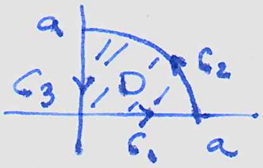
C3 **Lista 6**

(1) Encontrar o centroide da região, no 1º quadrante, limitada pela circunf. $x^2+y^2=a^2$

Solução




$$\partial D = C_1 \cup C_2 \cup C_3$$

$$C_1: \begin{cases} 0 \le x \le a \\ y = 0 \end{cases} \quad \alpha_1(t) = (t,0), \ t \in [0,a]$$

$$C_2: \begin{cases} x^2+y^2=a^2 \\ x \ge 0 \\ y \ge 0 \end{cases} \quad \alpha_2(t) = (a\cos t, a\,\text{sen}\,t) \quad t \in [0, \pi/2]$$

$$C_3: \begin{cases} x=0 \\ 0 \le y \le a \end{cases} \quad \alpha_3^-(t) = (0,t) \quad t \in [0,a]$$

$$\bar{x} = \frac{1}{2A} \oint_C x^2\,dy \ , \quad \bar{y} = \frac{1}{2A} \oint_C y^2\,dx \quad \left( A = \text{área}(D) = \frac{\pi a^2}{4} \right)$$

$$\int_{C_1} x^2\,dy = \int_0^a t^2 \cdot 0 = 0 \ , \quad \int_{C_2} x^2\,dy = \int_0^{\pi/2} a^2\cos^2 t \cdot a\cos t\,dt$$

$$\int_{C_2} x^2\,dy = a^3 \int_0^{\pi/2} \cos^3 t\,dt = a^3 \int_0^{\pi/2} (1-\text{sen}^2 t)\cos t\,dt$$

$$= a^3 \left( \text{sen}\,t \Big|_0^{\pi/2} \right) - \frac{a^3}{3} \text{sen}^3 t \Big|_0^{\pi/2}$$

$$= a^3(1-0) - \frac{a^3}{3}(1-0) = a^3 - \frac{a^3}{3} = \frac{2a^3}{3}$$

$$\int_{C_3} x^2\,dy = -\int_0^a 0\,dt = 0$$

$$\therefore \quad \bar{x} = \frac{2}{\pi a^2} \cdot \frac{2a^3}{3} = \frac{4}{3\pi} a$$

$$\bar{y} = \frac{1}{2A}\oint_C y^2\,dx$$

$$\int_{C_1} 0\,dt = 0 \quad , \quad \int_{C_2} y^2\,dx = \int_0^{\pi/2} a^2 \operatorname{sen}^2 t\,(-a\operatorname{sen} t)\,dt$$

$$= -a^3\int_0^{\pi/2} \operatorname{sen}^3 t\,dt$$

$$= -a^3\int_0^{\pi/2} (1-\cos^2 t)\operatorname{sen} t\,dt$$

$$= a^3\cos t\Big|_0^{\pi/2} - \frac{a^3}{3}\cos^3 t\Big|_0^{\pi/2}$$

$$= a^3 - a^3/3 = \frac{2a^3}{3}$$

$$\int_{C_3} t^2\cdot 0 = 0 \qquad \therefore \qquad \bar{y} = \frac{2}{\pi a^2}\cdot\frac{2a^3}{3} = \frac{4a}{3\pi}$$

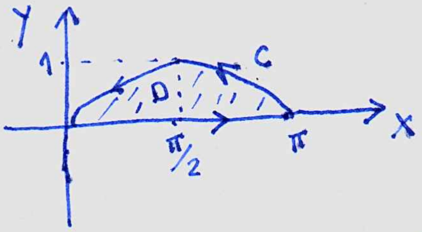
$$(\bar{x},\bar{y}) = \left(\frac{4a}{3\pi}, \frac{4a}{3\pi}\right)$$

(3) Use Green para calcular $\int_C \vec{F}\cdot d\vec{r}$.

$\vec{F} = (\sqrt{x}+y^3)\,\vec{i} + (x^2+\sqrt{y})\,\vec{j}$

C é o arco da curva $y = \operatorname{sen} x$ de $(0,0)$ até $(\pi,0)$ e o segmento de reta $(\pi,0)$ até $(0,0)$

Solução




orientando C positivamente,

$$\oint_C \vec{F}\cdot d\vec{r} = \int_D (Q_x - P_y)\,dA$$

D é o domínio t.q. $\partial D = C$

$P = \sqrt{x}+y^3$ e $Q = x^2+\sqrt{y}$

$$D: \begin{cases} 0 \leq x \leq \pi \\ 0 \leq y \leq \operatorname{sen} x \end{cases}$$

$$Q_x = 2x, \quad \cancel{Q_y} \quad P_y = 3y^2$$

$$\iint_D (Q_x - P_y)\, dA = \int_0^{\pi} \int_0^{\operatorname{sen} x} (2x - 3y^2)\, dy\, dx$$

$$= \int_0^{\pi} (2x \operatorname{sen} x - \operatorname{sen}^3 x)\, dx$$

$$= \underbrace{\int_0^{\pi} 2x \operatorname{sen} x\, dx}_{I_1} - \underbrace{\int_0^{\pi} \operatorname{sen}^3 x\, dx}_{I_2}$$

$I_1 = 2\pi$ (calcule por partes)

$I_2 = 4/3$ $\left( \operatorname{sen}^3 x = \operatorname{sen}^2 x \cdot \operatorname{sen} x = (1 - \cos^2 x) \operatorname{sen} x \right)$

$$\therefore \iint_D (Q_x - P_y)\, dA = 2\pi - 4/3$$

Agora note que mudamos a orientação de C ~~para~~ para ter D orientado positivamente.

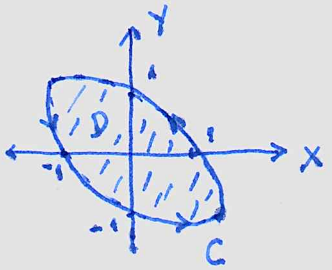
Então $$\oint_C \vec{F} \cdot d\vec{r} = 4/3 - 2\pi$$

④ Usar o Teor. de Green para calcular

$$\oint_C \operatorname{sen} y\, dx + x\cos y\, dy$$

ao longo da elipse $x^2 + xy + y^2 = 1$
com orientação positiva.

Solução




$$\oint_C P\,dx + Q\,dy = \int_D (Q_x - P_y)\,dA$$

$\partial D = C$

$P = \operatorname{sen} y \implies P_y = \cos y$

$Q = x\cos y \implies Q_x = \cos y$

$\therefore \quad Q_x - P_y = 0$

Daí

$$\oint_C \operatorname{sen} y\, dx + x\cos y\, dy = \int_D 0 \cdot dA = 0$$

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⑥ Área da região $\mathcal{R}$ limitada pela cardioide

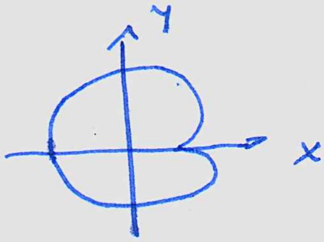
$\gamma(t) = \big(x(t), y(t)\big)$, $t \in [0, 2\pi]$

onde $\begin{cases} x = 2\cos t - \cos 2t \\ y = 2\operatorname{sen} t - \operatorname{sen} 2t \end{cases}$

Solução:

$$\text{área}(\mathcal{R}) = \frac{1}{2}\int_{\partial R} x\,dy - y\,dx$$

$$x = 2\cos t - \cos 2t$$




$$dx = (-2\,\text{sen}\,t + 2\,\text{sen}\,2t)\,dt$$

$$y = 2\,\text{sen}\,t - \text{sen}\,2t$$

$$dy = (2\cos t - 2\cos 2t)\,dt$$

$$x\,dy = (4\cos^2 t - 4\cos t\cos 2t - 2\cos t\cos 2t + 2\cos^2 2t)\,dt$$

$$y\,dx = (-4\,\text{sen}^2 t + 4\,\text{sen}\,t\,\text{sen}\,2t + 2\,\text{sen}\,t\,\text{sen}\,2t - 2\,\text{sen}^2 2t)\,dt$$

$$x\,dy - y\,dx = \left[4 + 2 - 6(\cos t\cos 2t + \text{sen}\,t\,\text{sen}\,2t)\right] dt$$

$$= 6(1 - \cos t\cos 2t - \text{sen}\,t\,\text{sen}\,2t)\,dt$$

$$\cos t \cdot \cos 2t = \cos t(\cos^2 t - \text{sen}^2 t) = \cos^3 t - \text{sen}^2 t\cos t$$

$$= (1 - \text{sen}^2 t)\cos t - \text{sen}^2 t\cos t$$

$$= \cos t - 2\,\text{sen}^2 t\cos t$$

Daí $\displaystyle\int_0^{2\pi} \cos t\cos 2t\,dt = \int_0^{2\pi} (\cos t - 2\,\text{sen}^2 t\cos t)\,dt$

$$= -\text{sen}\,t\Big|_0^{2\pi} + \frac{2}{3}\,\text{sen}^3 t\Big|_0^{2\pi}$$

$$= \underline{\underline{0}}$$

$$\text{sen}\,t\,\text{sen}\,2t = \text{sen}\,t \cdot 2\,\text{sen}\,t\cos t$$

$$= 2\,\text{sen}^2 t\cos t$$

$$\therefore \int_0^{2\pi} \text{sen}\,t\,\text{sen}\,2t\,dt = 0$$

Logo,

$$\text{área}(R) = \frac{1}{2}\int_0^{2\pi} (6 - 6\cos t\cos 2t - 6\,\text{sen}\,t\,\text{sen}\,2t)\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} 6\,dt = 3 \cdot 2\pi = \underline{\underline{6\pi}}$$