Hierarchic control for the one-dimensional wave equation in domains with moving boundary *

Isaias Pereira de Jesus †

Abstract

This work addresses the study of the controllability for a one-dimensional wave equation in domains with moving boundary. This equation models the motion of a string where an endpoint is fixed and the other one is moving. When the speed of the moving endpoint is less than $1 - \frac{2}{1+e^2}$, the controllability of this equation is established.

1 Introduction

As in [1], given $T > 0$, we consider the non-cylindrical domain defined by

$$\widehat{Q} = \left\{(x,t) \in \mathbb{R}^2;\ 0 < x < \alpha_k(t),\ \ t \in (0,T)\right\},$$

where

$$\alpha_k(t) = 1 + kt, \qquad 0 < k < 1.$$

Its lateral boundary is defined by $\widehat{\Sigma} = \widehat{\Sigma}_0 \cup \widehat{\Sigma}_0^*$, with

$$\widehat{\Sigma}_0 = \{(0,t);\ t \in (0,T)\} \quad \text{and} \quad \widehat{\Sigma}_0^* = \widehat{\Sigma} \backslash \widehat{\Sigma}_0 = \{(\alpha_k(t),t);\ t \in (0,T)\}.$$

We also represent by Ω_t and Ω_0 the intervals $(0,\alpha_k(t))$ and $(0,1)$, respectively. Consider the following wave equation in the non-cylindrical domain $\widehat{Q}$:

$$\left|\begin{array}{l} u'' - u_{xx} = 0 \quad \text{in} \quad \widehat{Q}, \\ u(x,t) = \begin{cases} \widetilde{w}(t) & \text{on} \quad \widehat{\Sigma}_0, \\ 0 \quad \text{on} & \widehat{\Sigma}_0^*, \end{cases} \\ u(x,0) = u_0(x),\ \ u'(x,0) = u_1(x) \ \ \text{in}\ \ \Omega_0, \end{array}\right. \tag{1.1}$$

To obtain a result of controllability the idea is to transform the problem (1.1) from a non-cylindrical domain into a cylindrical domain by a change of variable; see [2] for more details.

2 Main Results

Associated with the solution $u = u(x,t)$ of (1.1), we will consider the (secondary) functional

$$\widetilde{J}_2(\widetilde{w}_1, \widetilde{w}_2) = \frac{1}{2}\iint_{\widehat{Q}} (u(\widetilde{w}_1,\widetilde{w}_2) - \widetilde{u}_2)^2\, dxdt + \frac{\widetilde{\sigma}}{2}\int_{\widehat{\Sigma}_2} \widetilde{w}_2^2\, d\widehat{\Sigma}, \tag{2.1}$$

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†DM, UFPI, PI, Brasil, isaias@ufpi.edu.br

and the (main) functional

$$\widetilde{J}(\widetilde{w}_1) = \frac{1}{2}\int_{\widehat{\Sigma}_1} \widetilde{w}_1^2 \, d\widehat{\Sigma}, \tag{2.2}$$

where $\widetilde{\sigma} > 0$ is a constant and $\widetilde{u}_2$ is a given function in $L^2(\widehat{Q})$.

The control problem that we will consider is as follows: the follower $\widetilde{w}_2$ assumes that the leader $\widetilde{w}_1$ has made a choice. Then, it tries to find an equilibrium of the cost $\widetilde{J}_2$, that is, it looks for a control $\widetilde{w}_2 = \mathfrak{F}(\widetilde{w}_1)$ (depending on $\widetilde{w}_1$), satisfying:

$$\widetilde{J}_2(\widetilde{w}_1, \widetilde{w}_2) \leq \widetilde{J}_2(\widetilde{w}_1, \widehat{w}_2), \quad \forall\, \widehat{w}_2 \in L^2(\widehat{\Sigma}_2). \tag{2.3}$$

In another way, if the leader $\widetilde{w}_1$ makes a choice, then the follower $\widetilde{w}_2$ makes also a choice, depending on $\widetilde{w}_1$, which minimizes the cost $\widetilde{J}_2$, that is,

$$\widetilde{J}_2(\widetilde{w}_1, \widetilde{w}_2) = \inf_{\widehat{w}_2 \in L^2(\widehat{\Sigma}_2)} \widetilde{J}_2(\widetilde{w}_1, \widehat{w}_2). \tag{2.4}$$

This is equivalent to (2.3). This process is called Stackelberg-Nash strategy; see Díaz and Lions [3].

As in [1], we assume that

$$T > \frac{e^{\frac{2kM}{(1-k)(1-kM)}} - 1}{k} \qquad \text{where} \qquad M = \frac{1}{2k} ln\Big(\frac{1+k}{1-k}\Big), \tag{2.5}$$

and

$$0 < k < 1 - \frac{2}{1+e^2}. \tag{2.6}$$

Theorem 2.1. *Assume that* (2.5) *and* (2.6) *hold. Let us consider* $w_1 \in L^2(\Sigma_1)$ *and* w_2 *a Nash equilibrium in the sense* (2.4). *Then* $(v(T), v'(T)) = (v(., T, w_1, w_2), v'(., T, w_1, w_2))$, *where* v *solves the optimality system, generates a dense subset of* $L^2(0, 1) \times H^{-1}(0, 1)$.

Proof To prove theorem, we apply Holmgren's Uniqueness Theorem (cf. [4]; and see also [1] for additional discussions). ∎

References

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