C3 - LISTA 2

(1) D Limitada por $\begin{cases} x = 1, \ x = 2 \\ y = 0, \ y = \frac{\sqrt{3}}{3}x \end{cases}$

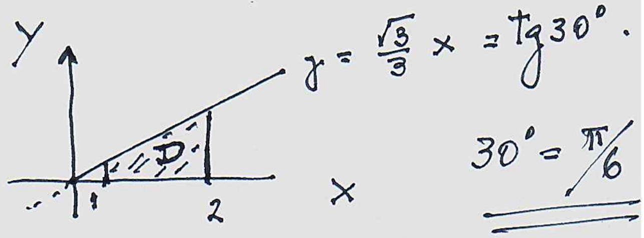
Densidade de D inversamente proporcional à distância do pto $(x,y) \in D$ à origem

$$\left(f(x,y) = \frac{k}{\sqrt{x^2+y^2}} \right)$$

Determinar a massa de D, M(D)

Solução:

$$M(D) = \int_D f \, dA.$$



$30° = \pi/6$

$$= \int_1^2 \int_0^{\frac{\sqrt{3}}{3}x} \frac{k}{\sqrt{x^2+y^2}} \, dy dx$$

.... Coord. Polares...

$\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \end{cases}$ "$dxdy = r\,dr\,d\theta$"

$x^2+y^2 = r^2$

$$D_{r\theta} : \begin{cases} 0 \leq \theta \leq \pi/6 \\ \frac{1}{\cos\theta} \leq r \leq \frac{2}{\cos\theta} \end{cases}$$

$\left(\begin{array}{l} x = 1 \Rightarrow r\cos\theta = 1 \\ \quad \Rightarrow r = 1/\cos\theta = \sec\theta \end{array} \right)$

$$M(D) = \int_0^{\pi/6} \int_{\sec\theta}^{2\sec\theta} \frac{k}{\sqrt{r^2}} \cdot r\,dr\,d\theta = k \int_0^{\pi/6} \int_{\sec\theta}^{2\sec\theta} dr\,d\theta$$

$$M(D) = k\int_0^{\pi/6} \sec\theta \, d\theta$$

$$= k \ln|\sec\theta + \operatorname{tg}\theta|\Big|_0^{\pi/6}$$

$$= k\left(\ln\left|\sec \tfrac{\pi}{6} + \operatorname{tg}\tfrac{\pi}{6}\right| - \ln|1+0|\right)$$

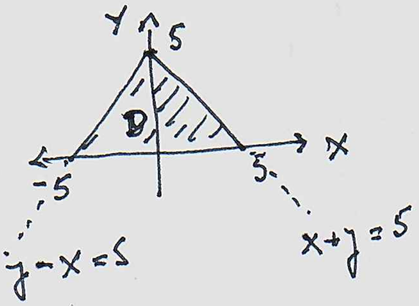
$$= k \ln\left|\frac{2}{\sqrt{3}} + \frac{\sqrt{3}}{3}\right| = k\ln\left(\frac{6+3}{3\sqrt{3}}\right)$$

$$= k\ln\sqrt{3} \qquad \frac{1}{2}k\ln 3$$

② Centro de massa de D, $(\bar{x}, \bar{y})$

D homogênea com a forma de um triângulo isosceles de base 10 cm e altura 5 cm.

Solução



$$D: \begin{cases} 0 \le y \le 5 \\ y-5 \le x \le 5-y \end{cases}$$

$$\bar{x} = \frac{M_y}{M(D)}, \quad \bar{y} = \frac{M_x}{M(D)}$$

$$M(D) = k \cdot \text{área}(D) = k \cdot \frac{10 \cdot 5}{2} = 25k$$

(D é homogên
k = densidad
de D)

$$\left(M(D) = \int_D k \cdot dA = k\int_D dA = k\,\text{área}(D)\right)$$

$$M_x = \int_D y \cdot f\, dA = \int_0^5 \int_{y-5}^{5-y} k y \, dx\, dy = \int_0^5 k y (5-y-(y-5))\, d$$

$$= k\int_0^5 y(10-2y)\,dy = k\int_0^5 (10y - 2y^2)\,dy$$

$$= k\left[5y^2 - \frac{2}{3}y^3\Big|_0^5\right] = k\left(125 - \frac{2}{3}\cdot 125\right) = \frac{125}{3}k$$

$$M_x = \frac{125}{3} k \qquad \therefore \quad \bar{y} = \frac{M_x}{M(D)} = \frac{125k/3}{25k} = 5/3$$

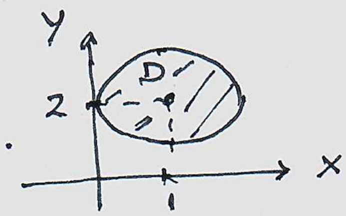
$$\boxed{\bar{y} = 5/3}$$

$$M_y = \int_D x\,k\,dA = \int_0^5 \int_{y-5}^{5-y} k x \,dx\,dy = k\int_0^5 \underbrace{\left(\int_{-(5-y)}^{5-y} x\,dx\right)}_{0} dy$$

$$= 0 \qquad \therefore \quad \boxed{\bar{x} = 0}$$

(3) $D \quad (x-1)^2 + (y-2)^2 \leq 1 \quad , \quad f(x,y) = k\, d((x,y),(1,2))$

$$= k\sqrt{(x-1)^2 + (y-2)^2}$$



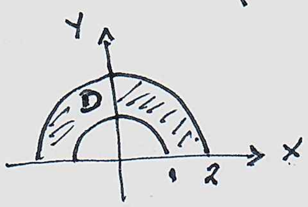
$$M(D) = \int_D f\,dA$$

$$\begin{cases} x - 1 = r\cos\theta \\ y - 2 = r\,\text{sen}\,\theta \end{cases} \qquad \text{"}dx\,dy = r\,dr\,d\theta\text{"} \qquad D_{r\theta}: \begin{cases} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{cases}$$

$$M(D) = \int_0^1 \int_0^{2\pi} k\sqrt{r^2}\,.\,r\,d\theta\,dr = 2\pi \int_0^1 k r^2\,dr$$

$$= \frac{2\pi k}{3}$$

(4) $D: \begin{cases} 1 \leq x^2 + y^2 \leq 4 \\ y \geq 0 \end{cases} \qquad f(x,y) = k\sqrt{x^2+y^2}$



$$\bar{x} = \frac{M_y}{M(D)} \quad , \quad \bar{y} = \frac{M_x}{M(D)}$$

$$M(D) = \int_D f\,dA$$

$$D_{r\theta}: \begin{cases} 1 \leq r \leq 2 \\ 0 \leq \theta \leq \pi \end{cases}$$

$$M(D) = \int_1^2 \int_0^{\pi} k\sqrt{r^2}\cdot r\, d\theta\, dr = \int_1^2 \int_0^{\pi} k r^2\, d\theta\, dr$$

$$= 2k\pi \int_1^2 r^2\, dr = 2k\pi \left.\frac{r^3}{3}\right|_1^2$$

$$= 2k\pi\left(\frac{8}{3} - \frac{1}{3}\right) = \underline{\underline{\frac{14}{3}k\pi}}$$

$$M_x = \int_D y f\, dA = \int_1^2 \int_0^{\pi} k r^2 \cdot r \operatorname{sen}\theta\, d\theta\, dr$$

$$= \int_1^2 k r^3 \left(-\cos\theta \Big|_0^{\pi}\right) dr = 2k \int_1^2 r^3\, dr$$

$$= 2k \left.\frac{r^4}{4}\right|_1^2 = 2k\left(\frac{16}{4} - \frac{1}{4}\right) = \underline{\underline{\frac{11}{2}k}}$$

$$\bar{y} = \frac{M_x}{M(D)} = \frac{11k/2}{\frac{14}{3}k\pi} = \underline{\underline{\frac{33}{28\pi}}}$$

$$M_y = \int_D x f\, dA = \int_1^2 \int_0^{\pi} r\cos\theta \cdot k r^2\, d\theta\, dr$$

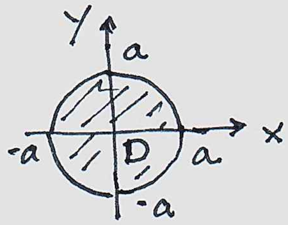
$$= k\int_1^2 r^3 \left(\operatorname{sen}\theta \Big|_0^{\pi}\right) dr = \underline{\underline{0}}$$

$$\underline{\left\{\ \bar{x} = 0\ \right\}}$$

5) $D: x^2+y^2 \leq a^2 \quad , \quad a>0$

$$\rho(x,y) = \frac{a^2}{a^2+x^2+y^2} \qquad \text{calcular } I_0 .$$

$$I_0 = I_x + I_y = \int_D (x^2+y^2)\cdot \rho(x,y)\, dx\, dy$$



coord. Polares: $\begin{cases} x = r\cos\theta \\ y = r \operatorname{sen}\theta \end{cases}$

$$D_{r\theta} : \begin{cases} 0 \leq r \leq a \\ 0 \leq \theta \leq 2\pi \end{cases}$$

$$I_0 = \int_{D_{r\theta}} r^2 \frac{a^2}{a^2+r^2}\, r\, dr\, d\theta = \int_0^a \int_0^{2\pi} \frac{a^2 r^3}{a^2+r^2}\, d\theta\, dr$$

$$= 2\pi \int_0^a a^2 \left(r - \frac{a^2 r}{a^2+r^2} \right) dr$$

$$= 2\pi a^2 \left(\frac{1}{2} r^2 - \frac{1}{2} a^2 \ln (a^2+r^2) \right) \Bigg|_0^a$$

$$= \pi a^2 \left(a^2 - a^2 \ln 2a^2 + a^2 \ln a^2 \right)$$

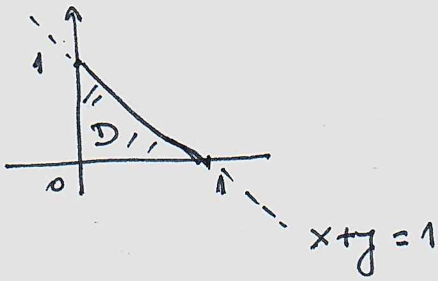
$$= \pi a^4 \left(1 - \ln 2 - \cancel{\ln a^2} + \cancel{\ln a^2} \right)$$

$$= \pi a^4 \left(1 - \ln 2 \right)$$

⑥ Calcular:

(a) $\displaystyle\int_D \frac{x-y}{x+y}\, dA$

D:



$D: \begin{cases} 0 \leq x \leq 1 \\ 0 \leq y \leq 1-x \end{cases}$

Fazemos a mudança de coord. $\begin{cases} u = x-y \\ v = x+y \end{cases}$

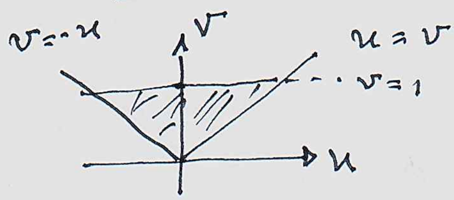
$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = 2 \qquad \therefore \quad \frac{\partial(x,y)}{\partial(u,v)} = 1/2$$

D_{uv}:

$x = 0 \Rightarrow \begin{cases} u = -y \\ v = y \end{cases} \Rightarrow v = -u$

$y = 0 \Rightarrow \begin{cases} u = x \\ v = x \end{cases} \Rightarrow u = v$

$x+y = 1 \Rightarrow v = 1$

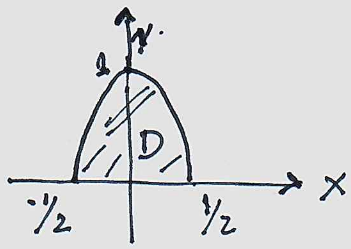


$D_{uv}: \begin{cases} 0 \leq v \leq 1 \\ -v \leq u \leq v \end{cases}$

$$\int_D \frac{x-y}{x+y}\, dA = \int_0^1 \int_{-v}^{v} \frac{u}{v}\cdot\frac{1}{2}\, du\, dv$$

$$= \int_0^1 \Big(\frac{1}{2v} \underbrace{\int_{-v}^{v} u\, du}_{0} \Big)\, dv = 0$$

(b) $\displaystyle\int_D \operatorname{sen}(4x^2+y^2)\,dA \qquad D: \begin{cases} 4x^2+y^2 \leq 1 \\ y \geq 0 \end{cases}$



$$\begin{cases} 2x = r\cos\theta \\ y = r\operatorname{sen}\theta \end{cases}$$

$$4x^2+y^2 = r^2$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix} \frac{1}{2}\cos\theta & -\frac{1}{2}r\operatorname{sen}\theta \\ \operatorname{sen}\theta & r\cos\theta \end{pmatrix} = \frac{1}{2}r$$

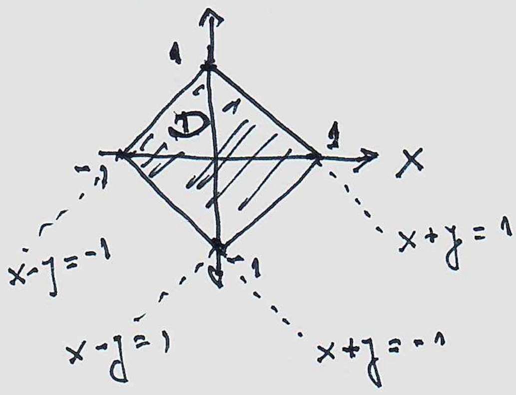
$$\int_D \operatorname{sen}(4x^2+y^2)\,dA = \int_0^1 \int_0^{\pi} \frac{1}{2} r \operatorname{sen} r^2 \, d\theta\, dr$$

$$= \frac{1}{2}\pi \int_0^1 r \operatorname{sen} r^2 \, dr = \frac{1}{2}\pi \int_0^1 \frac{1}{2} \operatorname{sen} t \, dt$$

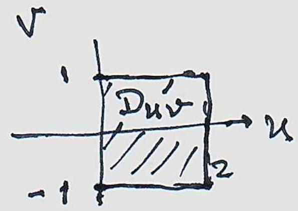
$$\begin{pmatrix} t = r^2 \\ dt = 2r\,dr \end{pmatrix}$$

$$= \frac{1}{4}\pi \left(-\cos t \Big|_0^1 \right) = \frac{1}{4}\pi (1-\cos 1)$$

(c) $\displaystyle\int_D (x+y+1)(x-y)^6 \, dA \qquad D: \ |x|+|y| \leq 1$



$$\begin{cases} u = x+y+1 \\ v = x-y \end{cases}$$



$x+y=1 \Rightarrow u=2$

$x+y=-1 \Rightarrow u=0$

$x-y=-1 \Rightarrow v=-1$

$x-y=1 = \quad v=1$

$$\int_D (x+y+1)(x-y)^6 \, dx\,dy = \int_{D_{uv}} u v^6 \, du\,dv \, |J|$$

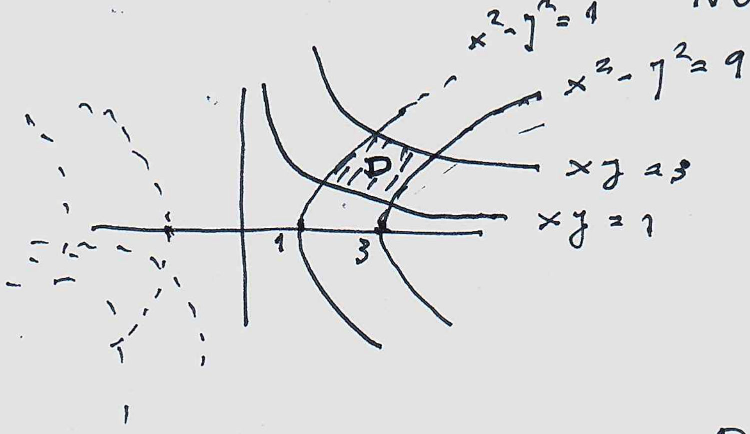
$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2 \qquad \frac{\partial(x,y)}{\partial(u,v)} = -1/2$$

$$\int_{D_{uv}} \frac{1}{2} u v^6 \, du\,dv = \int_0^2\int_{-1}^1 \frac{1}{2} u v^6 \, dv\,du$$

$$= \int_0^2 \frac{1}{14} u v^7 \Big|_{-1}^{1} du = \int_0^2 \frac{1}{7} u \, du = \frac{1}{14} u^2 \Big|_0^2$$

$$= 4/14 = \underline{\underline{2/7}}$$

(d) $\int_D (x^2+y^2)\,dA$ D limitada por $x^2-y^2=1$, $x^2-y^2=9$, $xy=1$ e $xy=3$. No 1º quadrante



$$\begin{cases} u = x^2-y^2 \\ v = xy \end{cases}$$

$$D_{uv}: \begin{cases} 1 \leq u \leq 9 \\ 1 \leq v \leq 3 \end{cases}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} 2x & -2y \\ y & x \end{pmatrix} = 2x^2+2y^2$$

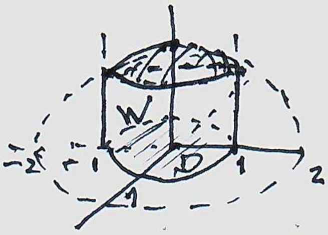
$$= 2(x^2+y^2)$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2(x^2+y^2)}$$

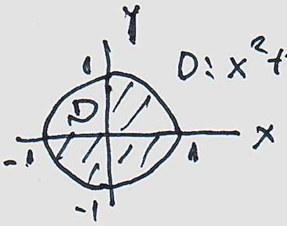
$$\int_D (x^2+y^2)\,dx\,dy = \int_1^3\int_1^9 \frac{x^2+y^2}{2(x^2+y^2)}\,dv\,du$$

$$= \frac{1}{2}\int_1^3\int_1^9 dv\,du = \underline{\underline{8}}$$

(7) Volume do sólido limitado, por cima por $x^2+y^2+z^2=4$ por baixo por $z=0$ e lateralmente por $x^2+y^2=1$



$$vol(W) = \int_D \sqrt{4-x^2-y^2}\, dA$$



$D: x^2+y^2 \leq 1$

$$D_{r\theta}: \begin{cases} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{cases} \qquad \begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \end{cases}$$

$$vol(W) = \int_0^1 \int_0^{2\pi} \sqrt{4-r^2}\cdot r\, d\theta\, dr$$

$$= 2\pi \int_0^1 r\sqrt{1-r^2}\, dr = \pi \int_0^1 2r\sqrt{1-r^2}\, dr$$

$$= \pi \left(-\frac{2}{3} (1-r^2)^{3/2} \Big|_0^1 \right)$$

$$= \frac{2}{3}\pi$$