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Title: Hierarchic control for the wave equation.

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Abstract: This paper deals with the hierarchical control of the wave PDE. We use Stackelberg-Nash strategies. As usual, we consider one leader and two followers. To each leader we associate a Nash equilibrium corresponding to a bi-objective optimal control problem; then, we look for a leader that solves an exact controllability problem. We consider linear and semilinear equations.

Statement of the problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary Γ of class C^2 and let us assume that $T > 0$. Let us consider the small open nonempty sets $\mathcal{O},\ \mathcal{O}_1,\ \mathcal{O}_2 \subset \Omega$. We consider the cylinder $Q = \Omega \times (0,T)$ with lateral boundary $\Sigma = \Gamma \times (0,T)$. By $\nu(x)$ we denote the outward unit normal to Ω at the point $x \in \Gamma$.

Let us consider the following system

$$\begin{cases} y_{tt} - \Delta y + a(x,t)y = F(y) + f1_{\mathcal{O}} + v^1 1_{\mathcal{O}_1} + v^2 1_{\mathcal{O}_2} & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(\cdot\,,0) = y^0,\ y_t(\cdot\,,0) = y^1 & \text{in} \quad \Omega, \end{cases} \tag{1}$$

where $a \in L^\infty(Q)$, $f \in L^2(\mathcal{O}\times(0,T))$, $v^i \in L^2(\mathcal{O}_i \times (0,T))$ $(i = 1,2)$, $F : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz-continuous function, $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and the notation 1_A indicates the characteristic function of A.

The main goal of this article is to analyze the hierarchic control of (1) and, in particular, to prove that the Stackelberg–Nash strategy allows to solve the exact controllability problem.

Main results

Let $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$ be given and let us consider the following set

$$\Gamma_+ := \{x \in \Gamma;\ (x - x_0)\cdot \nu(x) > 0\}$$

and the function $d : \overline{\Omega} \to \mathbb{R}$, with $d(x) = |x - x_0|^2$ for all $x \in \overline{\Omega}$. We also define

$$R_0 := \min\{\sqrt{d(x)} : x \in \overline{\Omega}\} \quad \text{and} \quad R_1 := \max\{\sqrt{d(x)} : x \in \overline{\Omega}\}. \tag{2}$$

We will impose the following assumption:

$$\exists \delta > 0 \ \text{ such that } \ \mathcal{O} \supset \mathcal{O}_\delta(\Gamma_+) \cap \Omega, \tag{3}$$

where

$$\mathcal{O}_\delta(\Gamma_+) = \{x \in \mathbb{R}^n;\ |x - x'| < \delta,\ x' \in \Gamma_+\}.$$

A trajectory to (1) is a solution to the system

$$\begin{cases} \bar{y}_{tt} - \Delta \bar{y} + a(x,t)\bar{y} = F(\bar{y}) & \text{in} \quad Q, \\ \bar{y} = 0 & \text{on} \quad \Sigma, \\ \bar{y}(\cdot\,,0) = \bar{y}^0,\ y_t(\cdot\,,0) = \bar{y}^1 & \text{in} \quad \Omega. \end{cases} \tag{4}$$

In the linear case ($F \equiv 0$), we have the following result on the exact controllability of (1):

Theorem 1. *Suppose that $T > 2R_1$, $F \equiv 0$ and the constants $\mu_i > 0$ $(i = 1,2)$ are large enough, depending on Ω, $\mathcal{O}$, the $\mathcal{O}_i$, the $\mathcal{O}_{i,d}$, T and $\|a\|_{L^\infty(Q)}$. Then, for any data $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$, there exist a control $f \in L^2(\mathcal{O} \times (0,T))$ and an associated Nash equilibrium pair $(v^1, v^2) = (v^1(f), v^2(f))$ such that the corresponding solution to* (1) *satisfies $y(\cdot, T) = \bar{y}(\cdot, T)$.*

The following results hold in the semilinear case:

Theorem 2. *Assume that $T > 2R_1$, $F \in W^{1,\infty}(\mathbb{R})$ and the $\mu_i > 0$ $(i = 1,2)$ are sufficiently large, depending on Ω, $\mathcal{O}$, the $\mathcal{O}_i$, the $\mathcal{O}_{i,d}$, T and $\|F\|_{W^{1,\infty}}$. Then, for any data $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$, there exist a control $f \in L^2(\mathcal{O} \times (0,T))$ and an associated Nash quasi-equilibrium pair $(v^1, v^2) = (v^1(f), v^2(f))$ such that the corresponding solution to* (1) *satisfies $y(\cdot, T) = \bar{y}(\cdot, T)$.*

References

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