UNIVERSIDADE FEDERAL DA PARAÍBA
CCEN - Departamento de Matemática
http://www.mat.ufpb.br

Cálculo III - 1^a Prova

João Pessoa, 21 de fevereiro de 2024
Professor: Pedro A. Hinojosa

Nome: ______________________________ Matrícula: ________________

Questão 1 *(2.0 pts) Uma lâmina no plano* XY *é limitada dentro da circunferência* $x^2 + (y-2)^2 = 4$ *e fora da circunferência* $x^2 + y^2 = 4$*. Calcule a massa da lâmina se a densidade da mesma é dada por* $f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$.

Questão 2 *(6.0 pts.) Calcule as integrais abaixo:*

(a) $\int_D (x^2 - y^2) sen(x+y) dA, \quad D = \{(x,y) \in \mathbb{R}^2 : 0 \le x+y \le \pi,\ 0 \le x-y \le \pi\}$

(b) $\int_W (x^2 + y^2 + z^2) dV, \quad W$ *é a região limitada superiormente pela esfera* $x^2 + y^2 + z^2 = 16$ *e inferiormente pelo cone* $z = \sqrt{x^2 + y^2}$;

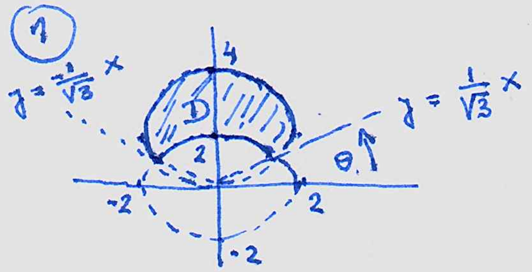
(c) $\int_W (x^2 + y^2) dV, \quad W$ *é a região interior ao cilindro* $x^2 + y^2 = 1$ *e à esfera* $x^2 + y^2 + z^2 = 4$.

Questão 3 *(2.0 pts) Faça um esboço do sólido* W *cujo volume é dado pela integral,* $\int_0^{\frac{\pi}{3}} \int_0^{2\pi} \int_0^{sec(\phi)} \rho^2 sen(\phi) d\rho d\theta d\phi$ *e calcule essa integral.*

Boa Prova !!

P1 - C3 2023.2

①



$\operatorname{tg}\theta = \frac{1}{\sqrt{3}}$

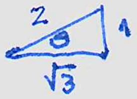
$$\begin{cases} x^2+y^2 = 4 \\ x^2+(y-2)^2 = 4 \end{cases}$$

$$\underbrace{x^2+y^2}_{4} - 4y + \cancel{4} = \cancel{4}$$

$4 = 4y$

$\boxed{y = 1}$

$$\left.\begin{matrix} y = 1 \\ x^2+y^2 = 4 \end{matrix}\right\} \Rightarrow x = \pm\sqrt{3}$$



$\operatorname{tg}\theta = 1/\sqrt{3} \Rightarrow \theta = 30°$, $\boxed{\theta = \pi/6}$ $\left(\begin{matrix} \operatorname{sen}\pi/6 = 1/2 \\ \cos\pi/6 = \sqrt{3}/2 \end{matrix} \right)$

$$D_{r\theta}: \begin{cases} \frac{\pi}{6} \leq \theta \leq 5\pi/6 \\ 2 \leq r \leq 4\operatorname{sen}\theta \end{cases}$$

$$\underbrace{x^2+y^2}_{r^2} - 4y + \cancel{4} = \cancel{4}$$

$r^2 - 4r\operatorname{sen}\theta = 0$

$\Rightarrow \underline{r = 4\operatorname{sen}\theta}$

densidade da placa : $f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$ $\left(f(r,\theta) = \frac{1}{r} \right)$

$$M = \int_D f\,dA = \int_{D_{r\theta}} f(r,\theta)\cdot r\,dr\,d\theta$$

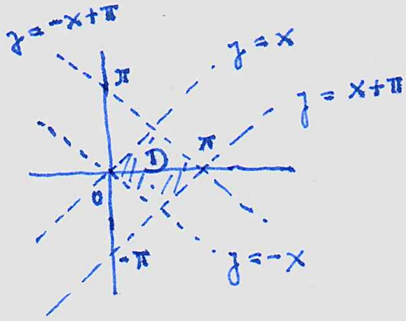
$$= \int_{\pi/6}^{5\pi/6}\int_2^{4\operatorname{sen}\theta} \frac{1}{r}\cdot r\,dr\,d\theta = \int_{\pi/6}^{5\pi/6} (4\operatorname{sen}\theta - 2)\,d\theta$$

$$= -4\cos\theta\Big|_{\pi/6}^{5\pi/6} - 2\left(\frac{5\pi}{6} - \frac{\pi}{6}\right)$$

$$= -4\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) - 2\cdot\frac{4\pi}{6} = 4\sqrt{3} - \frac{4}{3}\pi$$

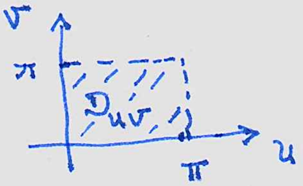
② (a) $\int_D (x^2-y^2)\,\text{sen}(x+y)\,dA$

$D = \{(x,y)\in\mathbb{R}^2 : \; 0 \leq x+y \leq \pi \;,\; 0 \leq x-y \leq \pi\}$



$=$

$$\begin{cases} u = x+y \\ v = x-y \end{cases}$$



$$\begin{cases} 0 \leq u \leq \pi \\ 0 \leq v \leq \pi \end{cases}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2 \qquad \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = 1/2$$

$$x^2-y^2 = (x+y)(x-y) = u\cdot v$$

$$\int_D (x^2-y^2)\,\text{sen}(x+y)\,dA = \int_0^\pi\int_0^\pi u\cdot v\,\text{sen}\,u\,dv\,du$$

$$= \int_0^\pi u\,\text{sen}\,u\left(\frac{1}{2}v^2\Big|_0^\pi\right)du = \frac{\pi^2}{2}\int_0^\pi u\,\text{sen}\,u\,du$$

$U = u \longrightarrow dU = du$
$dV = \text{sen}\,u\,du \rightarrow V = -\cos u$

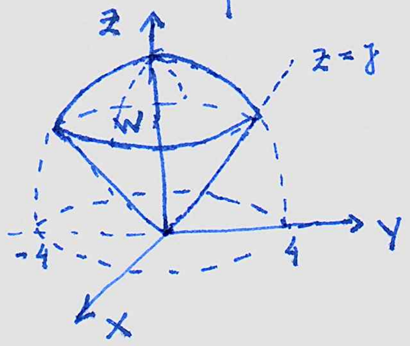
$$\int_0^\pi u\,\text{sen}\,u\,du = -u\cos u\Big|_0^\pi - \int_0^\pi -\cos u\,du = \pi + \int_0^\pi \cos u\,du$$

$$= \pi + \cancelto{0}{\text{sen}\,u\Big|_0^\pi} = \pi$$

$$\therefore \int_D (x^2-y^2)\,\text{sen}(x+y)\,dA = \frac{\pi^2}{2}\cdot\pi = \frac{\pi^3}{2}$$

(b) $\int_W (x^2+y^2+z^2)\,dV$

W região limitada por cima pela esfera $x^2+y^2+z^2=16$
e por baixo pelo cone $z=\sqrt{x^2+y^2}$



Coord. Esféricas

$$W_{\rho\phi\theta}: \begin{cases} 0 \leq \rho \leq 4 \\ 0 \leq \phi \leq \pi/4 \\ 0 \leq \theta \leq 2\pi \end{cases}$$

$$\begin{cases} x = \rho \,\text{sen}\,\phi \cos\theta \\ y = \rho\,\text{sen}\,\phi\,\text{sen}\,\theta \\ z = \rho\cos\phi \end{cases} \qquad J = \rho^2\,\text{sen}\,\phi \qquad x^2+y^2+z^2=\rho^2$$

$$\int_W (x^2+y^2+z^2)\,dV = \int_0^4\int_0^{2\pi}\int_0^{\pi/4} \rho^2\cdot\rho^2\,\text{sen}\,\phi\, d\phi\, d\theta\, d\rho$$

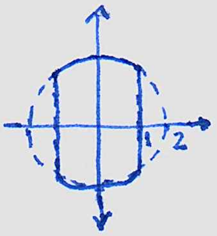
$$= 2\pi\int_0^4 \rho^4\left(-\cos\phi\,\Big|_0^{\pi/4}\right)d\rho = 2\pi\int_0^4 -\left(\frac{\sqrt{2}}{2}-1\right)\rho^4\,d\rho$$

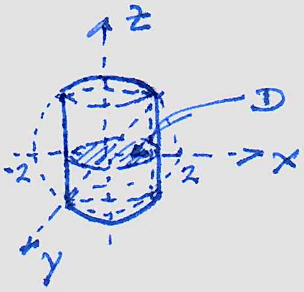
$$= \left(1-\frac{\sqrt{2}}{2}\right)\cdot 2\pi\int_0^4 \rho^4\,d\rho = \left(1-\frac{\sqrt{2}}{2}\right)\cdot 2\pi\cdot\frac{1}{5}\,4^5$$

$$= \frac{(2-\sqrt{2})\,4^5}{5}\,\pi$$

(c) $\int_W (x^2+y^2)\,dV$ W interior ao cilindro $x^2+y^2=1$ e à esfera $x^2+y^2+z^2=4$.

$=$





coord. cilíndricas

$$\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \\ z = z \end{cases}$$

$J = r$, $x^2+y^2=r^2$

$$W_{r\theta z} : \begin{cases} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \\ -\sqrt{4-r^2} \leq z \leq \sqrt{4-r^2} \end{cases}$$

$$W : \begin{cases} (x,y) \in D \\ -\sqrt{4-x^2-y^2} \leq z \leq \sqrt{4-x^2-y^2} \end{cases}$$

$$\int_W (x^2+y^2)\,dV = \int_0^1 \int_0^{2\pi} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r^2 \cdot r \, dz\, d\theta\, dr$$

$$= 2\pi \int_0^1 r^3 \cdot 2\sqrt{4-r^2}\, dr$$

$$= -2\pi \int_0^1 r^2 \left(-2r\sqrt{4-r^2}\right) dr$$

$$= -2\pi \int_0^1 r^2 \sqrt{4-r^2} \cdot (-2r\,dr)$$

$$\left[\begin{array}{l} u = 4-r^2 \\ du = -2r\,dr \\ \hookrightarrow r^2 = 4-u \\ r = 0 \Rightarrow u = 4 \\ r = 1 \Rightarrow u = 3 \end{array}\right\}$$

$$-2\pi \int_4^3 (4-u)\sqrt{u}\, du = \cancel{-}\, 2\pi \int_3^4 \left(4u^{1/2} - u^{3/2}\right) du$$

$$= 2\pi \left[4 \cdot \frac{2}{3} u^{3/2} - \frac{2}{5} u^{5/2}\right] \Bigg|_3^{\substack{3\\4}}$$

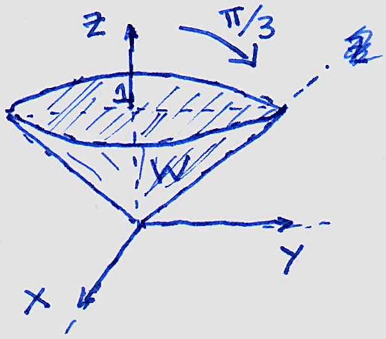
$$= 2\pi \left(\left[\frac{2}{3}\, 4^{5/2} - \frac{2}{5}\, 4^{5/2}\right] - \left[4 \cdot \frac{2}{3}\, 3^{3/2} - \frac{2}{5}\, 3^{5/2}\right) \right]$$

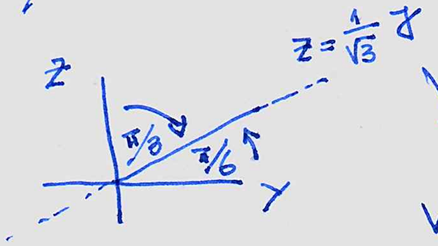
$$= 2\pi \left(\frac{4}{15} \cdot 4^{5/2} - \frac{11}{15} \cdot 2 \cdot 3^{3/2}\right) \ldots \text{etc} \quad \frac{4}{15}\left(2 \cdot 4^{5/2} - 11 \cdot 3^{3/2}\right)\pi$$

③ $$\int_0^{\pi/3}\int_0^{2\pi}\int_0^{\sec\phi} \rho^2 \operatorname{sen}\phi \, d\rho\, d\theta\, d\phi = \operatorname{vol}(W)$$

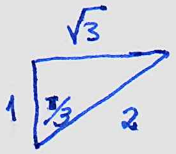
$$W_{\rho\phi\theta}: \begin{cases} 0 \leq \phi \leq \pi/3 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \rho \leq \sec\phi \end{cases} \qquad \begin{cases} x = \rho \operatorname{sen}\phi \cos\theta \\ y = \rho \operatorname{sen}\phi \operatorname{sen}\theta \\ z = \rho\cos\phi \end{cases}$$

$$z = 1 \Rightarrow \rho\cos\phi = 1$$
$$\Rightarrow \rho = \frac{1}{\cos\phi} = \sec\phi .$$





$$\tg \pi/6 = \frac{1}{\sqrt{3}}$$



$$\tg \pi/3 = \sqrt{3}$$

W é a região dentro do cone $z = \sqrt{\frac{1}{3}(x^2+y^2)}$ com $0 \leq z \leq 1$

$$\operatorname{vol}(W) = \frac{1}{3}\pi(\sqrt{3})^2 \cdot 1 = \pi$$

De fato,

$$\int_0^{\pi/3}\int_0^{2\pi}\int_0^{\sec\phi} \rho^2 \operatorname{sen}\phi\, d\rho\, d\theta\, d\phi = 2\pi\int_0^{\pi/3} \frac{1}{3}\rho^3\Big|_0^{\sec\phi} \operatorname{sen}\phi\, d\phi$$

$$= 2\pi\int_0^{\pi/3} \frac{1}{3}\sec^3\phi \operatorname{sen}\phi\, d\phi = \frac{2\pi}{3}\int_0^{\pi/3} \tg\phi \cdot \sec^2\phi\, d\phi$$

$$= \frac{2}{3}\pi \cdot \frac{1}{2}\tg^2\phi\Big|_0^{\pi/3} = \frac{\pi}{3}\left((\sqrt{3})^2 - 0\right) = \pi$$