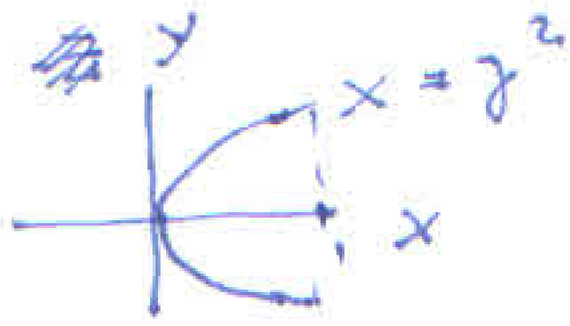
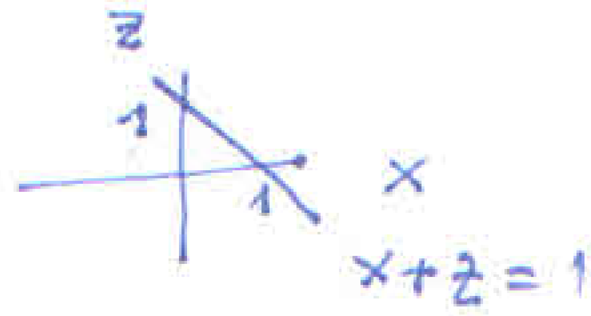
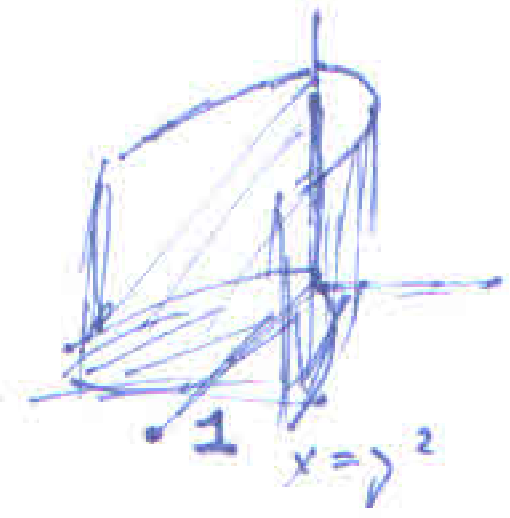
1 Determine o volume do sólido $W \subset \mathbb{R}^3$, onde

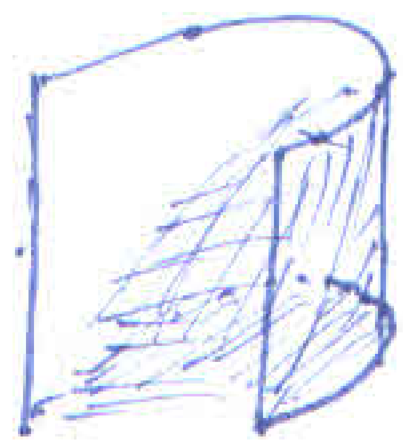
(a) W é limitado pelo cilindro $x = y^2$ e os planos $z = 0$ e $x + z = 1$

Solução:









$$\mathrm{vol}(W) = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} \int_0^{1-x} dz\, dy\, dx$$

$$W: \begin{cases} 0 \leq x \leq 1 \\ -\sqrt{x} \leq y \leq \sqrt{x} \\ 0 \leq z \leq 1-x \end{cases} \qquad = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} (1-x)\, dy\, dx$$

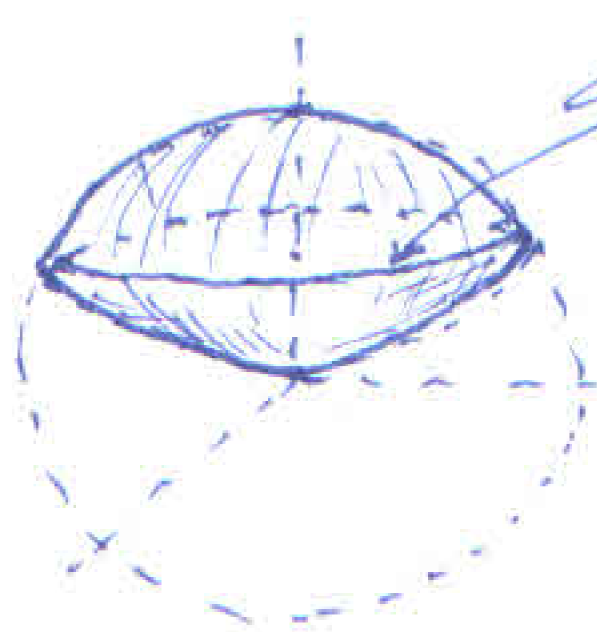
$$= \int_0^1 (1-x)\, 2\sqrt{x}\, dx$$

$$\mathrm{vol}(W) = \int_0^1 \left(2\sqrt{x} - 2x^{3/2}\right) dx = \left(\frac{4}{3}x^{3/2} - \frac{4}{5}x^{5/2}\right)\Big|_0^1$$

$$= \frac{4}{3} - \frac{4}{5} = \frac{8}{15} \text{ unidades de volume.}$$

(c) W é limitado pela esfera $x^2 + y^2 + z^2 = 4$ e pelo parabolóide $x^2 + y^2 = 3z$

Solução



$$\begin{cases} z = 1 \\ x^2 + y^2 = 3 \end{cases}$$

Em coordenadas cilíndricas W é descrito como:

$$W: \begin{cases} 0 \leq r \leq \sqrt{3} \\ 0 \leq \theta \leq 2\pi \\ \frac{r^2}{3} \leq z \leq \sqrt{4 - r^2} \end{cases}$$

$$W : \begin{cases} 0 \leq r \leq \sqrt{3} \\ 0 \leq \theta \leq 2\pi \\ \dfrac{r^2}{3} \leq z \leq \sqrt{4-r^2} \end{cases}$$

$$\text{vol}(W) = \int_0^{\sqrt{3}} \int_0^{2\pi} \int_{r^2/3}^{\sqrt{4-r^2}} r\,dz\,d\theta\,dr = \int_0^{\sqrt{3}} \int_0^{2\pi} \left(r\sqrt{4-r^2} - r^3/3 \right) d\theta\,dr$$

$$= 2\pi \int_0^{2} \left(r\sqrt{4-r^2} - \frac{r^3}{3} \right) dr =$$

$$= 2\pi \left(-\frac{1}{3}(4-r^2)^{3/2} - \frac{r^4}{12} \right) \Big|_0^2 = 2\pi \left(-\frac{16}{12} + \frac{4^{3/2}}{3} \right)$$

$$= 2\pi \left(-\frac{4}{3} + \frac{8}{3} \right) = \frac{8\pi}{3} \quad \text{u.v.}$$

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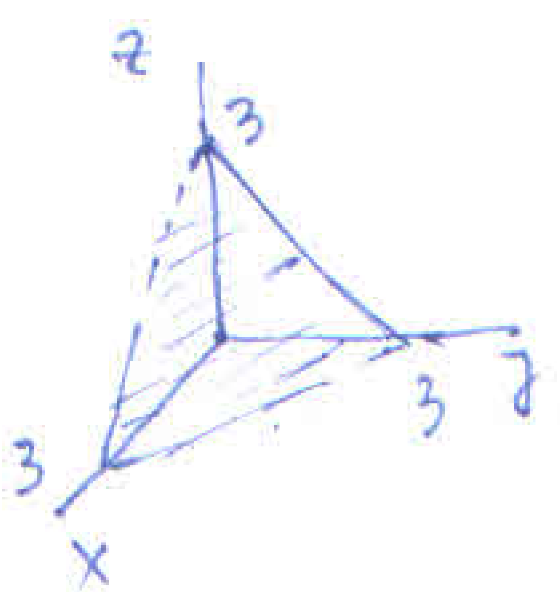
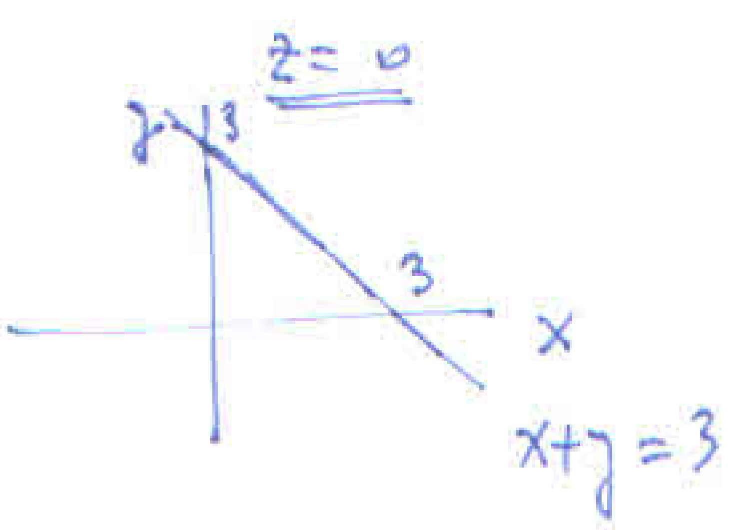
(2) Calcule $\iiint_W f\,dV$, onde:

(a) $f(x,y,z) = x - y$ e W é o tetraedro limitado pelos planos coord. e pelo plano $x+y+z=3$

Solução



$$W : \begin{cases} 0 \leq x \leq 3 \\ 0 \leq y \leq 3-x \\ 0 \leq z \leq 3-(x+y) \end{cases}$$



$$\iiint_W f\,dV = \int_0^3 \int_0^{3-x} \int_0^{3-(x+y)} (x-y)\,dz\,dy\,dx$$

$$= \int_0^3 \int_0^{3-x} \left[3(x-y) - (x+y)(x-y) \right] dy\,dx = \int_0^3 \int_0^{3-x} \left(-x^2 + y^2 + 3x - 3y \right) dy\,dx$$

$$\iiint_W f\,dV = \int_0^3 \left(-x^2y + \frac{y^3}{3} + 3xy - \frac{3y^2}{2}\right)\Bigg|_{y=0}^{y=3-x} dx$$

$$= \int_0^3 \left[-x^2(3-x) + \frac{(3-x)^3}{3} + 3x(3-x) - \frac{3(3-x)^2}{2}\right] dx$$

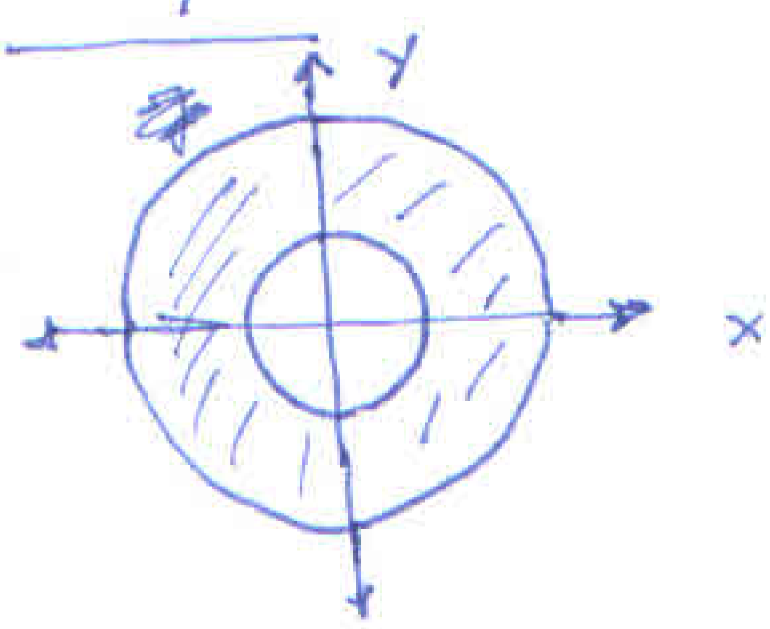
$$= \int_0^3 \left\{x(3-x)^2 + \frac{1}{3}(3-x)^3 - \frac{3}{2}(3-x)^2\right\} dx$$

$$= \int_0^3 \left(\frac{2}{3}x - \frac{1}{2}\right)(3-x)^2 dx$$

$$= \ldots\ldots = 0$$

(d) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$, W é a coroa esférica limitada por $x^2+y^2+z^2 = 1$ e $x^2+y^2+z^2 = 4$

Solução



Em coord esféricas W é descrito por:

$$W : \begin{cases} 1 \leq \rho \leq 2 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi \end{cases} \qquad \begin{aligned} dx\,dy\,dz &= \rho^2 \operatorname{sen}\phi \\ x^2+y^2+z^2 &= \rho^2 \end{aligned}$$

$$\iiint_W f\,dV = \int_1^2 \int_0^{2\pi} \int_0^{\pi} \rho \cdot \rho^2 \operatorname{sen}\phi \, d\phi \, d\theta \, d\rho$$

$$= \int_1^2 \int_0^{2\pi} -\rho^3 \cos\phi \Big|_0^{\pi} d\theta \, d\rho = \int_1^2 \int_0^{2\pi} 2\rho^3 \, d\theta \, d\rho$$

$$\int_1^2 \int_0^{2\pi} 2\rho^3 \, d\theta \, d\rho = 4\pi \int_1^2 \rho^3 \, d\rho = \pi \rho^4 \Big|_1^2$$

$$= 16\pi - \pi = \underline{\underline{15\pi}}$$

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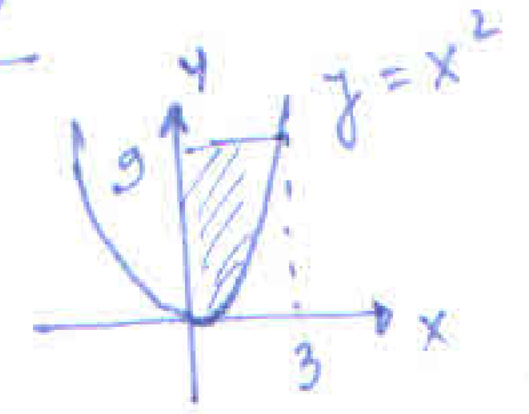
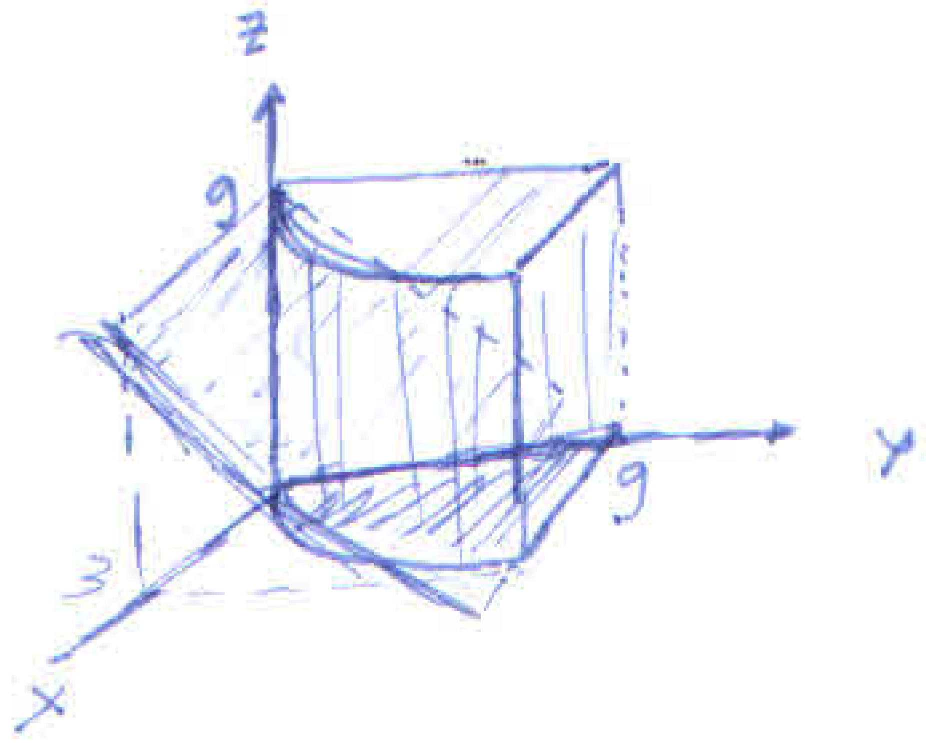
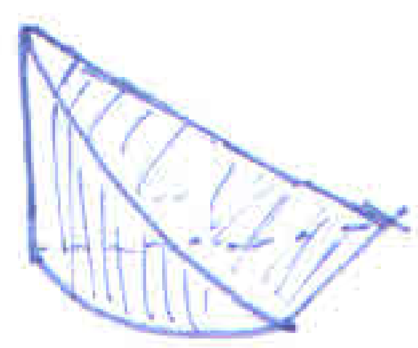
(3) Determine a massa de W. W é o sólido, no primeiro octante, limitado por $y = x^2$, $y = 9$, $z = 0$, $x = 0$ e $y + z = 9$.
A densidade é dada por $\delta(x,y,z) = x + y$

Solução







$$W: \begin{cases} 0 \leq y \leq 9 \\ 0 \leq x \leq \sqrt{y} \\ 0 \leq z \leq 9 - y \end{cases}$$

ou também:

$$W: \begin{cases} 0 \leq x \leq 3 \\ x^2 \leq y \leq 9 \\ 0 \leq z \leq 9 - y \end{cases}$$

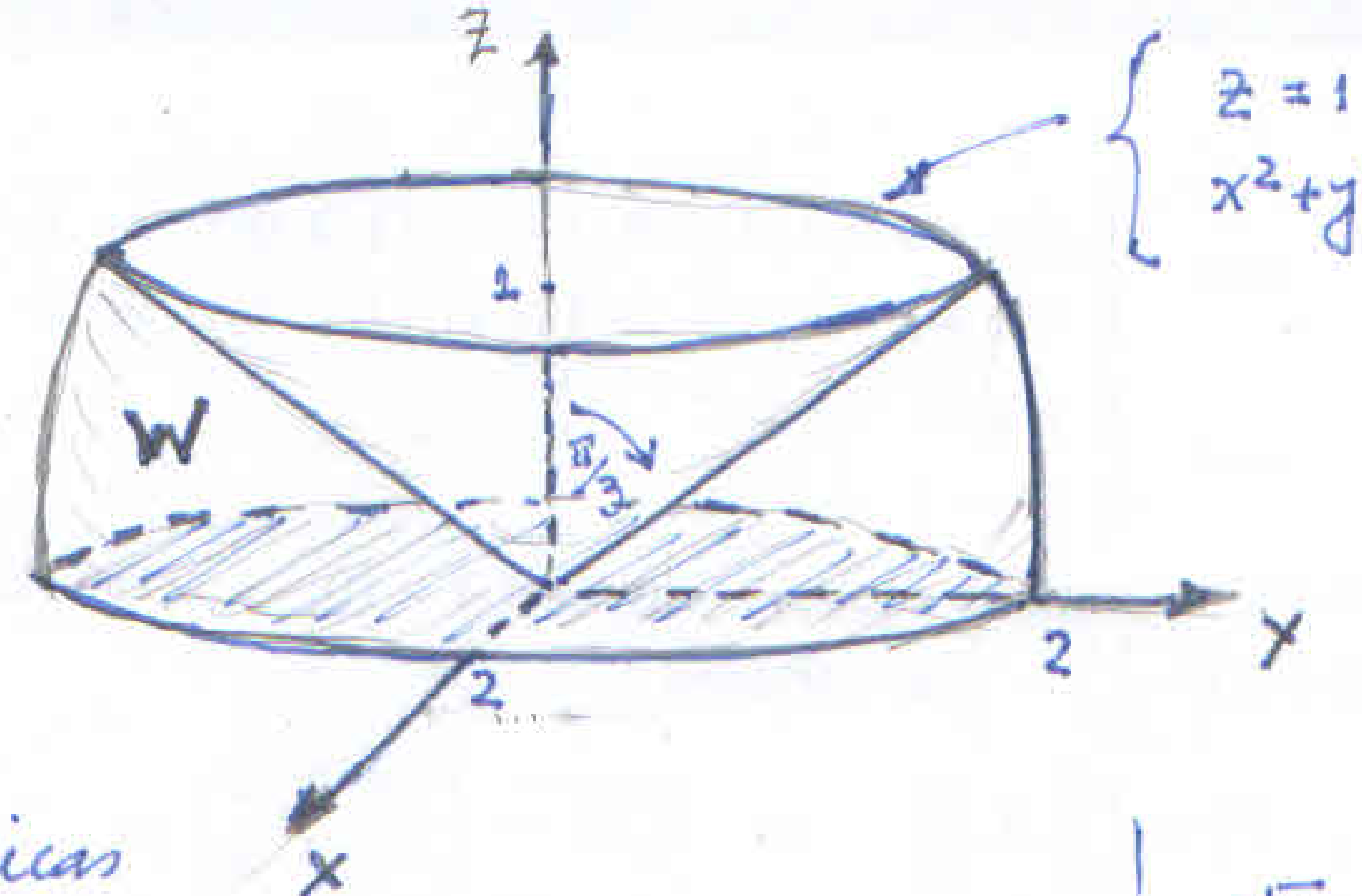
$$\text{Massa de W} = M(W) = \int_0^3 \int_{x^2}^9 \int_0^{9-y} (x+y) \, dz \, dy \, dx$$

$$= \int_0^3 \int_{x^2}^9 (x+y)(9-y) \, dy \, dx = \int_0^3 \left(9xy + \frac{9-x}{2} y^2 - \frac{y^3}{3} \right) \Bigg|_{x^2}^9$$

$$= \ldots\ldots = \frac{53.703}{140}$$

9 Calcule o volume do sólido acima do plano $z=0$, dentro da esfera $x^2+y^2+z^2=4$ e abaixo do cone $z=\sqrt{\dfrac{x^2+y^2}{3}}$.

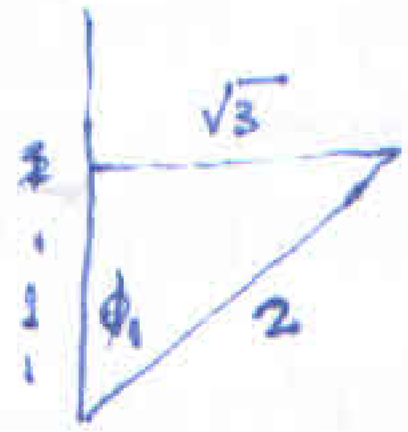
Solução



$$\begin{cases} z=1 \\ x^2+y^2=3 \end{cases}$$

W em coord. esféricas

$$W: \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le \rho \le 2 \\ \pi/3 \le \phi \le \pi/2 \end{cases}$$



$tg\, \phi_1 = \sqrt{3}$

$sen\, \phi_1 = \sqrt{3}/2$

$\cos \phi_1 = 1/2$

$\boxed{\phi_1 = \pi/3}$

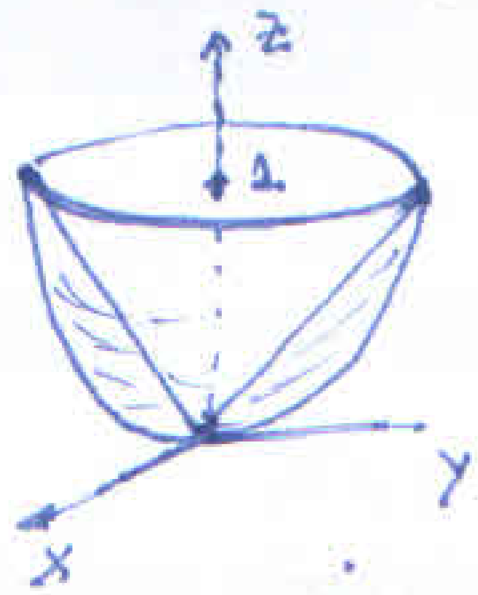
$$vol(W) = \int_0^{2\pi}\int_0^{2}\int_{\pi/3}^{\pi/2} \rho^2\, sen\phi \; d\phi\, d\rho\, d\theta$$

$$= \int_0^{2\pi}\int_0^{2} -\rho^2 \cos\phi \Big|_{\pi/3}^{\pi/2} d\rho\, d\theta = \int_0^{2\pi}\int_0^{2} -\rho^2\left(0-\tfrac{1}{2}\right) d\rho\, d\theta$$

$$= \int_0^{2\pi}\int_0^{2} \tfrac{1}{2}\rho^2\, d\rho\, d\theta = \frac{\pi}{3}\rho^3\Big]_0^2 = \frac{8\pi}{3} \text{ u.v.}$$

(10) Calcule o volume do sólido W acima do parabolóide $z = x^2+y^2$ e abaixo do cone $z = \sqrt{x^2+y^2}$

Solução:



$$W: \begin{cases} x^2+y^2 \leq 1 \\ x^2+y^2 \leq z \leq \sqrt{x^2+y^2} \end{cases}$$

$$vol(W) = \iiint_W dV$$

W em coord. cilíndricas

$$W: \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 1 \\ r^2 \leq z \leq r \end{cases}$$

$x = r\cos\theta$
$y = r\,\text{sen}\,\theta$
$z = z$
$dx\,dy\,dz = dV = r\,dr\,d\theta\,dz$

$$vol(W) = \int_0^1 \int_{r^2}^{r} \int_0^{2\pi} r\,d\theta\,dz\,dr = \int_0^1 \int_{r^2}^{r} 2\pi r\,dz\,dr$$

$$= \int_0^1 2\pi r(r-r^2)\,dr = 2\pi \int_0^1 (r^2-r^3)\,dr$$

$$= 2\pi \left(\frac{r^3}{3} - \frac{r^4}{4}\right)\Big|_0^1 = 2\pi\left(\frac{1}{3} - \frac{1}{4}\right)$$

$$= \pi/6 \quad \text{u.v.}$$