Universidade Federal da Paraíba
CCEN - Departamento de matemática
http://www.mat.ufpb.br

2^a Prova: Cálculo Vetorial e Geometria Analítica

12 de setembro de 2024

Prof: Pedro A. Hinojosa

Nome: ______________________________________ Matrícula: ____________________

1 (3,0 pts.) *Determine a equação cartesiana do plano* π *descrito nos casos abaixo.*

(a) π : *perpendicular ao vetor* $\vec{v} = [2, -1, -2]$ *cuja distância ao ponto* $P = (1, 1, 1)$ *é 4 unidades;*

(b) π : *contém os pontos* $A = (0, 1, 0)$, $B = (1, 2, 0)$ *e* $C = (1, 1, 3)$;

(c) π : *contem a reta* r *de equação* $r : \begin{cases} x &= 1 - 2t \\ y &= 2 + t \\ z &= -3 - t \end{cases} \quad t \in \mathbb{R}$

e é paralel0 ao vetor $[1, 2, -3]$

2 (2,0 pts.) *Considere o ponto* $P = (1, -2, -1)$ *e as retas*

$$r_1 : \begin{cases} x &= -2 - t \\ y &= 1 + 3t \\ z &= 4 - 2t \end{cases} \quad t \in \mathbb{R} \qquad e \qquad r_2 : \begin{cases} x &= 2 - s \\ y &= -3 + 2s \\ z &= 5 - s \end{cases} \quad s \in \mathbb{R}.$$

(a) Calcule $d(P, r_1)$, *a distância do ponto* P *à reta* r_1;

(b) calcule $d(r_1, r_2)$, *a distância entre as retas* r_1 *e* r_2.

3 (3,0 pts.) *dadas as retas* $r_1 : \begin{cases} x &= -1 - 2s \\ y &= 5 + 4s \\ z &= 3 + 2s \end{cases}$ *e* $r_2 : \begin{cases} x &= 1 + t \\ y &= 3 + t \\ z &= -7 + t \end{cases} \quad s,\ t \in \mathbb{R}.$

(a) Verifique que r_1 *e* r_2 *são retas reversas;*

(b) Determine as equações paramétricas da reta r *que interseta perpendicularmente as retas* r_1 *e* r_2.

4 (2,0 pts.) *Dados os pontos* $A = (1, 2, 0)$, $B = (1, 2, 3)$ *e* $C = (-1, -2, 2)$, *determine as coordenadas de um ponto* D *de modo que os pontos* A, B, C *e* D *sejam coplanares, o vetor* $\overrightarrow{AD}$ *seja ortogonal ao vetor* $\overrightarrow{AB}$ *e* $||\overrightarrow{AD}|| = 5$.

Boa Prova

CVGA Prova 2 2024.2

① Eq. cartesiana do plano π

(a) $\pi \perp \vec{v} = [2,-1,-2]$, $d(\pi, P) = 4$, $P = (1,1,1)$

$\pi \perp \vec{v} \Rightarrow \pi: 2x - y - 2z = d$

$d(\pi, P) = 4 \Rightarrow \dfrac{|2-1-2-d|}{\sqrt{4+1+4}} = 4$

$\Rightarrow \dfrac{|-d-1|}{3} = 4 \quad \therefore \quad |d+1| = 12$

$\Rightarrow d = 11$ ou $d = -13$

$\therefore \quad \pi: 2x - y - 2z = 11$ ou $\pi: 2x - y - 2z = -13$

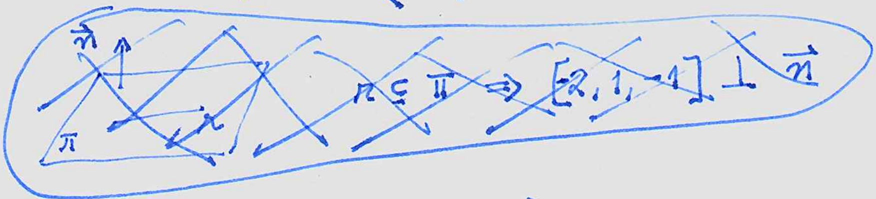
(b) $A, B, C \in \pi$.

$A = (0,1,0)$, $B = (1,2,0)$, $C = (1,1,3)$

$$\left.\begin{array}{l} \vec{AB} = [1,1,0] \\ \vec{AC} = [1,0,3] \\ \vec{AB} \times \vec{AC} = 3\vec{i} - 3\vec{j} - \vec{k} \end{array}\right\} \pi: 3x - 3y - z = 0 - 3 - 0 = -3$$

$3x - 3y - z = -3$

(c) $r \subseteq \pi$, $\pi \parallel [1,2,-3]$, $r: \begin{cases} x = 1-2t \\ y = 2+t \\ z = -3-t \end{cases}$



$$\left.\begin{array}{l} \vec{v} = [-2,1,-1] \parallel r \\ r \subseteq \pi \end{array}\right\} \Rightarrow [-2, 1, -1] \parallel \pi$$

$$\left.\begin{array}{l} [1,2,-3] \parallel \pi \\ [-2,1,-1] \parallel \pi \end{array}\right\} \Rightarrow [1,2,-3] \times [-2,1,-1] \perp \pi$$

$\Rightarrow [1, 7, 5] \perp \pi$

$\pi: x + 7y + 5z = 1 + 14 - 15 = 0$

$\pi: x + 7y + 5z = 0$

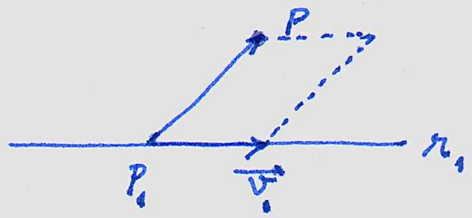
(2) $P = (1,-2,-1)$

$$r_1: \begin{cases} x = -2 - t \\ y = 1 + 3t \\ z = 4 - 2t \end{cases} \qquad r_2: \begin{cases} x = 2 - s \\ y = -3 + 2s \\ z = 5 - s \end{cases}$$

(a) $d(P, r_1) = ?$

$P \notin r_1$ ok! ✓

$(\vec{v}_1 \nparallel \overrightarrow{P_1P})$



$P_1 = (-2,1,4) \in r_1$

$\overrightarrow{P_1P} = [3,-3,-5]$

$\vec{v}_1 = [-1,3,-2]$

$$d(P,r_1) = \frac{\|\overrightarrow{P_1P} \times \vec{v}_1\|}{\|\vec{v}_1\|}$$

$\overrightarrow{P_1P} \times \vec{v}_1 = [21,11,6]$

$\|\vec{v}_1\| = \sqrt{1+9+4} = \sqrt{14}$

$\|\overrightarrow{P_1P} \times \vec{v}_1\| = \sqrt{441+121+36} = \sqrt{598}$

$$d(P,r_1) = \frac{\sqrt{598}}{\sqrt{14}} = \sqrt{\frac{299}{7}}$$

(b) $d(r_1,r_2) = ?$

$\vec{v}_1 = [-1,3,-2] \parallel r_1$

$\vec{v}_2 = [-1,2,-1] \parallel r_2$

$\vec{v}_1 \nparallel \vec{v}_2 \quad \therefore \quad r_1 \nparallel r_2$

$P_1 = (-2,1,4) \in r_1$

$P_2 = (2,-3,5) \in r_2$

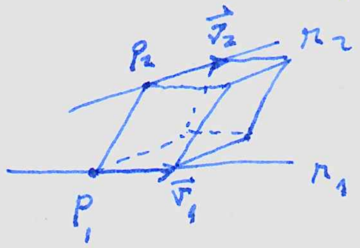
$\overrightarrow{P_1P_2} = [4,-4,1]$

$$\vec{v}_1 \times \vec{v}_2 \cdot \overrightarrow{P_1P_2} = \det\begin{pmatrix} -1 & 3 & -2 \\ -1 & 2 & -1 \\ 4 & -4 & 1 \end{pmatrix} = -2-8-12+16+4+3 = 1$$

$\neq 0.$

$\vec{v}_1 \times \vec{v}_2 \cdot \overrightarrow{P_1P_2} = 1 \neq 0$

$\vec{v}_1, \vec{v}_2$ e $\overrightarrow{P_1P_2}$ são l.i. $\therefore$ r_1 e r_2 são retas reversas.



$$d(r_1, r_2) = \frac{|\vec{v}_1 \times \vec{v}_2 \cdot \overrightarrow{P_1P_2}|}{\|\vec{v}_1 \times \vec{v}_2\|}$$

$\vec{v}_1 = [-1, 3, -2]$
$\vec{v}_2 = [-1, 2, -1]$

$\vec{v}_1 \times \vec{v}_2 = [1, 1, 1]$

$\|\vec{v}_1 \times \vec{v}_2\| = \sqrt{3}$.

$$\therefore \; d(r_1, r_2) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

(3) $r_1: \begin{cases} x = -1 - 2s \\ y = 5 + 4s \\ z = 3 + 2s \end{cases}$ $\quad r_2: \begin{cases} x = 1 + t \\ y = 3 + t \\ z = -7 + t \end{cases}$

(a) r_1 e r_2 são retas reversas

(b) Eq's paramétricas da reta r que corta perpend. as retas r_1 e r_2

solução:

(a) $\vec{v}_1 = [-2, 4, 2] \parallel r_1$
$\vec{v}_2 = [1, 1, 1] \parallel r_2$

$\vec{v}_1 \nparallel \vec{v}_2 \;\therefore\; r_1 \nparallel r_2$

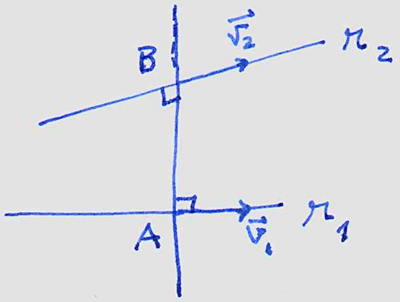
$P_1 = (-1, 5, 3) \in r_1$
$P_2 = (1, 3, -7) \in r_2$

$\overrightarrow{P_1P_2} = [2, -2, -10]$

$$\vec{v}_1 \times \vec{v}_2 \cdot \overrightarrow{P_1P_2} = \det\begin{pmatrix} -2 & 4 & 2 \\ 1 & 1 & 1 \\ 2 & -2 & -10 \end{pmatrix} = 20 - 4 + 8 - 4 - 4 + 40 = 56 \neq 0$$

$\vec{v}_1, \vec{v}_2$ e $\overrightarrow{P_1P_2}$ são l.i. $\therefore$ r_1 e r_2 são reversas

(b) $d(r_1, r_2) = ?$



$$A = (-1-2s,\ 5+4s,\ 3+2s)$$

$$B = (1+t,\ 3+t,\ -7+t)$$

$$\vec{AB} = [t+2s+2,\ t-4s-2,\ t-2s-10]$$

$$\vec{AB} \perp \vec{v_1} \Rightarrow -2(t+2s+2) + 4(t-4s-2) + 2(t-2s-10) = 0$$

$$\Rightarrow \quad 4t - 24s = 32$$

$$\boxed{t - 6s = 8}$$

$$\vec{AB} \perp \vec{v_2} \Rightarrow (t+2s+2) + (t-4s-2) + (t-2s-10) = 0$$

$$\boxed{3t - 4s = 10}$$

$$t = 6s + 8$$

$$3(6s+8) - 4s = 10$$

$$14s = -14 \quad \therefore \boxed{s = -1}$$

$$t = -6 + 8$$

$$\Rightarrow \boxed{t = 2}$$

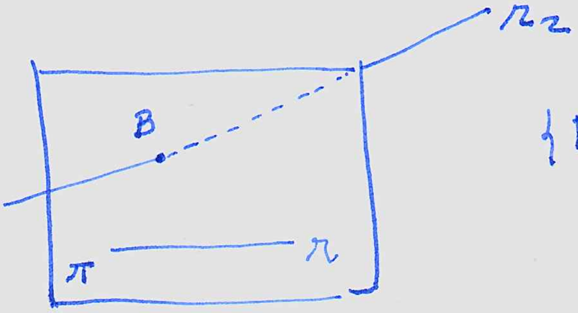
$$\left.\begin{array}{l} t = 2 \Rightarrow B = (3, 5, -5) \\ s = -1 \Rightarrow A = (1, 1, 1) \end{array}\right\} \Rightarrow \vec{AB} = [2, 4, -6]$$

$$\vec{AB} \parallel [1, 2, -3]$$

$$r: \begin{cases} x = 3 + \alpha \\ y = 5 + 2\alpha \\ z = -5 - 3\alpha, \quad \alpha \in \mathbb{R} \end{cases}$$

Outra solução:

Seja π o plano que contém a reta r_1 e é paralelo ao vetor $\vec{v_1} \times \vec{v_2}$



$$\{B\} = \pi \cap r_2$$

$$\vec{n} = (\vec{v}_1 \times \vec{v}_2) \times \vec{v}_1 \perp \pi$$

$$\vec{v}_1 = [-2, 4, 2]$$
$$\vec{v}_2 = [1, 1, 1]$$

$$\vec{v}_1 \times \vec{v}_2 = [2, 4, -6]$$

$$(\vec{v}_1 \times \vec{v}_2) \times \vec{v}_1 = [-32, -8, -16] \parallel [4, 1, \ ]$$

$$\pi : 4x + y + 2z = -4 + 5 + 6 = 7$$

$\uparrow$

$(-1, 5, 3) \in r_1$

$\therefore \ \pi : 4x + y + 2z = 7$

$\pi \cap r_2$

$$4(1+t) + (3+t) + 2(-7+t) = 7$$
$$7t - 7 = 7$$
$$7t = 14 \qquad \therefore \boxed{t = 2}$$

$t = 2 \Rightarrow B = (3, 5, -5)$

$r : \ldots$ etc

4 $A=(1,2,0)$, $B=(1,2,3)$, $C=(-1,-2,2)$

$D=(x,y,z)$

A, B, C e D coplanares

$\vec{AD} \perp \vec{AB}$

$\|\vec{AD}\| = 5$

$\vec{AB} = [0,0,3]$ $\vec{AC} = [-2,-4,2]$, $\vec{AD} = [x-1, y-2, z]$

$\vec{AB}$, $\vec{AC}$ e $\vec{AD}$ são l.d.

$$\therefore \det\begin{pmatrix} 0 & 0 & 3 \\ -2 & -4 & 2 \\ x-1 & y-2 & z \end{pmatrix} = 0$$

$$\Rightarrow -6(y-2) + 12(x-1) = 0$$

$$-6y + 12 + 12x - 12 = 0$$

$$2x - y = 0 \quad \therefore \boxed{y = 2x}$$

$$\vec{AD} \perp \vec{AB} \Rightarrow \vec{AB} \cdot \vec{AD} = 0$$

$$\Rightarrow 3z = 0 \quad \therefore \boxed{z = 0}$$

Daí $D = (x, 2x, 0)$ e $\vec{AD} = [x-1, 2x-2, 0]$

$$\|\vec{AD}\| = 5 \Rightarrow \sqrt{(x-1)^2 + 4(x-1)^2} = 5$$

$$\Rightarrow \sqrt{5(x-1)^2} = 5$$

$$\Rightarrow (x-1)^2 = 5$$

$$\Rightarrow x-1 = \pm\sqrt{5}$$

$$\therefore \boxed{x = 1 \pm \sqrt{5}}$$

~~$D = (\sqrt{5}, 2\sqrt{5}, 0)$ ou $D = (-\sqrt{5}$~~

$D = (1+\sqrt{5}, 2+2\sqrt{5}, 0)$ ou $D = (1-\sqrt{5}, 2-2\sqrt{5}, 0)$