UNIVERSIDADE FEDERAL DA PARAÍBA
CCEN - Departamento de Matemática
http://www.mat.ufpb.br

Cálculo III

2ª Prova, João Pessoa, 17 de abril de 2013

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Questão 1 *(2.0 pts) Considere a superfície S como sendo a parte do cone de equação $z^2 = x^2 + y^2$ que se encontra entre os planos $z = 1$ e $z = 2$. Determine o momento de inércia da superfície em relação ao eixo da mesma supondo que a densidade é constante.*

Questão 2 *(2.5 pts) Calcule $\int\int_S (\text{rot}\,\vec{F}) \cdot \vec{n}\, dS$, onde $\vec{F}(x,y,z) = (e^x - y)\vec{i} + (xz + y^2)\vec{j} + 2yz\vec{k}$ e $S : x^2 + y^2 + z^2 - 4z = 0, \quad z \leq 1$.*

Questão 3 *(2.0 pts) Calcule $\int_C -2xydx + (x^2 + y^2)dy$ onde C é a parte da elipse $x^2 + 4y^2 = 2x$ correspondente a $y \geq 0$ orientada no sentido anti-horário.*

Questão 4 *(2.0 pts) Calcule $\int\int_S \vec{F} \cdot \vec{n}\, dS$, onde: $\vec{F}(x,y,z) = x\vec{i} + y\vec{j} - \frac{z^2}{2}\vec{k}$ e S é a superfície de revolução obtida girando-se o segmento de reta AB, com $A = (0,1,2)$ e $B = (0,2,4)$, em torno do eixo Z e com vetor normal $\vec{n}$ exterior a S.*

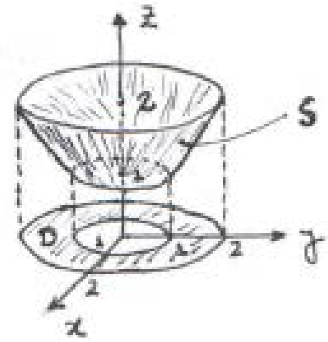
Questão 5 *(2.5 pts) Use o Teorema de Gauss para calcular o fluxo do campo*

$$\vec{F}(x,y,z) = z\arctan(y^2)\vec{i} + z^3\ln(x^2+1)\vec{j} + z\vec{k}$$

através da superfície $S : z = 2 - x^2 - y^2,\ 1 < z < 2.$ com normal $\vec{n}$ exterior.

Boa Prova !!

Questão 1



$S: \; z^2 = x^2 + y^2 \;, \quad 1 \le z \le 2$

$D: \; 1 \le x^2 + y^2 \le 2^2 = 4$

O eixo da superf. S é o eixo Z. Então

$$I_z = \iint_S (x^2+y^2) \cdot \rho(x,y,z)\, dS \qquad , \; \rho(x,y,z) = \text{densidade} = k = \text{cte.}$$

$$= k \iint_S (x^2+y^2)\, dS$$

A superf. S pode ser descrita como gráfico da função

$z = f(x,y) = \sqrt{x^2+y^2}$ com $(x,y) \in D: \; 1 \le x^2+y^2 \le 4$

temos $\quad f_x = \dfrac{x}{\sqrt{x^2+y^2}} \quad , \quad f_y = \dfrac{y}{\sqrt{x^2+y^2}}$

$$1 + \|\nabla f\|^2 = 1 + \left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2$$

$$= 1 + \frac{x^2+y^2}{x^2+y^2} = 2$$

$$dS = \sqrt{1 + \|\nabla f\|^2}\, dx\,dy = \sqrt{2}\, dx\,dy.$$

$$I_z = k \iint_S (x^2+y^2)\, dS = k \iint_D (x^2+y^2)\sqrt{2}\, dx\,dy$$

[redacted]

Em coordenadas polares temos

$$\begin{cases} x = r\cos\theta \\ y = r\,\mathrm{sen}\,\theta \\ dx\,dy = r\,dr\,d\theta \\ x^2+y^2 = r^2 \end{cases} \qquad D_{r\theta}: \begin{cases} 0 \leq \theta \leq 2\pi \\ 1 \leq r \leq 2 \end{cases}$$

$$I_z = \sqrt{2}\,k \iint_{D_{r,\theta}} r^2 . r\,dr\,d\theta$$

$$= \sqrt{2}\,k \int_1^2 \int_0^{2\pi} r^3\,d\theta\,dr$$

$$= 2\sqrt{2}\,k\pi \int_1^2 r^3\,dr$$

$$= 2\sqrt{2}\,k\pi \left.\frac{r^4}{4}\right|_1^2 = \frac{\sqrt{2}}{2}\,k\pi\,(16-1)$$

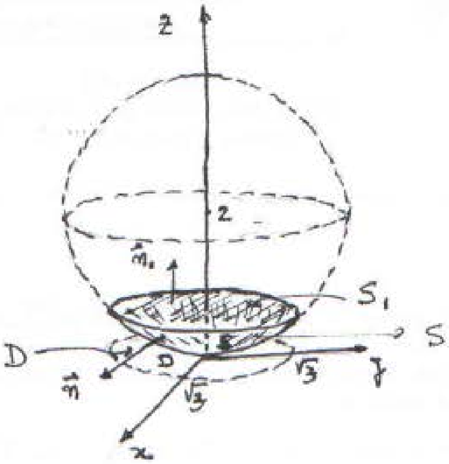
$$= \frac{15\sqrt{2}\,k\pi}{2}$$

Questão 2 $\displaystyle\iint_S (\text{rot}\,\vec{F})\cdot\vec{n}\,dS$

$\vec{F}(x,y,z) = (e^x - y)\,\vec{i} + (xz + y^2)\,\vec{j} + 2yz\,\vec{k}$

$S:\ x^2 + y^2 + z^2 - 4z = 0 \quad , \quad z \leq 1$

$D: \left\{ x^2 + y^2 \leq 3 \right.$



Seja $\tilde{S} = S \cup S_1$ onde $S_1: \begin{cases} z = 1 \\ x^2 + y^2 \leq 3 \end{cases}$

S_1 orientada com $\vec{n}_1 = \vec{k}$

$dS_1 = \sqrt{1 + z_x^2 + z_y^2}\,dx\,dy = \sqrt{1+0+0}\,dx\,dy = dx\,dy$

Seja W o sólido limitado por $\tilde{S}$

Teo de Gauss $\Rightarrow$ $\displaystyle\iint_{\tilde{S} = \partial W} (\text{rot}\,\vec{F})\cdot\vec{n}\,d\tilde{S} = \iiint_W \text{div}(\text{rot}\,\vec{F})\,dV$

$\text{div}(\text{rot}\,\vec{F}) = 0$, logo

$$\iint_{\tilde{S} = \partial W} (\text{rot}\,\vec{F})\cdot n\,d\tilde{S} = 0$$

Por outro lado,

$$0 = \iint_{\tilde{S}} (\text{rot}\,\vec{F}) \cdot n\, dS = \iint_{S} (\text{rot}\,\vec{F}) \cdot \vec{n}\, dS + \iint_{S_1} (\text{rot}\,\vec{F}) \cdot \vec{n}_1 \cdot dS_1$$

Daí

$$\iint_{S} (\text{rot}\,\vec{F}) \cdot \vec{n}\, dS = -\iint_{S_1} (\text{rot}\,\vec{F}) \cdot \vec{n}_1\, dS_1$$

$$\text{rot}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x - y & xz + y^2 & 2yz \end{vmatrix} = (2z - x)\vec{i} + 0\,\vec{j} + (z+1)\vec{k}$$

$$(\text{rot}\,\vec{F}) \cdot \vec{n}_1 = z + 1$$

Logo

$$\iint_{S_1} (\text{rot}\,\vec{F}) \cdot \vec{n}_1\, dS_1 = \iint_{D} (1+1)\, dx\, dy$$

$$= 2 \iint_{D} dx\, dy = 2\, \text{área}(D) = 2 \cdot \pi (\sqrt{3})^2 = 6\pi.$$

$$\therefore \quad \iint_{S} (\text{rot}\,\vec{F}) \cdot \vec{n}\, dS = -6\pi$$

Questão 3

$$\int_C -2xy\,dx + (x^2+y^2)\,dy$$

C: $x^2+4y^2=2x$, $y \geq 0$ sentido anti-horário

$x^2+4y^2=2x \longrightarrow (x-1)^2+4y^2=1$

ou $(x-1)^2 + \dfrac{y^2}{1/4} = 1$ (com $y \geq 0$)

Uma parametrização de C é

$$\alpha(t) = \left(1+\cos t, \tfrac{1}{2}\operatorname{sen} t\right), \quad 0 \leq t \leq \pi$$

$$\left(\begin{cases} x-1 = \cos t \\ \dfrac{y}{1/2} = \operatorname{sen} t \end{cases} \qquad (x-1)^2 + \left(\dfrac{y}{1/2}\right)^2 = 1 \right)$$

Logo, $\alpha'(t) = \left(-\operatorname{sen} t, \tfrac{1}{2}\cos t\right)$ e

$$\vec{F}(\alpha(t)) \cdot \alpha'(t) = \vec{F}\left(1+\cos t, \tfrac{1}{2}\operatorname{sen} t\right) \cdot \left(-\operatorname{sen} t, \tfrac{1}{2}\cos t\right)$$

$$= \left(-2(1+\cos t)\tfrac{1}{2}\operatorname{sen} t, (1+\cos t)^2 + \tfrac{1}{4}\operatorname{sen}^2 t\right) \cdot \left(-\operatorname{sen} t, \tfrac{1}{2}\cos t\right)$$

$$= (1+\cos t)\operatorname{sen}^2 t + \frac{1}{2}\left[(1+\cos t)^2 + \frac{1}{4}\operatorname{sen}^2 t\right]\cos t$$

$$\vec{F}(\alpha(t))\,\alpha'(t) = (1+\cos t)\,\mathrm{sen}^2 t + \frac{1}{2}\left(1+2\cos t+\cos^2 t+\frac{1}{4}\mathrm{sen}^2 t\right)\cos t$$

$$= \underbrace{\mathrm{sen}^2 t + \mathrm{sen}^2 t\cos t + \frac{1}{2}\cos t + \cos^2 t}_{1} + \frac{1}{2}\underbrace{\cos^2 t}_{(1-\mathrm{sen}^2 t)}\cos t + \frac{1}{8}\mathrm{sen}^2 t\cos t$$

$$= 1 + \frac{9}{8}\mathrm{sen}^2 t\cos t + \frac{1}{2}\cos t + \frac{1}{2}\cos t - \frac{1}{2}\mathrm{sen}^2 t\cos t$$

$$= 1 + \frac{5}{8}\mathrm{sen}^2 t\cos t + \cos t$$

Logo,

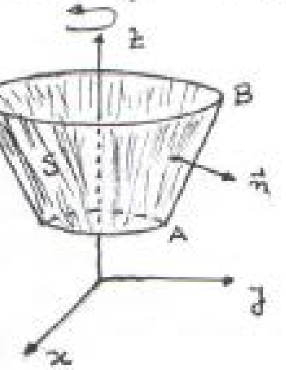
$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{\pi} \vec{F}(\alpha(t))\cdot\alpha'(t)\,dt$$

$$= \int_0^{\pi}\left(1+\frac{5}{8}\mathrm{sen}^2 t\cos t + \cos t\right)dt$$

$$= \left(t + \frac{5}{8}\cdot\frac{1}{3}\mathrm{sen}^3 t + \mathrm{sen}\, t\right)\Big|_0^{\pi} = \pi$$

Questão 4

$$\iint_S \vec{F} \cdot \vec{n}\, dS \quad , \quad \vec{F} = x\vec{i} + y\vec{j} - \frac{z^2}{2}\vec{k}$$



$A = (0,1,2)$

$B = (0,2,4)$

Uma parametrização para o segmento AB é

$$\vec{r}(t) = A + t(B-A) \qquad 0 \leq t \leq 1$$
$$= (0,\ 1+t,\ 2+2t)$$

Então, uma parametrização para S é

$$\begin{cases} x = (1+t)\cos\theta \\ y = (1+t)\,\text{sen}\,\theta \\ z = 2+2t \end{cases} \quad \text{com} \quad (t,\theta) \in D: \begin{cases} 0 \leq t \leq 1 \\ 0 \leq \theta \leq 2\pi \end{cases}$$

Um vetor normal a S é

$$\vec{N} = (\cos\theta, \text{sen}\,\theta, 2) \times (-(1+t)\,\text{sen}\,\theta, (1+t)\cos\theta, 0)$$

$$\vec{N} = (1+t)(-2\cos\theta, -2\,\text{sen}\,\theta, 1)$$

Como $\vec{n}$ aponta para fora, $\vec{n} = \dfrac{-\vec{N}}{\|\vec{N}\|}$.

$$dS = \|\vec{N}\|\, dt\, d\theta$$

Então $\iint_S \vec{F} \cdot \vec{n} \, dS =$

$$= \iint_D \left((1+t)\cos\theta, (1+t)\,\text{sen}\,\theta, -\frac{(2+2t)^2}{2} \right) \bullet (1+t)\left(2\cos\theta, 2\,\text{sen}\,\theta, -1\right) dt\,d\theta$$

$$= \iint_D \left[2(1+t)^2\cos^2\theta + 2(1+t^2)\,\text{sen}^2\theta + \frac{4}{2}(1+t)^3 \right] dt\,d\theta$$

$$= \iint_D \left[2(1+t)^2 + 2(1+t)^3 \right] dt\,d\theta$$

$$= \int_0^{2\pi} \int_0^1 \left[2(1+t)^2 + 2(1+t)^3 \right] dt\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{2}{3}(1+t)^3 + \frac{1}{2}(1+t)^4 \right)\Big|_0^1 d\theta = \left[\left(\frac{16}{3} - \frac{2}{3} \right) + \left(\frac{16}{2} - \frac{1}{2} \right) \right] 2\pi$$

$$= \left(\frac{16}{3} + \frac{16}{2} \right) \cdot 2\pi = \frac{80}{3}\pi$$

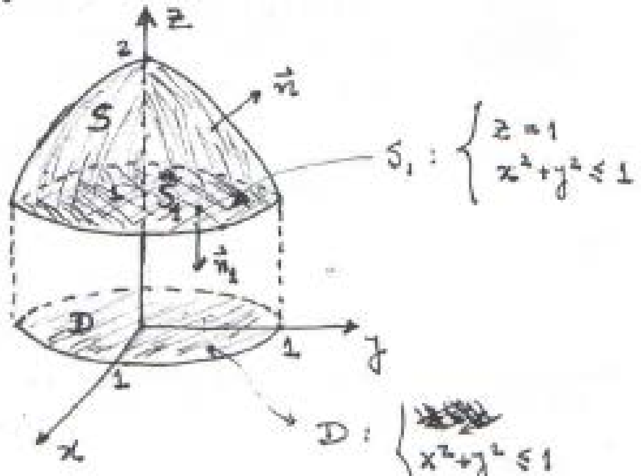
$$\left(\frac{14}{3} + \frac{15}{2} \right) 2\pi$$

$$= \frac{73}{6} \cdot 2\pi = \frac{73}{3}\pi$$

Questão 5

$$\vec{F}(x,y,z) = z\,\text{arctg}(y^2)\,\vec{i} + z^3 \ln(x^2+1)\,\vec{j} + z\,\vec{k}$$

$S: \; z = 2 - x^2 - y^2 \;, \quad 1 \leq z \leq 2 \,.$



Seja $\quad \tilde{S} := S \cup S_1$

S_1 orientada por $\vec{n}_1 = -\vec{k} = (0,0,-1)$

$dS_1 = \sqrt{1 + z_x^2 + z_y^2}\; dx\,dy = dx\,dy$

seja $\quad W$ o sólido limitado por $\tilde{S}$.

$\tilde{S} = \partial W$ está orientada positivamente e $\vec{F}$ é de classe C^1, então vale o teor. de Gauss,

$$\iint_{\tilde{S}} \vec{F} \cdot \vec{n}\, d\tilde{S} = \iiint_W \text{div}\, \vec{F}\, dV \,.$$

$\text{div}\, \vec{F} = 0 + 0 + 1 = 1$

logo $\quad \displaystyle\iint_{\tilde{S}} \vec{F} \cdot \vec{n}\, d\tilde{S} = \iiint_W dV$

Para calcular $\iiint_W dV$ usamos coord. cilíndricas

$$\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \\ z = z \end{cases} \qquad dV = r\,dr\,d\theta\,dz$$

$$W_{r,\theta,z} : \begin{cases} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \\ 1 \leq z \leq 2 - r^2 \end{cases} \qquad (1 \leq z \leq 2 - x^2 - y^2)$$

Então,

$$\iiint_W dV = \int_0^1 \int_0^{2\pi} \int_1^{2-r^2} r\,dz\,d\theta\,dr$$

$$= \int_0^1 \int_0^{2\pi} r(2 - r^2 - 1)\,d\theta\,dr = \int_0^1 \int_0^{2\pi} (r - r^3)\,d\theta\,dr$$

$$= \int_0^1 2\pi(r - r^3)\,dr$$

$$= 2\pi\left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Big|_0^1 = 2\pi\left(\frac{1}{2} - \frac{1}{4}\right)$$

$$= \pi/2$$

$$\therefore \quad \iiint_W dV = \iiint_W \operatorname{div} \vec{F}\,dV = \pi/2$$

Logo, $\displaystyle\iint_{\tilde{S}} \vec{F} \cdot \vec{n}\,d\tilde{S} = \pi/2$

$$\iint_{\tilde{S}} \vec{F}\cdot\vec{n}\, d\tilde{S} = \iint_{S} \vec{F}\cdot\vec{n}\, dS + \iint_{S_1} \vec{F}\cdot\vec{n}_1\, dS_1$$

e $$\iint_{S_1} \vec{F}\cdot\vec{n}_1\, dS_1 = \iint_{D} \left(1\cdot \operatorname{arctg}(y^2), 1^3 \ln(x^2+1), 1\right)\cdot(0,0,-1)\, dx\,dy$$

$$= \iint_{D} -dx\,dy = -\iint_{D} dx\,dy$$

$$= -\text{área}(D) = -\pi\cdot 1^2 = -\pi$$

$$\therefore \boxed{\iint_{S_1} \vec{F}\cdot\vec{n}_1\, dS_1 = -\pi}$$

Daí

$$\iint_{S} \vec{F}\cdot\vec{n}\, dS - \pi = \pi/2$$

$$\therefore \boxed{\iint_{S} \vec{F}\cdot\vec{n}\, dS = \frac{3\pi}{2}}$$