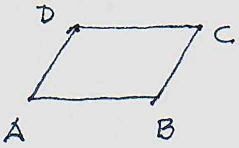
CVGA - LISTA 3

① $\vec{AB} = \vec{i} + \vec{j} - \vec{k}$, $\vec{AD} = 2\vec{i} + \vec{j} + 4\vec{k}$



Área do paralelogramo ABCD = ?

Solução: Área do Paralelogramo $= \| \vec{AB} \times \vec{AD} \|$

$$\vec{AB} \times \vec{AD} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 2 & 1 & 4 \end{pmatrix} = 5\vec{i} - 6\vec{j} - \vec{k}$$

$$\| \vec{AB} \times \vec{AD} \| = \sqrt{25 + 36 + 1} = \sqrt{62}$$

② $\vec{AB} = -\vec{i} + \vec{j}$, $\vec{AC} = \vec{j} + 3\vec{k}$

área do ΔABC

Solução:

$$\text{Área } (\Delta ABC) = \frac{1}{2} \| \vec{AB} \times \vec{AC} \|$$

$$\vec{AB} \times \vec{AC} = 3\vec{i} - 3\vec{j} + \vec{k} , \quad \| \vec{AB} \times \vec{AC} \| = \sqrt{9 + 9 + 1} = \sqrt{19}$$

③ Dados $\vec{u}, \vec{v}, \vec{w}$ t.q. $[\vec{u}, \vec{v}, \vec{w}] = 6$ calcular

$$[2\vec{u} - 3\vec{v} + \vec{w}, -\vec{u} + \vec{v}, \vec{v} - 3\vec{w}]$$

Solução

$$(2\vec{u} - 3\vec{v} + \vec{w}) \times (-\vec{u} + \vec{v}) = -2 \cancel{\vec{u} \times \vec{u}} + 2\vec{u} \times \vec{v} + 3\vec{v} \times \vec{u} - 3 \cancel{\vec{v} \times \vec{v}} - \vec{w} \times \vec{u} + \vec{w} \times \vec{v}$$

$$= 2\vec{u} \times \vec{v} - 3\vec{u} \times \vec{v} + \vec{u} \times \vec{w} - \vec{v} \times \vec{w}$$

$$= -\vec{u} \times \vec{v} + \vec{u} \times \vec{w} - \vec{v} \times \vec{w}$$

$$(2\vec{u} - 3\vec{v} + \vec{w}) \times (-\vec{u} + \vec{v}) \cdot (\vec{v} - 3\vec{w}) = -\cancel{\vec{u} \times \vec{v} \cdot \vec{v}} + \vec{u} \times \vec{w} \cdot \vec{v} - \cancel{\vec{v} \times \vec{w} \cdot \vec{v}} - \vec{u} \times \vec{v} \cdot 3\vec{w} - \vec{u} \times \vec{v} \cdot 3w + \cancel{\vec{v} \times \vec{w} \cdot 3\vec{w}}$$

$$= \vec{u} \times \vec{w} \cdot \vec{v} - 3\vec{u} \times \vec{v} \cdot \vec{w} - 3\vec{u} \times \vec{v} \cdot \vec{w}$$

$$= -\vec{u} \times \vec{v} \cdot \vec{w} - 6\vec{u} \times \vec{v} \cdot \vec{w} = 7\vec{u} \times \vec{v} \cdot \vec{w}$$

$$= 7[\vec{u}, \vec{v}, \vec{w}] = 7 \cdot 6 = 42$$

(4) Decompor $\vec{w}$ como soma de dois vetores $\vec{u}$ e $\vec{v}$ $(\vec{w} = \vec{u} + \vec{v})$ com $\vec{u}$ paralelo ao vetor $[0,1,3]$ e $\vec{v}$ ortogonal a $[0,1,3]$

<u>Solução</u> $\quad (\vec{w} = [-1,-3,2])$

$$\vec{w} = \vec{u} + \vec{v}$$

$$\vec{u} \parallel [0,1,3] \Rightarrow \vec{u} = \lambda[0,1,3], \quad \lambda \in \mathbb{R}$$

$$\left.\begin{array}{l} \vec{v} \perp [0,1,3] \\ \vec{v} = [v_1, v_2, v_3] \end{array}\right\} \Rightarrow v_2 + 3v_3 = 0$$

$$\vec{u} + \vec{v} = [v_1, v_2 + \lambda, v_3 + 3\lambda] = \vec{w} = [-1,-3,2]$$

$$\Rightarrow \underline{\underline{v_1 = -1}}, \quad v_2 + \lambda = -3, \quad v_3 + 3\lambda = 2$$

$$\boxed{\lambda = -3 - v_2}, \quad v_3 + 3(-3 - v_2) = 2$$

$$v_3 - 3v_2 = 11$$

$$\begin{array}{l} v_2 + 3v_3 = 0 \\ -3v_2 + v_3 = 11 \end{array} \qquad \boxed{\begin{array}{l} v_2 = -33/10 \\ v_3 = 11/10 \end{array}} \qquad \lambda = -3 + \frac{33}{10} = \frac{3}{10}$$

Daí

$$\vec{u} = [0, 3/10, 9/10], \quad \vec{v} = [-1, -33/10, 11/10]$$

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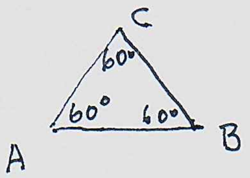
(5)



ΔABC Equilátero

$$\|\vec{AB}\| = \|\vec{BC}\| = \|\vec{AC}\| = 1$$

Calcular $\vec{AB} \cdot \vec{AC} + \vec{BA} \cdot \vec{BC} + \vec{CA} \cdot \vec{CB}$

<u>Solução</u>

$$\vec{AB} \cdot \vec{AC} = \underbrace{\|\vec{AB}\|}_{1}\underbrace{\|\vec{AC}\|}_{1} \cos \underbrace{\measuredangle(AB, AC)}_{60^\circ} \qquad \cos 60^\circ = 1/2$$

$$= 1/2$$

Analogamente, $\vec{BA} \cdot \vec{BC} = 1/2$ e $\vec{CA} \cdot \vec{CB} = 1/2$

$$\therefore \vec{AB} \cdot \vec{AC} + \vec{BA} \cdot \vec{BC} + \vec{CA} \cdot \vec{CB} = 3/2$$

(6) $A = (1,0,1)$
$B = (-1,0,2)$
$C = (1,1,1)$

$\triangle ABC$ é retângulo ?

Solução :

$\vec{AB} = [-2, 0, 1]$
$\vec{AC} = [0, 1, 0]$

$\vec{AB} \perp \vec{AC}$ $(\vec{AB} \cdot \vec{AC} = 0)$

$\therefore$ $\triangle ABC$ é ~~...~~ retângulo

(7) $\vec{u} = 2\vec{i} + \vec{j} - \vec{k}$
$\vec{v} = -\vec{i} + 2\vec{k}$
$\vec{w} = 2\vec{j} - 3\vec{k}$

$\vec{u}$, $\vec{v}$ e $\vec{w}$ ~~det~~ determinam um paralelepípedo. Calcular seu volume.

Solução

$\vec{u}$, $\vec{v}$ e $\vec{w}$ não ~~são~~ coplanares, de fato :

$$\det\begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 2 \\ -1 & 2 & -3 \end{pmatrix} = 0 + 0 + 2 - 0 - 8 - 3 = -9 \neq 0 .$$

Assim, $\vec{u}$, $\vec{v}$ e $\vec{w}$ determinam um paralelepípedo.

$$\text{volume} = \left| \vec{u} \times \vec{v} \cdot \vec{w} \right|$$

$$\vec{u} \times \vec{v} = 2\vec{i} - 3\vec{j} + \vec{k}$$

$$\vec{u} \times \vec{v} \cdot \vec{w} = 0 - 6 - 3 = -9$$

alguma relação?

... Toda !!

$$\text{volume} = |-9| = 9$$

(8) $A = (1,2,0)$
$B = (1,2,3)$
$C = (-1,-2,2)$

Determinar D de modo que A, B, C e D sejam coplanares e $\vec{AD} \perp \vec{AB}$

Solução:

A, B, C, D são coplanares $\iff$ $\vec{AB}, \vec{AC}$ e $\vec{AD}$ são l.d.

$\vec{AD} \perp \vec{AB} \iff \vec{AD} \bullet \vec{AB} = 0$

Seja $D = (x, y, z)$.

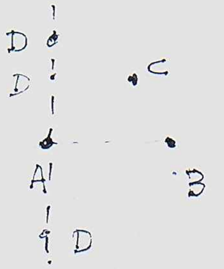
$\vec{AB} = [0, 0, 3]$
$\vec{AC} = [-2, -4, 2]$
$\vec{AD} = [x-1, y-2, z]$

$$\det \begin{pmatrix} 0 & -2 & x-1 \\ 0 & -4 & y-2 \\ 3 & 2 & z \end{pmatrix} = 3[-2(y-2)+4(x-1)]$$

$$= -6y - 6 + 4x - 4 = 4x - 6y - 10$$

$$\boxed{4x - 6y - 10 = 0}$$

$\vec{AD} \bullet \vec{AB} = 0 \Rightarrow 3z = 0$
$= \boxed{z = 0}$



As coord. (x, y, z) do pto D verificam $4x - 6y = 10$, $z = 0$
por exemplo $D = (1, -1, 0)$

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(9) $\vec{u} = \frac{1}{\sqrt{3}}(\vec{i} + \vec{j} - \vec{k})$ $\vec{v} = \frac{1}{\sqrt{2}}(\vec{j} + \vec{k})$, $\vec{w} = \frac{1}{\sqrt{6}}(2\vec{i} - \vec{j} + \vec{k})$

$\{\vec{u}, \vec{v}, \vec{w}\}$ é base ortonormal de $\mathbb{R}^3$. Determinar as coord. de $\vec{a} = 3\vec{i} - 5\vec{j} + 2\vec{k}$ nessa base.

Solução

$\vec{u} \cdot \vec{v} = \frac{1}{\sqrt{3}\sqrt{2}}(1 - 1) = 0 \quad \therefore \quad \vec{u} \perp \vec{v}$

$\vec{u} \cdot \vec{w} = \frac{1}{\sqrt{3}\sqrt{6}}(2 - 1 - 1) = 0 \quad \therefore \quad \vec{u} \perp \vec{w}$

$$\vec{v} \cdot \vec{w} = \frac{1}{\sqrt{2}\sqrt{6}}(-1+1) = 0.$$

$\therefore \ \{\vec{u}, \vec{v}, \vec{w}\}$ é uma base ortogonal de $\mathbb{R}^3$.

Além disso,

$$\|\vec{u}\|^2 = \frac{1}{3}(1+1+1) = 1, \quad \|\vec{v}\|^2 = \frac{1}{2}(1+1) = 1$$

$$\|w\|^2 = \frac{1}{6}(4+1+1) = 1$$

$\therefore \ \{\vec{u}, \vec{v}, \vec{w}\}$ é uma base ortonormal de $\mathbb{R}^3$

coord. de $\vec{a} = 3\vec{i} - 5\vec{j} + 2\vec{k}$ nessa base.

$$\vec{a} = x\vec{u} + y\vec{v} + z\vec{w}.$$

$$x = \vec{a} \cdot \vec{u} = \frac{1}{\sqrt{3}}(3-5-2) = -4/\sqrt{3}$$

$$y = \vec{a} \cdot \vec{v} = \frac{1}{\sqrt{2}}(-5+2) = 7/\sqrt{2}$$

$$z = \vec{a} \cdot \vec{w} = \frac{1}{\sqrt{6}}(6+5+2) = 13/\sqrt{6}$$

... etc

(10) $\vec{v} \perp [1,1,0]$

$\vec{v} \perp [-1,0,1]$ $\quad \|\vec{v}\| = 2, \quad \cos\measuredangle(\vec{v}, \vec{j}) > 0$

~~cos ∡(v, j)~~

Determinar o vetor $\vec{v}$

Solução

$$\begin{array}{l} v \perp [1,1,0] \\ \vec{v} \perp [-1,0,1] \end{array} \Rightarrow \vec{v} \parallel [1,1,0] \times [-1,0,1]$$

$$[1,1,0] \times [-1,0,1] = [1,-1,1] \quad \therefore \vec{v} = \lambda[1,-1,1]$$

$$\|\vec{v}\| = \sqrt{\lambda^2 + \lambda^2 + \lambda^2} = \sqrt{3}\,|\lambda| = 2 \Rightarrow |\lambda| = 2/\sqrt{3}$$

$$\cos\measuredangle(\vec{v}, \vec{j}) > 0 \Leftrightarrow \vec{v} \cdot \vec{j} > 0 \Rightarrow -\lambda > 0$$

$$\Rightarrow \lambda < 0 \quad \therefore \lambda = \frac{-2}{\sqrt{3}}.$$

Daí $\vec{v} = \frac{-2}{\sqrt{3}}[1, -1, 1]$

(11) $\vec{AB} = [0, 1, 3]$

$\vec{AC} = [-1, 1, 0]$

área $(\Delta ABC) = ?$

Solução

área$(\Delta ABC) = \frac{1}{2} \| \vec{AB} \times \vec{AC} \|$

$\vec{AB} \times \vec{AC} = [-3, -3, 1]$, $\| \vec{AB} \times \vec{AC} \| = \sqrt{9+9+1} = \sqrt{19}$

(12) $\measuredangle(\vec{u}, \vec{v}) = \pi/6$ $\quad \|\vec{u}\| = 1$, $\|\vec{v}\| = 7$

calcular $\| \vec{u} \times \vec{v} \|$ e $\| (2\vec{u}) \times (3\vec{v}) \|$

Solução

$$\| \vec{u} \times \vec{v} \| = \|\vec{u}\| \|\vec{v}\| \left| \operatorname{sen} \measuredangle(u, v) \right|$$

$$= 1 \cdot 7 \cdot \operatorname{sen} \pi/6 = 7 \cdot 1/2 \qquad (\pi/6 \simeq 30^\circ)$$

$$= 7/2$$

$$(2\vec{u}) \times (3\vec{v}) = 6 \, \vec{u} \times \vec{v}$$

$$\| (2\vec{u}) \times (3\vec{v}) \| = 6 \| \vec{u} \times \vec{v} \| = 6 \cdot 7/2 = 21$$