Quasilinear Schrödinger equations with unbounded or decaying potentials *

de Carvalho, G. M.; † & Severo, U. B. ‡

1 Introduction

In this work we are concerned with the existence of solution for *quasilinear Schrödinger equations* of the form

$$i\frac{\partial \psi}{\partial t} = -\Delta\psi + W(x)\psi - \eta(x,|\psi|^2)\psi - \kappa\left[\Delta\rho(|\psi|^2)\right]\rho'(|\psi|^2)\psi, \tag{1.1}$$

where $\psi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$, κ is a real constant, $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential and $\eta : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}$, $\rho : \mathbb{R}_+ \to \mathbb{R}$ are suitable functions. Quasilinear equations of the form (1.1) appear naturally in mathematical physics and have been derived as models of several physical phenomena corresponding to various types of nonlinear terms ρ. Here, we consider the case where $\rho(s) = s$, $\kappa > 0$ and our main interest is in the existence of *standing wave solutions*, that is, solutions of type

$$\psi(x,t) = \exp(-i\mathcal{E}t)u(x),$$

where $\mathcal{E} \in \mathbb{R}$ and $u \geq 0$ is a real function. A simple computation shows that ψ satisfies (1.1) if and only if the function $u(x)$ solves the quasilinear equation

$$-\Delta u + V(x)u - \kappa[\Delta(u^2)]u = g(x,u), \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $V(x) := W(x) - \mathcal{E}$ is the new potential and $g(x,u) := \eta(x,u^2)u$ is the new nonlinear term.

The equation (1.2) has attracted a lot of attention of many researchers and some existence and multiplicity results have been obtained. We want to deal with equation (1.2) where the potential V verifies the conditions:

i) $\limsup_{|x|\to 0} V(x) = +\infty$ (singular at origin);

ii) $\liminf_{|x|\to\infty} V(x) = 0$ (vanishing at infinity).

More precisely, we are concerned with problems of the form

$$\begin{cases} -\Delta u + V(|x|)u - \kappa[\Delta(u^2)]u = Q(|x|)g(u), & x \in \mathbb{R}^N, \\ u(x) \to 0 \quad \text{as} \quad |x| \to \infty. \end{cases} \tag{P}$$

The main purpose is to show that, using a variational framework based on a suitable weighted Orlicz space, it is possible to find sufficient conditions for existence of nonnegative and nonzero solutions for (P), where $N \geq 3$, $V, Q : (0,\infty) \to \mathbb{R}$ are continuous and satisfy convenient assumptions at the origin and infinity, and the nonlinear term $g : \mathbb{R} \to \mathbb{R}$ is continuous and has adequate growth conditions for this class of problems.

More precisely we make the following assumptions on the potentials V and Q:

(V_1) $V : (0,\infty) \to \mathbb{R}$ is continuous, $V(r) \geq 0$ for all $r > 0$ and there exist $a \in \mathbb{R}$ and $a_0 \geq -2$ such that

$$0 < \liminf_{r\to 0^+} \frac{V(r)}{r^{a_0}} \leq \limsup_{r\to 0^+} \frac{V(r)}{r^{a_0}} < \infty \quad \text{and} \quad 0 < \liminf_{r\to +\infty} \frac{V(r)}{r^{a}};$$

*Poster submitted to VII WENLU

†Department of Mathematics, Rural Federal University of Pernambuco, gilson.carvalho@ufrpe.br

‡Department of Mathematics, Federal University of Paraíba,uberlandio@mat.ufpb.br

(Q_1) $Q:(0,\infty)\to\mathbb{R}$ is continuous, $Q(r)>0$ for all $r>0$ and there exist $b,b_0\in\mathbb{R}$ such that

$$0<\liminf_{r\to 0^+}\frac{Q(r)}{r^{b_0}}\le\limsup_{r\to 0^+}\frac{Q(r)}{r^{b_0}}<\infty \quad\text{and}\quad \limsup_{r\to+\infty}\frac{Q(r)}{r^{b}}<\infty,$$

where

(i) $b_0>b$ if $b\ge -2\ge a$;

(ii) $b_0\ge b$ and $b_0>-2$ if $b\le\max\{a,-2\}$.

By (V_1) and (Q_1) the potentials V and Q can be singular at the origin and vanishing at infinity, as well as can be unbounded at infinity.

In order to establish the hypotheses on the nonlinearity $g(s)$, we introduce the following numbers:

$$\alpha:=\begin{cases}\dfrac{2(N+b)}{N-2}, & \text{if } b\ge -2 \text{ and } a\le -2;\\ 2, & \text{if } b\le\max\{a,-2\};\end{cases}$$

and

$$\beta:=\frac{2(N+b_0)}{N-2}.$$

The numbers α and β are related with some embedding results, which are important for us to use the variational methods. Throughout this work the following hypotheses on $g(s)$ will be assumed:

(g_1) $g:\mathbb{R}\to\mathbb{R}$ is continuous and $g(s)=o(|s|^{\alpha-1})$ as $s\to 0$;

(g_2) there exist $p\in(\alpha,2\beta)$ and $C_1>0$ such that

$$|g(s)|\le C_1(1+|s|^{p-1}),\ \forall\ s\in\mathbb{R};$$

(g_3) there exists $\mu>2$ such that

$$0<2\mu G(s):=2\mu\int_0^s g(t)dt\le sg(s),\ \forall\ s>0.$$

Under these conditions, we have our main result.

Theorem 1.1. *Suppose that (V_1), (Q_1), $(g_1)-(g_3)$ are satisfied with $b\ge -\frac{N+2}{N}$ in $(Q_1)_{(i)}$, $b\neq -2$ in $(Q_1)_{(ii)}$, $a_0\ge b_0$ and $p>\max\{4,\alpha+1\}$. Then, problem (P) has a nonnegative and nonzero solution $u\in X_{rad}=\left\{v\in D^{1,2}_{rad}(\mathbb{R}^N):\int_{\mathbb{R}^N}V(|x|)|v|^2\,dx<\infty\right\}$.*

References

[1] U. B. Severo and G. M. de Carvalho, *Quasilinear Schrödinger equations with unbounded or decaying potentials.* Mathematische Nachrichten, (2017).

[2] J. M. do Ó and U. B. Severo, *Quasilinear Schrödinger equations involving concave and convex nonlinearities.* Commun. Pure Appl. Anal. **8** (2009), 621–644.

[3] J. Su, Z.-Q. Wang and M. Willem, *Nonlinear Schrödinger equations with unbounded and decaying radial potentials.* Commun. Contemp. Math. **9** (2007), 571–583.