\, dV$$

Em coord. esféricas $\begin{cases} x = \rho \operatorname{sen}\phi \operatorname{sen}\theta \\ y = \rho \operatorname{sen}\phi \cos\theta \\ z = \rho \cos\phi \end{cases}$

temos $x^2+y^2+z^2 = \rho^2$, $dV = \rho^2 \operatorname{sen}\phi \, d\rho\, d\phi\, d\theta$

e W nas coord. (ρ, ϕ, θ) é dado por

$$W: \begin{cases} 0 \leq \rho \leq 1 \\ 0 \leq \phi \leq \pi/2 \\ 0 \leq \theta \leq 2\pi \end{cases}$$

Então

$$3 \iiint_W (x^2+y^2+z^2)\, dV = 3 \iiint_{W_{\rho\phi\theta}} \rho^2 \cdot \rho^2 \operatorname{sen}\phi \, d\rho\, d\theta\, d\phi$$

$$= 3 \int_0^{\pi/2} \int_0^1 \int_0^{2\pi} \rho^4 \operatorname{sen}\phi \, d\theta\, d\rho\, d\phi = 6\pi \int_0^{\pi/2} \int_0^1 \rho^4 \operatorname{sen}\phi \, d\rho\, d\phi$$

$$= \frac{6\pi}{5} \int_0^{\pi/2} \operatorname{sen}\phi \, d\phi = \frac{6\pi}{5} \left(-\cos\phi \Big|_0^{\pi/2} \right)$$

$$= -\frac{6\pi}{5} (0 - 1) = 6\pi/5$$

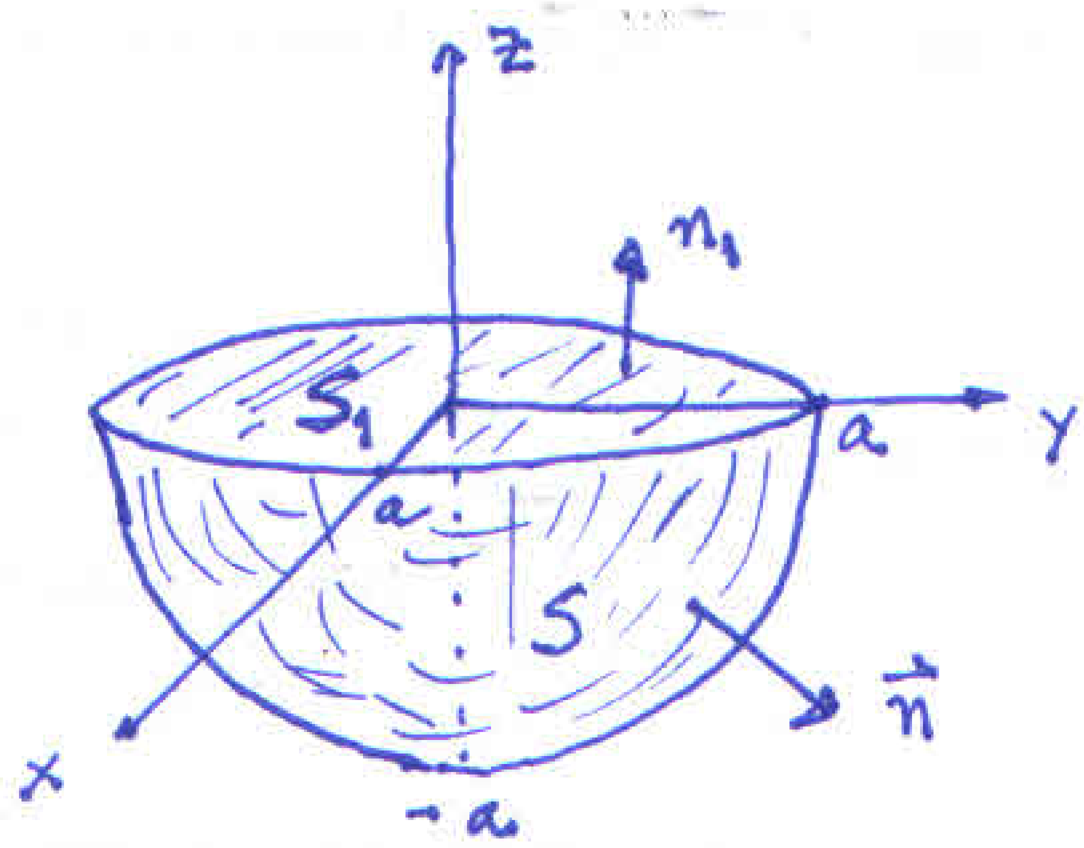
Exemplo 3

Calcule o fluxo do campo $\vec{F}$ através de S, com $\vec{n}$ exterior à superf. "fechada".

$$\vec{F}(x,y,z) = (x,y,z).$$

$$S: \quad x^2+y^2+z^2 = a^2 \quad , \quad z \leq 0$$

Solução:

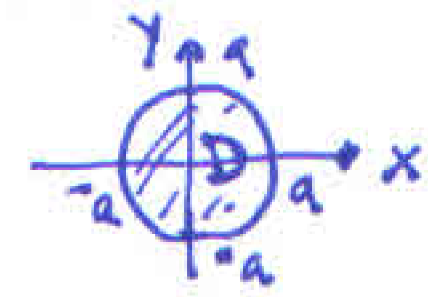


Seja $\tilde{S} = S \cup S_1$, onde

$$S_1: \begin{cases} z = 0 \\ x^2+y^2 \leq a^2 \end{cases}$$

$$S_1 = \{(x,y,0) : (x,y) \in D\}$$

$$D: \ x^2+y^2 \leq a^2$$



Seja W o sólido limitado por $\tilde{S}$.

Pelo Teor de Gauss temos:

$$\iint_{\tilde{S} = \partial W} \vec{F} \cdot \vec{n} \, dS = \iiint_W \operatorname{div} \vec{F} \, dV.$$

$\operatorname{div} \vec{F} = 1+1+1 = 3$, logo

$$\iint_{\tilde{S}} \vec{F} \cdot \vec{n} \, dS = \iiint_W 3 \, dV = 3 \operatorname{vol}(W) = 3\left(\frac{1}{2} \cdot \frac{4}{3} \pi a^3\right)$$

$$= 2\pi a^3$$

Daí, $$\iint_S \vec{F} \cdot \vec{n} \, dS + \iint_{S_1} \vec{F} \cdot \vec{n}_1 \, dS = 2\pi a^3$$

$$\iint_{S_1} \vec{F} \cdot \vec{n}\, ds = \iint_{D} (x, y, 0) \cdot (0, 0, 1)\, dx\, dy$$

$$= \iint_{D} 0\, dx\, dy = 0$$

$$\therefore \quad \iint_{S} \vec{F} \cdot \vec{n}\, dS = 2\pi a^3$$