UNIVERSIDADE FEDERAL DA PARAÍBA
CCEN - Departamento de Matemática
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Cálculo III - Manhã - 1^a Prova

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Questão 1 *(2.0 pts) Determine a massa do sólido W limitado dentro da semi esfera* $S_1 : \begin{cases} x^2 + y^2 + z^2 & = & 9 \\ z & \geq & 0 \end{cases}$ *e fora da esfera* $S_2 : x^2 + y^2 + (z-1)^2 = 1$, *sabendo-se que sua densidade é dada por* $f(x,y,z) = \frac{1}{x^2+y^2+z^2}$.

Questão 2 *(4.0 pts.) Calcule as integrais abaixo:*

(a) $\int_D \sqrt{(x^2+y^2)}dA, \quad D : x^2 + (y-1)^2 \leq 1$

(b) $\int_0^3 \int_x^{\sqrt{18-x^2}} sen(x^2+y^2+1)dydx,$

Questão 3 *(2.0 pts) Expresse a integral abaixo em coordenadas esféricas e calcule a integral obtida.*

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dzdydx}{1+x^2+y^2+z^2}$$

Questão 4 *(2.0 pts) Expresse a integral abaixo em coordenadas cilíndricas e calcule a integral obtida.*

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^4 \sqrt{1+x^2+y^2}dzdydx$$

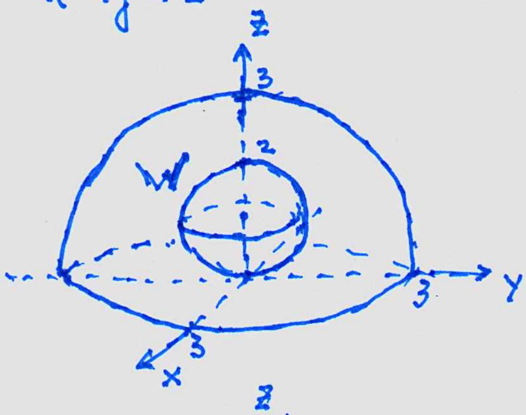
Boa Prova !!

C3 P1 - 2024.1 Manhã

(1) Massa de W

$$W : \begin{cases} \text{dentro da semi-esfera } \begin{cases} x^2+y^2+z^2 = 9 \\ z \geq 0 \end{cases} \\ \text{fora da esfera } x^2+y^2+(z-1)^2 = 1 \end{cases}$$

densidade: $f(x,y,z) = \dfrac{1}{x^2+y^2+z^2}$

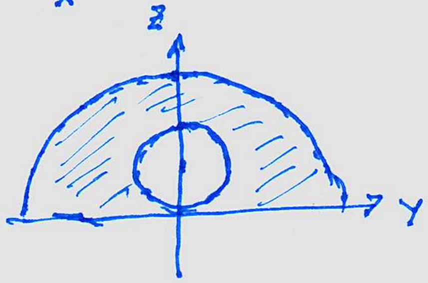


Solução:

$$M(W) = \int_W f\,dV$$

coord. Esféricas

$$\begin{cases} x = \rho \operatorname{sen}\phi \cos\theta \\ y = \rho \operatorname{sen}\phi \operatorname{sen}\theta \\ z = \rho\cos\phi \end{cases}$$



$$W_{\rho\phi\theta} : \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi/2 \\ 2\cos\phi \leq \rho \leq 3 \end{cases}$$

$$x^2+y^2+(z-1)^2 = 1$$
$$x^2+y^2+z^2-2z+1 = 1$$
$$\rho^2 - 2\rho\cos\phi = 0$$
$$\boxed{\rho = 2\cos\phi}$$

$$M(W) = \int_0^{2\pi}\int_0^{\pi/2}\int_{2\cos\phi}^{3} \frac{1}{\rho^2}\cdot\rho^2 \operatorname{sen}\phi \, d\rho\, d\phi\, d\theta$$

$$= 2\pi\int_0^{\pi/2} (3-2\cos\phi)\operatorname{sen}\phi\, d\phi$$

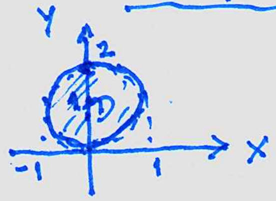
$$= 6\pi\left(-\cos\phi \Big|_0^{\pi/2}\right) - 4\pi\cdot\frac{1}{2}\operatorname{sen}^2\phi \Big|_0^{\pi/2}$$

$$= 6\pi\left(-(0-1)\right) - 2\pi(1-0) = 6\pi - 2\pi = \underline{\underline{4\pi}}$$

② Calcular:

(a) $\displaystyle\int_D \sqrt{x^2+y^2}\, dA \qquad D: x^2+(y-1)^2 \leq 1$

Solução



Coord. Polares
$\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \end{cases} \qquad D_{r\theta}: \begin{cases} 0 \leq \theta \leq \pi \\ 0 \leq r \leq 2\,\text{sen}\,\theta \end{cases}$

$$\int_D \sqrt{x^2+y^2}\, dA = \int_0^{\pi}\int_0^{2\,\text{sen}\,\theta} \sqrt{r^2}\cdot r\, dr\, d\theta$$

$x^2+(y-1)^2 = 1$
$x^2+y^2-2y = 0$
$r^2 - 2r\,\text{sen}\,\theta = 0$
$\Rightarrow \underline{r = 2\,\text{sen}\,\theta}$

$$= \int_0^{\pi}\int_0^{2\,\text{sen}\,\theta} r^2\, dr\, d\theta$$

$$= \int_0^{\pi} \frac{r^3}{3}\Big|_0^{2\,\text{sen}\,\theta} d\theta$$

$$= \frac{8}{3}\int_0^{\pi} \text{sen}^3\theta\, d\theta = \frac{8}{3}\int_0^{\pi}(1-\cos^2\theta)\,\text{sen}\,\theta\, d\theta$$

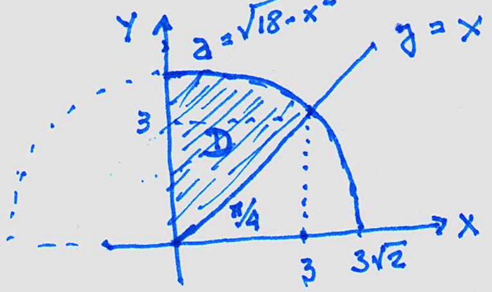
$$= \frac{8}{3}\left(-\cos\theta\Big|_0^{\pi}\right) + \frac{8}{3}\cdot\frac{1}{3}\cos^3\theta\Big|_0^{\pi}$$

$$= \frac{8}{3}\left(-(-1-1)\right) + \frac{8}{9}(-1-1) = \frac{16}{3} - \frac{8}{9}$$

$$= 2\cdot\frac{8}{3} - \frac{1}{3}\cdot\frac{8}{3} = \frac{8}{3}\left(2-\frac{1}{3}\right) = \frac{40}{9}$$

(b) $\displaystyle\int_0^3 \int_x^{\sqrt{18-x^2}} \operatorname{sen}(x^2+y^2+1)\,dy\,dx = I$

Solução



Coord. Polares

$$D_{r\theta}: \begin{cases} \pi/4 \leq \theta \leq \pi/2 \\ 0 \leq r \leq 3\sqrt{2} \end{cases}$$

$$\left(\sqrt{18} = 3\sqrt{2}\right)$$

$$I = \int_0^{3\sqrt{2}} \int_{\pi/4}^{\pi/2} \operatorname{sen}(r^2+1)\cdot r\,d\theta\,dr$$

$$= \frac{\pi}{4} \int_0^{3\sqrt{2}} \operatorname{sen}(r^2+1)\cdot r\,dr = \frac{\pi}{8} \int_0^{3\sqrt{2}} \operatorname{sen}(r^2+1)\cdot 2r\,dr$$

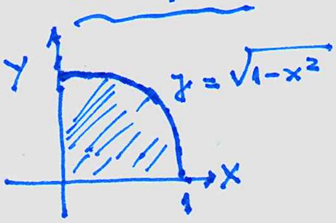
$$= \frac{\pi}{8}\left(-\cos(r^2+1)\Big|_0^{3\sqrt{2}}\right) = \frac{\pi}{8}\left(-\left(\cos 19 - \cos 1\right)\right)$$

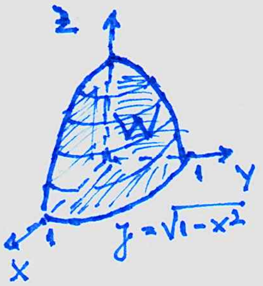
$$= \left(\cos 1 - \cos 19\right)\frac{\pi}{8}$$

(3) Calcular (usando coord. esféricas)

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{1+x^2+y^2+z^2}$$

Solução





$$W_{\rho\phi\theta}: \begin{cases} 0 \leq \rho \leq 1 \\ 0 \leq \theta \leq \pi/2 \\ 0 \leq \phi \leq \pi/2 \end{cases}$$

$$I = \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} \frac{1}{1+\rho^2} \cdot \rho^2 \operatorname{sen}\phi \, d\theta \, d\phi \, d\rho$$

$$= \frac{\pi}{2} \int_0^1 \int_0^{\pi/2} \frac{\rho^2}{1+\rho^2} \operatorname{sen}\phi \, d\phi \, d\rho$$

$$= \frac{\pi}{2} \int_0^1 \frac{\rho^2}{1+\rho^2} \underbrace{\left(-\cos\phi \Big|_0^{\pi/2} \right)}_{-(0-1)} d\rho$$

$$= \frac{\pi}{2} \int_0^1 \frac{\rho^2}{1+\rho^2} \, d\rho \qquad \left(\frac{\rho^2}{1+\rho^2} = 1 - \frac{1}{1+\rho^2} \right)$$

$$= \frac{\pi}{2} \int_0^1 \left(1 - \frac{1}{1+\rho^2} \right) d\rho$$

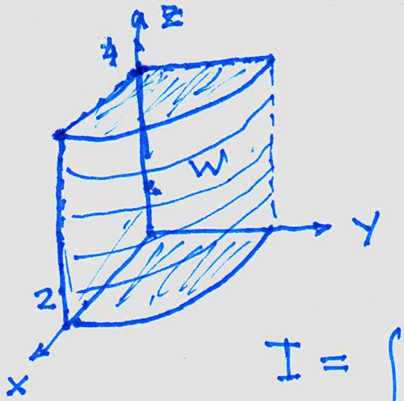
$$= \frac{\pi}{2} \cdot 1 - \frac{\pi}{2} \operatorname{arctg}\rho \Big|_0^1$$

$$= \pi/2 - \pi/2 \left(\pi/4 - 0 \right) = \frac{\pi}{2} - \frac{\pi^2}{8}$$

④ Calcular (usando coord. cilíndricas)

$$I = \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^4 \sqrt{1+x^2+y^2}\, dz\, dy\, dx$$

Solução:



$$W_{r\theta z} : \begin{cases} 0 \leq r \leq 2 \\ 0 \leq \theta \leq \pi/2 \\ 0 \leq z \leq 4 \end{cases}$$

$$I = \int_0^2 \int_0^{\pi/2} \int_0^4 \sqrt{1+r^2}\; r\, dz\, d\theta\, dr$$

$$= \int_0^2 \int_0^{\pi/2} 4\sqrt{1+r^2}\; r\, d\theta\, dr$$

$$= \pi \int_0^2 \sqrt{1+r^2} \cdot 2r\, dr$$

$$= \pi \cdot \frac{2}{3} (1+r^2)^{3/2} \Big|_0^2$$

$$= \frac{2\pi}{3} \left(5^{3/2} - 1 \right)$$