Universidade Federal da Paraíba
Centro de Ciências Exatas e da Natureza
Programa Associado de Pós-Graduação em Matemática UFPB/UFCG
Curso de Doutorado em Matemática

Capillary Problem and Mean Curvature Flow of Killing Graphs

por

Gabriela Albuquerque Wanderley

João Pessoa - PB

Maio/2013

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sob orientação do

Prof. Dr. Jorge Herbert Soares de Lira

Tese apresentada ao Corpo Docente do Programa Associado de Pós-Graduação em Matemática - UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

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Abstract

We study two types of Neumann problem related to Capillary problem and to the evolution of graphs under mean curvature flow in Riemannian manifolds endowed with a Killing vector field. In particular, we prove the existence of Killing graphs with prescribed mean curvature and prescribed boundary conditions.

Keywords: Neumann, Capillary, Mean curvature flow.

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Sumário

Introduction

Capillarity phenomena happens whenever two materials are situated adjacent to each other and do not mix. We use the term capillary surface to describe the free interface that occurs when one of the materials is a liquid and the other a liquid or gas.

We can observe the capillarity phenomena in various places, some are simple, others, such as the rise of liquid in a narrow tube, are more important and has been studied since the 17th century by an Italian scientis Nicoló Aggiunti. He wrote in his booklet a first description of the observation of that problem [1].

The modern theory of capillarity starts in the beginning of the 19th century and is mainly based on mathematical methods of calculus of variations, and on differential geometry. But the initial mathematical insights were introduced by Thomas Young, a medical physician and natural philosopher who in 1805 introduced the mathematical concept of mean curvature H of a surface and who showed its importance for capillarity by relating it to the pressure change across the surface [2].

It was Laplace [3] that derived a formal mathematical expression for the mean curvature H of a surface $u(x,t)$,

$$2H = divTu, \quad where \quad Tu = \frac{Du}{\sqrt{1+|Du|^2}}$$

The notion of mean curvature of a surface was introduced by T. Young (1805) and P. S. Laplace (1806) just for characterizing quantitatively the rise of liquid in a narrow tube. The Laplace or Young-Laplace equation can be written as

$$P = \sigma(\frac{1}{R_1} + \frac{1}{R_2}),$$

where P is the pressure, σ is the surface tension, R_1 and R_2 are the two principal radii of curvature, so for the height u of the surface above the level corresponding to atmospheric pressure we have

$$\frac{1}{2}ku = H(\frac{1}{R_1} + \frac{1}{R_2}),$$

where k is a physical constant.

One of the reasons for studying capillary problems is that the problem of finding a capillary surface is a purely geometric one, that is, to find a surface whose mean curvature is a prescribed function of position and which meets prescribed rigid boundary walls in a prescribed angle. That is if we assume that the surface can be described as a graph of a function u over a domain Ω then we have

$$\mathrm{div}\Big(\frac{\nabla u}{W}\Big) = ku. \quad \text{in} \quad \Omega \tag{1}$$

$$\langle N, \nu\rangle = \phi \quad \text{on} \quad \partial\Omega. \tag{2}$$

Now a large number of the modern results on capillary surfaces are devoted to establishing the existence of solutions for the problem (1), (2). The first general result was obtained only in 1973 using the variational approach [4].

Gauss unified the work of Young and Laplace in 1830, deriving both the differential equation and boundary conditions using Johann Bernoulli's virtual work principles, according to which the energy of a mechanical system in equilibrium is unvaried under arbitrary virtual displacements consistent with the constraints [5]. We observe that, the energy functional consists of a 'surface integral' plus a 'volume integral'. Now the problem is that the classical definition of surface area is rather inadequate for treating this type of problem. A satisfactory theory of surface area for a general class of surfaces of codimension one in R^n, $n \geq 2$, has been developed by E. De Giorgi in the fifties, and then by M. Miranda, M. Giaquinta, E. Giusti, and others [6]-[9]. Independently the ideas of geometric measure theory were developed by H. Fédérer, W. H. Fleming, F. J. Almgren, W. K. Allard, and others, and have been used effectively by Jean Taylor to consider boundary regularity for capillarity problems [10]-[14].

One of the problems that we will discuss in this thesis is the existence of solution to the problem of capillarity

$$\mathrm{div}\Big(\frac{\nabla u}{W}\Big) - \langle\frac{\nabla\gamma}{2\gamma}, \frac{\nabla u}{W}\rangle = \Psi. \tag{3}$$

Notice that equation (3.1) is the prescribed mean curvature equation for Killing graphs. The first general existence results for constant mean curvature graphs in Riemannian ambients as warped product spaces were treated in [15]. A general existence result for solutions of the Dirichlet problem for this equation may be found in [16]. There the authors used local perturbations of the Killing cylinders as barriers for obtaining height and gradient estimates. However this kind of barrier is not suitable to obtain *a priori* estimates for solutions of Neumann problems. For that reason we consider now local perturbations of the graph itself adapted from the original Korevaar's approach in [17] and its extension by M. Calle e L. Shahriyari [18].

Solutions of mean curvature equations can also be constructed as stationary limits of mean curvature flow with speed given by the difference of the actual and the desired mean curvature.

We say that a hypersurface M_t in a Riemannian manifold M is said to be evolving by mean curvature flow if each point of the surface moves, in time and space, in the direction of its unit normal N with speed equal to the mean curvature H at that point. For example, round spheres in Euclidean space evolve under mean curvature flow while concentrically shrinking inward until they collapse in finite time to a single point, the common center of the spheres. Equivalently if one considers the mean curvature flow of smooth family of immersions $F_t = F(\cdot, t) : M^n \longrightarrow \overline{M}^{n+1}$ this is given by

$$\frac{\partial}{\partial t}F(p,t) = nH(F(p,t))N(F(p,t)), \forall (p,t) \in M^n \times [0,T)$$

There are two approaches to the study of mean curvature flow. One may work directly with the immersions or if the hypersurfaces obey a graph condition, one may study mean curvature flow with classical techniques by considering it as a quasilinear parabolic partial differential equation.

Mean curvature flow is perhaps the most important geometric evolution equation of submanifolds in Riemannian manifolds and has been studied for some time, at least since 1956, when Mullins [19] considered a version of mean curvature flow in one dimension, were he proposed mean curvature flow to model the formation of grain boundaries in annealing metals. In 1978 Brakke [20] studied the mean curvature flow of surfaces from the point of view of geometric measure theory.

For closed convex surfaces in R^{n+1}, one result of great interest is that of Huisken

[21]. There the author proves that under mean curvature flow, compact, initially convex surfaces retain their convexity and becoming more and more spherical at the end of the evolution. In [22] this was extended to general Riemannian manifolds under the assumption that the initial hypersurface is sufficiently convex: Each principal curvature λ_i of the initial surface has to be bounded below by a constant depending on the curvature and the derivative of the curvature in the ambient manifold. The analogous result for the one dimensional case, the curve shortening flow, was obtained by Gage and Hamilton [23], [24], where it was proved that initially convex planar curves contract to points. This was later generalised by Grayson [25],[26] for all closed embedded planar curves. He proved that any embedded closed curve on a 2-surface of bounded geometry will either smoothly contract to a point in finite time or converge to a geodesic in in finite time.

In many contributions to the theory of mean curvature flow one assumes that M is a smooth closed manifold. The reason is, that one key technique in mean curvature flow (or more generally in the theory of parabolic geometric evolution equations) is the application of the maximum principle. But even for complete non-compact submanifolds there are powerful techniques, similar to the maximum principle, that can be applied in some situations. In the complete case one of the most important tools is the monotonicity formula found by Huisken [27], Ecker and Huisken [28] and Hamilton [29] and that equally well applies to mean curvature flow in higher codimension. A local monotonicity for evolving Riemannian manifolds has been found recently by Ecker, Knopf, Ni and Topping [30].

The non-parametric mean curvature flow of graphs with either a ninety degree contact angle or Dirichlet boundary condition on cylindrical domains has been studied by Huisken [31] and there proves a long time existence and convergence to minimal surfaces theorem. This was later generalised by Altschuler and Wu [32], where they allow arbitrary contact angles at the fixed boundary for two dimensional graphs. This in turn was also later generalised to arbitrary dimensions by Guan [33] in Euclidian space, and Calle [18] in Riemannian manifolds. From the point of view of immersions mean curvature flow with Dirichlet boundary data has been studied by Stone [34],[35] in Euclidean space and Priwitzer in [36] in the setting of Riemannian manifolds.

In this thesis we study the following Neumann problem in Riemannian manifold related to the evolution of Killing graphs under mean curvature flow.

$$\frac{\partial X}{\partial t} = (nH - \mathcal{H})N, \tag{4}$$

$$X(0,\cdot) = \vartheta(u_0(\cdot),\cdot), \tag{5}$$

with boundary condition

$$\langle N, \nu \rangle|_{\partial\Sigma_t} = \phi, \tag{6}$$

As an application we prove the existence of Killing graphs with prescribed mean curvature and prescribed boundary conditions. This problem is considered as a flow of immersions which have also the property of being graphs. This will allow us to transform the evolution equation for the immersion into that for a scalar function.

This equation is parabolic and quasilinear and standard theory guarantees that the problem of solving (1.6)-(1.8) is reduced to obtaing a priori height and gradient estimates for solutions the problem. This thesis is divided into four chapter as follows.

In Chapter 1 we give a brief explanation of the problems that we treat in this thesis, namely, Capillary Problem and Mean Curvature Flow of Killing Graphs, both with Neumann boundary conditions.

In Chapter 2 we present a set of theorems which concerns the theory parabolic, including maximum principle and short time existence.

In Chapter 3 and Chapter 4 we will proof the Capillary Problem and Mean Curvature Flow of Killing Graphs respectively.

Capítulo 1

The Problems

Let M be a $(n+1)$-dimensional Riemannian manifold endowed with a Killing vector field Y. Suppose that the distribution orthogonal to Y is of constant rank and integrable. Given an integral leaf P of that distribution, let $\Omega \subset P$ be a bounded domain with regular boundary $\Gamma = \partial\Omega$. Let $\vartheta : \mathbb{I} \times \bar{\Omega} \to M$ the flow generated by Y with initial values in M, where $\mathbb{I}$ is a maximal interval of definition. In geometric terms, the ambient manifold is a warped product $M = P \times_{1/\sqrt{\gamma}} \mathbb{I}$ where $\gamma = \epsilon/\langle Y, Y\rangle$.

Given $T \in [0, +\infty)$, let $u : \bar{\Omega} \times [0, T) \to \mathbb{I}$ be a smooth function. Fixed this notation, the Killing graph of $u(\cdot, t)$, $t \in [0, T)$, is the hypersurface $\Sigma_t \subset M$ parametrized by the map

$$X(t, x) = \vartheta(u(x, t), x), \quad x \in \bar{\Omega}.$$

Notice that this definition could be slightly more general if we suppose that the coordinates of $x \in \bar{\Omega}$ change with the parameter $t \in [0, T)$. To abolish this possibility is equivalent to rule out tangential diffeomorphisms of Ω.

The Killing cylinder K over Γ is by its turn defined by

$$K = \{\vartheta(s, x) : s \in \mathbb{I}, \, x \in \Gamma\}. \tag{1.1}$$

Let N be a unit normal vector field along Σ_t. In what follows, we denote by H the mean curvature of Σ_t with respect to the orientation given by N.

The height function with respect to the leaf P is measured by the arc lenght parameter ς of the flow lines of Y, that is,

$$\varsigma = \frac{1}{\sqrt{\gamma}} s.$$

In this thesis we work with two types of problems. The first is a capillary problem. We prove that there exist solutions for a non-parametric capillary problem in a wide class of Riemannian manifolds endowed with a Killing vector field. In other terms, we prove the existence of Killing graphs with prescribed mean curvature and prescribed contact angle along its boundary. For the second type of problem we consider the mean curvature flow of killing graphs with a Neumann boundary condition.

In what follows we present a brief presentation of the problems

1.1 Capillary problem

We formulate a capillary problem in this geometric context which model stationary graphs under a gravity force whose intensity depends on the point in the space. More precisely, given a *gravitational potential* $\Psi \in C^{1,\alpha}(\bar{\Omega}\times\mathbb{R})$ we define the functional

$$\mathcal{A}[u] = \int_\Sigma \left(1 + \int_0^{u/\sqrt{\gamma}} \Psi(x, s(\varsigma))\,\mathrm{d}\varsigma\right)\mathrm{d}\Sigma. \tag{1.2}$$

The volume element $\mathrm{d}\Sigma$ of Σ is given by

$$\frac{1}{\sqrt{\gamma}}\sqrt{\gamma + |\nabla u|^2}\,\mathrm{d}\sigma,$$

where $\mathrm{d}\sigma$ is the volume element in P.

The first variation formula of this functional may be deduced as follows. Given an arbitrary function $v \in C_c^\infty(\Omega)$ we compute

$$\begin{aligned}
\frac{d}{d\tau}\Big|_{\tau=0}\mathcal{A}[u+\tau v] &= \int_\Omega \left(\frac{1}{\sqrt{\gamma}}\frac{\langle\nabla u, \nabla v\rangle}{\sqrt{\gamma + |\nabla u^2|}} + \frac{1}{\sqrt{\gamma}}\Psi(x, u(x))v\right)\sqrt{\sigma}\mathrm{d}x\\
&= \int_\Omega \left(\mathrm{div}\Big(\frac{1}{\sqrt{\gamma}}\frac{\nabla u}{W}v\Big) - \mathrm{div}\Big(\frac{1}{\sqrt{\gamma}}\frac{\nabla u}{W}\Big)v + \frac{1}{\sqrt{\gamma}}\Psi(x, u(x))v\right)\sqrt{\sigma}\mathrm{d}x\\
&- \int_\Omega \left(\frac{1}{\sqrt{\gamma}}\mathrm{div}\Big(\frac{\nabla u}{W}\Big) - \frac{1}{\sqrt{\gamma}}\langle\frac{\nabla\gamma}{2\gamma}, \frac{\nabla u}{W}\rangle - \frac{1}{\sqrt{\gamma}}\Psi(x, u(x))\right)v\sqrt{\sigma}\mathrm{d}x,
\end{aligned}$$

where $\sqrt{\sigma}\mathrm{d}x$ is the volume element $\mathrm{d}\sigma$ expressed in terms of local coordinates in P. The differential operators div and ∇ are respectively the divergence and gradient in P with respect to the metric induced from M.

We conclude that stationary functions satisfy the capillary-type equation

$$\mathrm{div}\Big(\frac{\nabla u}{W}\Big) - \langle\frac{\nabla\gamma}{2\gamma}, \frac{\nabla u}{W}\rangle = \Psi. \tag{1.3}$$

Notice that a Neumann boundary condition arises naturally from this variational setting: given a $C^{2,\alpha}$ function $\Phi : K \to (-1,1)$, we impose the following prescribed angle condition

$$\langle N, \nu \rangle = \Phi \tag{1.4}$$

along $\partial\Sigma$, where

$$N = \frac{1}{W}\big(\gamma Y - \vartheta_* \nabla u\big) \tag{1.5}$$

is the unit normal vector field along Σ satisfying $\langle N, Y\rangle > 0$ and ν is the unit normal vector field along K pointing inwards the Killing cylinder over Ω.

Following [18] and [17] we suppose that the data Ψ and Φ satisfy

i. $|\Psi| + |\bar{\nabla}\Psi| \leq C_\Psi$ in $\bar{\Omega} \times \mathbb{R}$,

ii. $\langle \bar{\nabla}\Psi, Y\rangle \geq \beta > 0$ in $\bar{\Omega} \times \mathbb{R}$,

iii. $\langle \bar{\nabla}\Phi, Y\rangle \leq 0$,

iv. $(1-\Phi^2) \geq \beta'$,

v. $|\Phi|_2 \leq C_\Phi$ in K,

for some positive constants C_Ψ, C_Φ, β and β', where $\bar{\nabla}$ denotes the Riemannian connection in M. Assumption (ii) is classically referred to as the *positive gravity* condition. Even in the Euclidean space, it seems to be an essential assumption in order to obtain *a priori* height estimates. A very geometric discussion about this issue may be found at [37]. Condition (iii) is the same as in [18] and [17] since at those references N is chosen in such a way that $\langle N, Y\rangle > 0$.

We will prove the following result

Theorem 1 *Let Ω be a bounded $C^{3,\alpha}$ domain in P. Suppose that the $\Psi \in C^{1,\alpha}(\bar{\Omega}\times\mathbb{R})$ and $\Phi \in C^{2,\alpha}(K)$ with $|\Phi| \leq 1$ satisfy conditions* (i)-(v) *above. Then there exists a unique solution $u \in C^{3,\alpha}(\bar{\Omega})$ of the capillary problem* (3.1)-(3.2).

We observe that $\Psi = nH$, where H is the mean curvature of Σ calculated with respect to N. Therefore Theorem 13 establishes the existence of Killing graphs with prescribed mean curvature Ψ and prescribed contact angle with K along the boundary. Since the Riemannian product $P \times \mathbb{R}$ corresponds to the particular case where $\gamma = 1$, our result extends the main existence theorem in [18]. Space forms constitute other important examples of the kind of warped products we are considering. In particular, we encompass the case of Killing graphs over totally geodesic hypersurfaces in the hyperbolic space $\mathbb{H}^{n+1}$.

1.2 Mean curvature flow

We will establish conditions for longtime existence of a prescribed mean curvature flow of the form

$$\frac{\partial X}{\partial t} = (nH - \mathcal{H})N, \tag{1.6}$$
$$X(0,\cdot) = \vartheta(u_0(\cdot),\cdot), \tag{1.7}$$

for given functions $u_0 : \bar{\Omega} \to \mathbb{R}$ and $\mathcal{H} : \bar{\Omega} \to \mathbb{R}$. In order to define boundary conditions for the evolution problem (1.6) we consider a function $\phi \in C^\infty(\Gamma)$ such that $|\phi| \leq \phi_0 < 1$ for some positive constant ϕ_0. Let ν be the inward unit normal vector field along K. We impose the following Neumann condition associated to (1.6)

$$\langle N, \nu \rangle|_{\partial\Sigma_t} = \phi, \tag{1.8}$$

where $\langle \cdot, \cdot \rangle$ denotes the Riemannian metric in M.

Let $x^1, \ldots, x^n$ be local coordinates in P. This system is augmented to be a coordinate system in M by setting $x^0 = s$, the flow parameter of Y. The tangent space of Σ_t at a point $X(t,x)$, $x \in \bar{\Omega}$, is spanned by the coordinate vector fields

$$X_* \frac{\partial}{\partial x^i} = \vartheta_* \frac{\partial}{\partial x^i} + u_i \vartheta_* \frac{\partial}{\partial x^0} = \frac{\partial}{\partial x^i}\Big|_X + u_i \frac{\partial}{\partial x^0}\Big|_X. \tag{1.9}$$

In terms of these coordinates the induced metric in Σ_t is expressed in local components by

$$g_{ij} = \sigma_{ij} + \frac{1}{\gamma} u_i u_j, \tag{1.10}$$

where $\gamma = \frac{1}{|Y|^2}$ and σ_{ij} are the local components of the metric in P.

In order to compute the mean curvature of Σ_t, we fix N as the vector field

$$N = \frac{1}{W}\big(\gamma Y - \vartheta_* \nabla u\big), \tag{1.11}$$

where ∇u is the gradient of u in P and

$$W = \sqrt{\gamma + |\nabla u|^2}. \tag{1.12}$$

The second fundamental form of Σ_t calculated with respect to this choice of normal vector field has local components

$$a_{ij} = \langle \bar{\nabla}_{X_* \frac{\partial}{\partial x^i}} X_* \frac{\partial}{\partial x^j}, N \rangle, \tag{1.13}$$

where $\bar{\nabla}$ denotes the covariant derivative in M. We then compute

$$\begin{aligned} a_{ij} &= \langle \bar{\nabla}_{X_* \frac{\partial}{\partial x^i}} \vartheta_* \frac{\partial}{\partial x^j}, N \rangle + \langle \bar{\nabla}_{X_* \frac{\partial}{\partial x^i}} u_j \vartheta_* \frac{\partial}{\partial x^0}, N \rangle \\ &= \langle \bar{\nabla}_{\vartheta_* \frac{\partial}{\partial x^i}} \vartheta_* \frac{\partial}{\partial x^j}, N \rangle + u_i \langle \bar{\nabla}_{\vartheta_* \frac{\partial}{\partial x^0}} \vartheta_* \frac{\partial}{\partial x^j}, N \rangle + u_j \langle \bar{\nabla}_{\vartheta_* \frac{\partial}{\partial x^i}} \vartheta_* \frac{\partial}{\partial x^0}, N \rangle \\ &\quad + u_{i,j} \langle \vartheta_* \frac{\partial}{\partial x^0}, N \rangle + u_i u_j \langle \bar{\nabla}_{\vartheta_* \frac{\partial}{\partial x^0}} \vartheta_* \frac{\partial}{\partial x^0}, N \rangle. \end{aligned}$$

Hence using the fact that the maps $x \mapsto \vartheta(s, x)$ are isometries and that the hypersurfaces defined by $\{\vartheta(s, x) : x \in P\}$, $s \in \mathbb{I}$, are totally geodesic one concludes that

$$\begin{aligned} a_{ij} = \langle \bar{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, -\frac{1}{W} \nabla u \rangle + u_i \langle \bar{\nabla}_{\frac{\partial}{\partial x^j}} Y, \frac{1}{W} \gamma Y \rangle + u_j \langle \bar{\nabla}_{\frac{\partial}{\partial x^i}} Y, \frac{1}{W} \gamma Y \rangle \\ + u_{i,j} \langle Y, \frac{1}{W} \gamma Y \rangle + u_i u_j \langle \bar{\nabla}_Y Y, -\frac{1}{W} \nabla u \rangle. \end{aligned}$$

It follows from Killing's equation that

$$a_{ij} = \frac{u_{i;j}}{W} - \frac{u_i}{W} \frac{\gamma_j}{2\gamma} - \frac{u_j}{W} \frac{\gamma_i}{2\gamma} - \frac{u_i u_j}{2W} u^k \frac{\gamma_k}{\gamma^2}. \tag{1.14}$$

It turns out that a_{ij} could be also expressed by

$$a_{ij} = \frac{u_{i;j}}{W} - \frac{u_i}{W} \gamma \langle \bar{\nabla}_Y Y, \frac{\partial}{\partial x^j} \rangle - \frac{u_j}{W} \gamma \langle \bar{\nabla}_Y Y, \frac{\partial}{\partial x^i} \rangle - \frac{u_i u_j}{W} \langle \bar{\nabla}_Y Y, \nabla u \rangle. \tag{1.15}$$

Taking traces with respect to the induced metric one obtains the following expression for the mean curvature H of the hypersurface Σ_t

$$nH = \Big(\sigma^{ij} - \frac{u^i}{W} \frac{u^j}{W} \Big) \frac{u_{i;j}}{W} - \frac{2\gamma + |\nabla u|^2}{W^3} \langle \frac{\bar{\nabla} \gamma}{2\gamma}, \nabla u \rangle. \tag{1.16}$$

Alternatively one has

$$nH = \Big(\sigma^{ij} - \frac{u^i}{W} \frac{u^j}{W} \Big) \frac{u_{i;j}}{W} - \frac{2\gamma + |\nabla u|^2}{W^3} \gamma \langle \bar{\nabla}_Y Y, \nabla u \rangle. \tag{1.17}$$

At this point we recall that

$$\bar{\nabla}_{\frac{\partial}{\partial x^i}} Y = -\frac{1}{2} \frac{\gamma_i}{\gamma} Y \tag{1.18}$$

and

$$\bar{\nabla}_Y Y = \frac{1}{2} \frac{\nabla \gamma}{\gamma^2}. \tag{1.19}$$

what implies that

$$\langle \bar{\nabla}_Y Y, \nabla u\rangle = -\langle \bar{\nabla}_{\nabla u} Y, Y\rangle = \frac{1}{2\gamma^2}\langle \nabla\gamma, \nabla u\rangle. \tag{1.20}$$

Using this one easily verifies that (1.16) may be written in divergence form as

$$\operatorname{div}\frac{\nabla u}{W} - \frac{1}{2\gamma W}\langle \nabla\gamma, \nabla u\rangle = nH. \tag{1.21}$$

In fact we have

$$\Big(\frac{u^i}{W}\Big)_{;i} = \frac{1}{W}u^i_{;i} - \frac{1}{W^3}u^i u^j u_{i;j} - \frac{1}{2W^3}u^i\gamma_i.$$

It is worth to point out that (1.21) is equivalent to

$$\operatorname{div}\frac{\nabla u}{W} - \frac{\gamma}{W}\langle \bar{\nabla}_Y Y, \nabla u\rangle = nH. \tag{1.22}$$

We conclude that (1.6) may be written nonparametrically as

$$\frac{\partial u}{\partial t} = W\operatorname{div}\frac{\nabla u}{W} - W\mathcal{H} - \gamma\langle \bar{\nabla}_Y Y, \nabla u\rangle. \tag{1.23}$$

Indeed it holds that

$$nH - \mathcal{H} = \langle \frac{\partial X}{\partial t}, N\rangle = \langle \frac{\partial u}{\partial t}\vartheta_*\frac{\partial}{\partial x^0}, \frac{\gamma}{W}\vartheta_*\frac{\partial}{\partial x^0}\rangle = \frac{1}{W}\frac{\partial u}{\partial t}.$$

Using (1.16) one verifies that (1.23) is equivalent to

$$\frac{\partial u}{\partial t} = \Big(\sigma^{ij} - \frac{u^i}{W}\frac{u^j}{W}\Big)u_{i;j} - \frac{2\gamma + |\nabla u|^2}{W^2}\langle \frac{\bar{\nabla}\gamma}{2\gamma}, \nabla u\rangle - W\mathcal{H}. \tag{1.24}$$

We conclude that the Neumann problem (1.6)-(1.8) has the following nonparametric form

$$u_t = \Big(\sigma^{ij} - \frac{u^i}{W}\frac{u^j}{W}\Big)u_{i;j} - \Big(\frac{1}{2\gamma} + \frac{1}{2W^2}\Big)\gamma^i u_i - W\mathcal{H} \quad \text{in} \quad \Omega\times[0,T) \tag{1.25}$$

$$u(\cdot,0) = u_0(\cdot) \quad \text{in} \quad \Omega\times\{0\} \tag{1.26}$$

with boundary condition

$$\langle N, \nu\rangle = \phi \quad \text{on} \quad \partial\Omega\times[0,T). \tag{1.27}$$

This boundary value problem describes the evolution of the Killing graph of the function $u(\cdot,t)$ by its mean curvature in the direction of the unit normal N with prescribed contact angle at the boundary.

The standard theory for quasilinear parabolic equations [48] guarantees that the problem of solving (1.6)-(1.8) is reduced to obtaning *a priori* height and gradient estimates for solutions to (4.4)-(4.6).

We will prove the following result

Theorem 2 *There exists a unique solution* $u : \bar{\Omega} \times [0, \infty) \to \mathbb{I}$ *to the problem* (1.6)-(1.8). *Moreover, if* $\phi = 0$ *and* $\mathcal{H} = 0$ *the graphs* Σ_t *converge to a minimal graph which contacts the cylinder* K *orthogonally along its boundary.*

Theorem 2 extends Theorem 1.1 in [31] as well as Theorem 2.4 in [33] and Theorem 2.4 in [18] in a twofold way. The corresponding theorems in [31] and [33] concern evolution of graphs in Euclidean space whereas [18] deals with the case of graphs in Riemannian product spaces of the form $P \times \mathbb{R}$. Moreover those earlier results hold only for the case when the prescribed mean curvature is $\mathcal{H} = 0$. An existence result for evolution of graphs in Euclidean space by the Gauss-Kronecker curvature under Neumann boundary conditions is proved in [39]. We also mention that the Dirichlet problem for the evolution of graphs in warped spaces is extensively studied in [38].

Capítulo 2

Parabolic Theory

2.1 Maximum and comparison principles

The maximum principle is an important tool in the study of second order parabolic problems, in particular here for the study of mean curvature flow. In general the maximum principle states that the maximum of a solution of a homogeneous linear or quasilinear parabolic equation in a domain must occur on the boundary of that domain. In fact, this maximum must occur on a special subset called the parabolic boundary. The parabolic boundary includes the domain at initial time. The strong maximum principle asserts that the solution is constant if the maximum occurs anywhere other than on the parabolic boundary.

In this chapter we present a set of maximum principles for scalar functions which satisfy a parabolic evolution equation on a bounded domain in a Riemannian manifold (P^n, σ). We follow the PhD thesis of Valentina Mira [42] and Benjamin Lambert [43] which were based on Lieberman [48] . The comparison and maximum principles will be used to obtain interior estimates, and since we have a boundary value problem, the estimates we give here will depend upon the boundary values.

Let $\Omega \subset P^n$ be a domain with a smooth boundary $\partial\Omega$. We define our parabolic domain to be

$$\widetilde{\Omega} = \Omega \times [0, T).$$

The parabolic boundary $\mathcal{P}\widetilde{\Omega}$ is the union of the following three components: $B\widetilde{\Omega} = \Omega \times \{0\}$, $S\widetilde{\Omega} = \partial\Omega \times (0, T)$ and $C\widetilde{\Omega} = \partial\Omega \times \{0\}$. We denote an arbitrary point $(x, t) \in \widetilde{\Omega}$ by X.

Consider the quasilinear operator P defined by

$$Pu = a^{ij}(x, \nabla u)u_{i;j} + a(x, \nabla u) - u_t,$$

for some $u \in C^{2,1}(\widetilde{\Omega})$ where the coefficients are given by

$$a^{ij} = \sigma^{ij} - \frac{u^i}{W}\frac{u^j}{W} \tag{2.1}$$

and

$$a = -\Big(\frac{1}{2\gamma} + \frac{1}{2W^2}\Big)\gamma^i u_i - W\mathcal{H} \tag{2.2}$$

in terms of the notation fixed in Chapter I.

Notice that the coefficients do not depend *explicitly* on the variables t or u but only implicitly through the terms involving ∇u. This operator is parabolic in the sense that the matrix a^{ij} is positive definite. Indeed, for any $(x, p) \in T\Omega$ and $\zeta \in T^*_x P$ we have

$$\frac{\gamma}{\gamma + |p|^2}|\zeta|^2 \leq a^{ij}(x, p)\zeta_i\zeta_j \leq |\zeta|^2,$$

We conclude that restricted to the points of the form $(x, \nabla u)$ the extremal eigenvalues are given by

$$\lambda = \frac{\gamma}{\gamma + |\nabla u|^2} \quad \text{and} \quad \Lambda = 1.$$

Hence P is parabolic in u. However, the ratio $\frac{\Lambda}{\lambda} = 1 + \frac{1}{\gamma}|\nabla u|^2$ is uniformly bounded if and only if $|\nabla u|$ is uniformly bounded in $\widetilde{\Omega}$. This means that P is uniformly parabolic in u if and only if ∇u is uniformly bounded.

In order to prove a maximum principle we prove first a comparison principle as follows.

Theorem 3 *(Comparison principle) Let P be the quasilinear operator as above. Suppose that there exists an increasing positive constant k such that $a(x, p) + k(M)z$ is a decreasing function of z on $T\widetilde{\Omega} \times [-M, M]$ for any $M > 0$. If u and v are functions in $C^{2,1}(\overline{\widetilde{\Omega}}\backslash P\widetilde{\Omega}) \cap C(\overline{\widetilde{\Omega}})$ such that P is parabolic with respect to u or v, $Pu \geq Pv$ in $\overline{\widetilde{\Omega}}\backslash P\widetilde{\Omega}$, and $u \leq v$ in $\mathcal{P}\tilde{\Omega}$, then $u \leq v$ in $\overline{\tilde{\Omega}}$.*

Proof. We define $w = (u - v)e^{\lambda t}$, where λ is a constant to be chosen later. Let $M = \max\{\sup|u|, \sup|v|\}$. We have that $u \leq v$ in $\mathcal{P}\tilde{\Omega}$, then, $w \leq 0$ in $\mathcal{P}\tilde{\Omega}$. Let

$X_0 = (x_0, t_0)$ be a point where w attains its first positive maximum. At this point, we have,

$$\begin{aligned} &Du - Dv = Dw = 0 \\ &(D^2u - D^2v)e^{\lambda t} = D^2w \leq 0, and \\ &(u_t - v_t)e^{\lambda t} + \lambda(u - v)e^{\lambda t} = w_t > 0. \end{aligned} \tag{2.3}$$

Now let $\alpha = (X_0, u(X_0), Du(X_0))$ and $\beta = (X_0, v(X_0), Dv(X_0))$, then

$$Lu(X_0) - Lv(X_0) = a^{ij}(\alpha)D^2_{ij}(u - v) + (a(\alpha) - a(\beta)) - \frac{\partial}{\partial t}(u - v).$$

It follows by (2.3) and the hypothesis on the existence of the constant k that

$$Pu(X_0) - Pv(X_0) \leq (k(M) + \lambda)(u - v).$$

Now if we have $u > v$ choosing $\lambda < -k(M)$ we conclude that

$$Pu(X_0) - Pv(X_0) < 0,$$

which contradicts the hypothesis that $Lu \geq Lv$. So we cannot have an interior positive maximum of w, which gives us $u \leq v$ in $\bar{\tilde{\Omega}}$.

□

The uniqueness of a solution for a parabolic boundary value problem follows directly from the comparison principle above.

Corollary 4 *(Uniqueness) Suppose that P is as in Theorem 3 and that u and v belong to $C^{2,1}(\tilde{\Omega}) \cap C(\bar{\tilde{\Omega}})$. If $Pu = Pv$ in $\tilde{\Omega}$ and $u = v$ on $\mathcal{P}\tilde{\Omega}$, then $u = v$ in $\tilde{\Omega}$.*

Now, we prove a maximum principle using the comparison principle above.

Theorem 5 *(Maximum Principle) Let P be a parabolic operator whose coefficients a^{ij} and a do not depend on z. If $Pu \geq 0$ in $\tilde{\Omega}$ then*

$$\sup_{\tilde{\Omega}} u \leq \sup_{\mathcal{P}\tilde{\Omega}} u.$$

Proof. Let $v = \sup_{\mathcal{P}\tilde{\Omega}} u$. Observe that $Pv = 0$ then $Pu \geq 0 = Pv$. And $u \leq \sup_{\mathcal{P}\tilde{\Omega}} u = v$ in $\mathcal{P}\tilde{\Omega}$. It follows by Theorem 3 that $u \leq v$ in $\bar{\tilde{\Omega}}$. Then $\sup_{\tilde{\Omega}} u \leq v$, this completes the prove. □

We will need the boundary point lemma of E. Hopf , which is normally referred to as the Hopf Lemma. At a maximum point of a scalar function on a domain the directional derivative towards that point is non-negative. If this point is a boundary point and the scalar function satisfies a parabolic inequality, then the following result gives us a strict sign on the derivative in a direction away from the boundary. Here we prove a Hopf Lemma where the parabolic boundary is assumed to be at least C^1. This result can be found throughout the literature, for example in [44].

Lemma 6 *(Hopf Lemma) Let $\tilde{\Omega}$ be a space-time domain with C^1-boundary in which u is a solution of the parabolic inequality*

$$Pu \geq 0$$

where P is a quasilinear parabolic operator with smooth coefficients. Suppose that $X_0 = (x_0, t_0)$ is a point on the boundary $\partial\tilde{\Omega}$ where the maximum value M of u occurs. Assume that there exists a sphere through X_0 whose interior lies entirely in $\tilde{\Omega}$ and in which $u < M$. Also suppose that the radial direction from the centre of the sphere to X is not parallel to the times axis. Then if $\frac{\partial}{\partial \nu}$ denotes any directional derivative away from the boundary, we have

$$\tfrac{\partial u}{\partial \nu} > 0 \text{ at } X_0.$$

Remark 7 *In the proof we use a local system of coordinates and then we assimilate the distance sphere to an Euclidean sphere for sake of simplicity.*

Proof. Observe that in X_0 any directional derivative of u in a direction pointing towards the point X_0 will be non-negative. So in order to obtain the strict sign we will consider a perturbation of the solution u to which we apply the maximum principle.

Let $S \subset \tilde{\Omega}$ the sphere that appear in the hypothesis, with boundary $\partial\Omega$ and centre at $X_s = (x_s, t_s)$. Consider now another sphere K centered at X_0 and with boundary ∂K and with radius smaller than $|X_0 - X_s|_{\mathbb{R}^{n+1}} = \sqrt{|x_0 - x_s|^2_{\mathbb{R}^n} + |t_0 - t_s|^2}$.

Now denote by C_1 and C_2 the portion of ∂K which is included in S, respectively the portion of ∂S included in K. We also add the end points of the arcs C_1 and C_2 to obtain a closed lens-shaped domais which we denote by D. Then we have

(i) $u < M$ on C_2 except at X_0. If S does not satisfy this then a slightly smaller sphere osculating the boundary at X_0 will be contained in the interior of S, and so the condition $u < M$ will be satisfied everywhere on the arc C_2 except the point X_0.

(ii) $u = M$ at X_0. It is because the hypothesis.

(iii) There exists a sufficiently small constant $\mu > 0$ such that $u \leq M - \mu$ on C_1. Since $u < M$ everywhere in the interior of S and C_1 is a closed subset of S.

Define the function

$$v(x,t) = e^{-\alpha|X-X_s|^2_{\mathbb{R}^{n+1}}} - e^{-\alpha|X_0-X_s|^2_{\mathbb{R}^{n+1}}},$$

and choose α large enough such that

$$Pv(x,t) > 0 \text{ for all } (x,t) \text{ on } D \cup \partial D.$$

Now, consider the function

$$w = u + \epsilon v.$$

Observe that for every positive ϵ, $Pw = Pu + \epsilon Pv > 0$ everywhere in D. It follows by (iii) that there exists an ϵ so small that we have

$$w < M \quad on \quad C_1. \tag{2.4}$$

Now $v = 0$ on ∂S, also on the arc C_2. This together with relation (i) gives

$$w < M \quad on \quad C_2 \quad except \quad at \quad X_0, \tag{2.5}$$

and

$$w = M \quad at \quad X_0. \tag{2.6}$$

Applying the maximum principle for the function w and using (2.4), (2.5) and (2.6) we conclude that the maximum of the function w occurs only at the boundary point X_0. It follows that

$$\frac{\partial w}{\partial \nu} = \frac{\partial u}{\partial \nu} + \epsilon \frac{\partial v}{\partial \nu} \geq 0 \tag{2.7}$$

for any outward pointing direction ν of the set D. Denote by η the outer pointing unit normal to the boundary $\partial\tilde{\Omega}$ at X_0. We have that $\langle \nu, \eta \rangle > 0$ since ν is also outward pointing. Choose a coordinate system such that X_s is the origin and let $r(X) = |X - X_s|_{\mathbb{R}^{n+1}}$. We may rewrite v as

$$v(x,t) = e^{-\alpha r^2} - e^{-\alpha |X_0 - X_s|^2_{\mathbb{R}^{n+1}}},$$

than we have

$$\frac{\partial v}{\partial x_i} = -2\alpha x_i e^{-\alpha r^2}.$$

It follows that

$$\frac{\partial u}{\partial \nu} = -2\alpha r e^{-\alpha r^2} \langle \nu, \eta \rangle < 0.$$

Using this and (2.7) we conclude that

$$\frac{\partial u}{\partial \nu} > 0 \quad at \quad X_0.$$

□

2.2 Short and longtime existence results

The Neumann problem (1.6)-(1.8) we stated in Chapter I may be rewritten as follows

$$\begin{aligned} Pu &= 0 \quad \text{in} \quad \widetilde{\Omega}, \\ Mu &= -\phi \quad \text{in} \quad S\widetilde{\Omega}, \\ u &= u_0 \quad \text{in} \quad B\widetilde{\Omega} \cap C\widetilde{\Omega}. \end{aligned} \tag{2.8}$$

where

$$Pu = \Big(\sigma^{ij} - \frac{u^i}{W}\frac{u^j}{W}\Big)u_{i;j} - \Big(\frac{1}{2\gamma} + \frac{1}{2W^2}\Big)\gamma^i u_i - W\mathcal{H} - u_t \tag{2.9}$$

and the boundary operator is given by

$$Mu = \langle N, -\nu \rangle = \langle \frac{\nabla u}{W}, \nu \rangle. \tag{2.10}$$

We also assume that $Mu_0 = -\phi$ in $C\widetilde{\Omega}$ in order to have compatibility between the boundary and initial value conditions.

The first step towards solving (2.8) is to show that a solution exists for a short interval of time. After that we proof that given uniform bounds on $|\nabla u|$ and u we have a bound on $|u|_\delta$ for some $\delta \in (1,2)$. This Hölder norm will be defined in the sequel.

Then, the short time existence, under the assumption that $|u|_\delta$ is bounded implies the existence of solutions for $t \in [0, +\infty)$.

We now define the above mentioned Hölder space H_δ and other further spaces and norms needed for what follows. For $\alpha \in (0, 1]$, we say that a real function $f : \widetilde{\Omega} \to \mathbb{R}$ is Hölder continuous at X_0 with exponent α if the quantity

$$[f]_{\alpha;X_0} = \sup_{X \in \widetilde{\Omega} \setminus \{X_0\}} \frac{|f(X) - f(X_0)|}{|X - X_0|^\alpha}$$

is finite. Here and from now on we are considering the following parabolic distance between points $X = (x, t)$ and $X_0 = (x_0, t_0)$:

$$|X - X_0| = \max\{\mathrm{d}(x, x_0), \sqrt{t - t_0}\}. \tag{2.11}$$

If $[f]_{1:X_0}$ is finite , we say that f is Lipschitz continuous at X_0. Also it is easy to see that if f is Hölder continuous at a point, then it is also continuous there. If the semi-norm

$$[f]_{\alpha;\widetilde{\Omega}} = \sup_{X_0 \in \widetilde{\Omega}} [f]_{\alpha;X_0}$$

is finite, we say that f is uniformly Hölder continuous in $\widetilde{\Omega}$. Finally, if f is differentiable then it is Lipschitz.

We also define a kind of temporal Hölder quotient

$$\langle f \rangle_{\beta;X_0} = \sup \left\{ \frac{|f(x_0, t) - f(x_0, t_0)|}{|t - t_0|^{\frac{\beta}{2}}} : (x_0, t) \in \tilde{\Omega} \setminus \{(x_0, t_0)\} \right\}$$

with the corresponding semi-norm defined by

$$\langle f \rangle_{\beta;\widetilde{\Omega}} = \sup_{X_0 \in \widetilde{\Omega}} \langle f \rangle_{\beta;X_0}.$$

Then for any $a > 0$ such that $a = k + \alpha$, where k is a non-negative integer and $\alpha \in (0, 1]$, we can define

$$\langle f \rangle_{a;\widetilde{\Omega}} = \sum_{|\beta| + 2j = k-1} \langle \nabla^\beta \partial_t^j f \rangle_{\alpha+1},$$

$$[f]_{a;\widetilde{\Omega}} = \sum_{|\beta| + 2j = k} \left[\nabla^\beta \partial_t^j f \right]_\alpha,$$

$$|f|_{a;\widetilde{\Omega}} = \sum_{|\beta|+2j\leq k} \sup \left|\nabla^\beta \partial_t^j f\right| + [f]_{a;\widetilde{\Omega}} + \langle f\rangle_{a;\widetilde{\Omega}}.$$

We may verify that $|f|_\alpha$ defines a norm on $H_a(\widetilde{\Omega}) = \{f : \widetilde{\Omega} \to \mathbb{R}; |f|_a < \infty\}$ which makes $H_a(\widetilde{\Omega})$ a Banach space.

The smoothness of $\partial\Omega \subset P^n$ implies that given any sistem of local coordinates $(x^1, \ldots, x^{n-1}, x^n)$ which flatten out $\partial\Omega$ locally, we may describe $S\widetilde{\Omega}$ in terms of the augmented coordinate system $(x^1, \ldots, x^{n-1}, x^n, t)$ as a graph of the form

$$x^n = f(x^1, \ldots, x^{n-1}, t),$$

for some function $f \in H_\delta(Q)$, where $Q = B(0,r) \times [0,\varepsilon) \in \mathbb{R}^{n-1} \times \mathbb{R}$, for some $r > 0, \varepsilon > 0$ and for any $\delta \geq 1$. In particular we conclude from the very definition that the parabolic boundary $P\widetilde{\Omega}$ has H_δ regularity, for any $\delta \in (1,2)$.

The proof of short time existence for quasilinear partial equations follows in two steps. First we obtain the existence of a solution for an associated linear problem, and then extend the existence to the quasilinear case through a fixed point argument.

Given $\epsilon \in (0,T)$, we denote $\widetilde{\Omega}_\epsilon = \{X = (x,t) \in \widetilde{\Omega} : t < \epsilon\}$. Then, fixed a function u, we consider the linear problem

$$\begin{aligned}
&L_u v = a^{ij}(x, \nabla u)v_{i;j} - \left(\frac{1}{2\gamma^2} + \frac{1}{2(\gamma + |\nabla u|^2)}\right)\gamma^i v_i - v_t = \mathcal{H}\sqrt{\gamma + |\nabla u|^2} \\
&\quad \text{in} \quad P\widetilde{\Omega}_\epsilon, \\
&M_u v = \langle \frac{\nabla v}{\sqrt{\gamma + |\nabla u|^2}}, \nu\rangle = -\phi \quad \text{on} \quad S\widetilde{\Omega}_\epsilon, \\
&v = u_0 \quad \text{on} \quad B\widetilde{\Omega}_\epsilon \cup C\widetilde{\Omega}_\varepsilon.
\end{aligned} \tag{2.12}$$

Fixed $\delta \in (1,2)$ and $\theta \in (1,\delta)$, denote $B_0 = 1 + |u_0|_\theta$. Then define

$$\mathcal{S} = \{u \in H_\theta(\tilde{\Omega}_\epsilon); |u|_\theta \leq B_0\},$$

where $\epsilon > 0$ will be chosen later. We define the map $J : \mathcal{S} \to H_\theta$ by declaring that $Ju = v$ if v is *the* solution of the problem (2.12). We claim that J is well defined.

For that we use the next result which may be found in [48] and that yields a short time solution for the linear problem (2.12) under some requirements on the boundary and initial conditions as well as on the regularity of the parabolic boundary.

Theorem 8 *([48], Th. 5.18). Given a linear parabolic operator of the form*

$$Lv = \bar{a}^{ij}(x)v_{i;j} + \bar{a}^{i}(x)v_i - v_t$$

and the boundary operator

$$Nv = \langle \nabla v, \beta \rangle,$$

for a given vector field β, suppose that there exists $\alpha_0 \in (0,1)$ such that $P\widetilde{\Omega}$ is H_δ where $\delta = 2 + \alpha_0$ and that

- *L is uniformly parabolic, that is, that there exists $\bar{\lambda}$ and $\bar{\Lambda}$ so that*

$$\bar{\lambda}|\zeta|^2 \le \bar{a}^{ij}\zeta_i\zeta_j \le \bar{\Lambda}|\zeta|^2;$$

- *there exist positive constants A e B such that $|\bar{a}^{ij}|_{\alpha_0} \le A$, $|\bar{a}^{i}|_{\alpha_0} \le B$;*

- *there exist constants $\chi > 0$ and $B_1 > 0$ such that $\langle \beta, \nu \rangle \ge \chi$ and $|\beta|_{1+\alpha_0} \le B_1\chi$.*

Then for all $f \in H_{\alpha_0}$, $\phi \in H_{1+\alpha_0}$ and for any initial data $u_0 \in H_{2+\alpha_0}(\Omega) \cap C(\bar{\Omega})$ such that $Nu_0 = -\phi$, there exists a unique solution $v \in H_{2+\alpha_0}$ of the problem

$$\begin{aligned} Lv = f \quad & \text{in} \quad P\widetilde{\Omega}, \\ Nv = -\phi \quad & \text{on} \quad S\widetilde{\Omega}, \\ v = u_0 \quad & \text{on} \quad B\widetilde{\Omega} \cup C\widetilde{\Omega}. \end{aligned}$$

and there is a constant C determined only by A, B, B_1, C_1, n, α, γ, δ and $\tilde{\Omega}$ such that

$$|v|_{2+\alpha} \le C\big(|f|_{\alpha_0} + |\phi|_{1+\alpha_0}/\chi + |u_0|_{2+\alpha_0}\big).$$

Remark 9 *Since*

$$\beta = \frac{\nu}{\sqrt{\gamma + |\nabla u|^2}}.$$

it follows that

$$\langle \beta, \nu \rangle = \frac{1}{\sqrt{\gamma + |\nabla u|^2}} = |\beta|.$$

Hence we fix $\mu = 1$ in the original notation of Theorem 5.18 in [?]. Now we observe that if ∇u is uniformly bounded by a constant C_0 then we obtain we fix the constant R in the statement of the Theorem 5.18 as

$$\frac{2R}{\inf_\Omega \gamma - R\inf_\Omega |\nabla\gamma|} = \frac{\inf_\Omega \gamma}{2\sup_\Omega \gamma^2 + \sup_\Omega |\nabla\gamma|} \frac{\inf_\Omega \gamma}{\sup_\Omega \gamma + C_0^2}$$

Then we obtain

$$2R + 2R\sup|(\bar{a}^1, \ldots, \bar{a}^n)| \le \bar{\lambda}$$

since in our case

$$\bar{\lambda} = \frac{\gamma}{\gamma + |\nabla u|^2}$$

and

$$\bar{a}^i = -\left(\frac{1}{2\gamma^2} + \frac{1}{2(\gamma + |\nabla u|^2)}\right)\gamma^i$$

and $c = 0$.

To pass from the linear results to the quasilinear ones we need the following Brouwer fixed point theorem. The proof can be found in [48].

Theorem 10 *(Lieberman [48], 1996). Let S be a compact, convex subset of a Banach space $\mathcal{B}$ and let J be a continuous map of S into itself. Then J has a fixed point.*

Now we can state the result of short time existence for quasilinear problems.

Theorem 11 *Under the hypothesis of the Theorem 2, there exists a positive constant $\epsilon > 0$ such that the problem (2.8) has a unique solution $u \in H_{2+\alpha}$ defined in $\widetilde{\Omega}_\epsilon$.*

Proof. As we mentioned above, $P\widetilde{\Omega}_\epsilon$ is H_δ regular for *any* $\delta \in (1, 2)$. We proceed with the proof observing that the the gradient estimates we will obtain in the subsequent chapters for the quasilinear problem are uniform in $\widetilde{\Omega}_\epsilon$ (they are global in space, only local in time). Using the classical work by Ladyzhenskaia and Uraltseva [40] we prove that there exists a (locally defined) Hölder exponent α_0 such that u is bounded in the parabolic Hölder norm with such exponent. The compactness of Ω and the fact that ϵ may be taken small enough imply that we may choose the same α_0 for the whole domain $\widetilde{\Omega}_\epsilon$.

Hence given a prospective solution u of the quasilinear problem (2.8) there exists C_0 such that $|u|_{1+\alpha_0} \leq C_0$. Hence if we define

$$\bar{a}^{ij}(x) := \sigma^{ij} + \frac{u^i u^j}{\gamma + |\nabla u|^2}, \tag{2.13}$$

$$\bar{a}^i(x) := \left(\frac{1}{2\gamma^2} + \frac{1}{2(\gamma + |\nabla u|^2)}\right)\gamma^i, \tag{2.14}$$

$$f(x) := \mathcal{H}\sqrt{\gamma + |\nabla u|^2}, \tag{2.15}$$

$$\beta(x) := \frac{1}{\sqrt{\gamma + |\nabla u|^2}}\,\nu. \tag{2.16}$$

we obtain constants A, B, B_1 and χ depending on the $H_{1+\alpha_0}$ norm of u and on the geometry of Ω. Hence Theorem 8 implies that there exists a solution $v \in H_{2+\alpha_0}$ to the problem (2.12).

The same reasoning may be replicated starting with arbitrary functions u in the set

$$\mathcal{S} = \{u \in H_\theta(\tilde{\Omega}_\epsilon) : |u|_\theta \leq B_0\},$$

where $\theta \in (1, \delta)$ with $\delta = 1 + \alpha_0$ chosen in such a way that $H_{1+\alpha_0} \subset H_\theta$ continuously for any $\theta \in (1, \delta = 1 + \alpha_0)$. The time interval $[0, \epsilon)$ will be chosen later.

We conclude that the map $J : \mathcal{S} \to H_\theta$ is well-defined. In order to use the Brouwer fixed point theorem, we have to prove that J map $\mathcal{S}$ into itself. For proving this, we observe that the the Schauder-type estimate in Theorem 8 implies that

$$|v|_1 \leq |v|_{\delta=1+\alpha_0} \leq C|v|_{2+\alpha_0} \leq \hat{C}, \tag{2.17}$$

where C comes from Theorem 8 and depends on all the inicial data and boundary coefficients and also on $\hat{C} = \hat{C}(A, B, n, \alpha, \delta, \gamma, \tilde{\Omega}_\epsilon) < \infty$. In particular, we have $v \in H_\theta$. Now we will prove that $v \in \mathcal{S}$. Denoting $\theta = 1 + \alpha$ we have

$$|v - u_0|_\theta = \sup |\nabla v - \nabla u_0| + [\nabla v - \nabla u_0]_\alpha + \sup |v - u_0| + \langle v - u_0\rangle_{1+\alpha}.$$

The terms $|\nabla_x v|$ and $|v_t|$ are estimated by $\hat{C}$ since they are summands in the norm $|v|_1$. Then the first and third terms are controlled. Now since $v(\cdot, 0) = u_0$ we have

$$|\nabla v(x,t) - \nabla u_0(x)| = |\nabla v(x,t) - \nabla v(x,0)| \leq \hat{C}\epsilon^{\frac{1+\alpha}{2}},$$

where ∇ indicates both space and time derivatives, and

$$|v(x,t) - u_0(x)| = |v(x,t) - v(x,0)| \leq \hat{C}\epsilon.$$

With respect to the last term it follows from $|v|_t \leq \hat{C}$ that denoting $g = v - u_0$ we have

$$\langle g\rangle_{1+\alpha} = \sup_{s \neq t} \frac{g(x,s) - g(x,t)}{|s-t|^{\frac{1+\alpha}{2}}} \leq \hat{C} \sup_{s\neq t} |s-t|^{\frac{1-\alpha}{2}} \leq C\epsilon^{\frac{1-\alpha}{2}}.$$

Finally observing that $[\nabla v]_{1+\alpha}$ is estimated by $|v|_{2+\alpha}$ and then by $\hat{C}$ it results that

$$[\nabla g]_{1+\alpha} = \sup_{X,Y\in\tilde{\Omega},\, X\neq Y} \frac{|\nabla g(X) - \nabla g(Y)|}{(\max \mathrm{d}(x,y), \sqrt{s-t})^\alpha} \leq \hat{C}\epsilon^{1-\frac{\alpha}{2}}.$$

We conclude that choosing $\epsilon > 1$ there exists $C = C(n) > 0$ such that

$$|v - u_0|_\theta \leq C\hat{C}\epsilon^{\frac{1-\alpha}{2}}.$$

Then we choose ϵ small enough in order to guarantee that

$$|v|_\theta \leq |u_0|_\theta + |v - u_0|_\theta \leq |u_0|_\theta + C\hat{C}\epsilon^{\frac{1-\alpha}{2}} \leq B_0.$$

We conclude that J maps $\mathcal{S}$ into itself and we can apply the Theorem 10, since the set $\mathcal{S}$ is a ball in the function space $H_\theta(\widetilde{\Omega}_\epsilon)$, and so a convex set. Then the map J has a fixed point u, which is in $H_{2+\alpha(\theta-1)}$ and which solves our quasilinear problem. This completes the proof. □

Finally we have the following longtime existence theorem.

Theorem 12 *Suppose that we have short-time existence to problem* (2.8) *and that there exist constants* $\delta \in (1, 2)$ *and* $C_\delta > 0$ *such that*

$$|u|_\delta \leq C_\delta$$

in the maximal interval of definition. Then there exists a solution to (2.8) *defined in* $[0, +\infty)$.

Proof. Suppose that there exists a solution u to problem 2.8 defined in some maximal open time interval $[0, T)$ where T is finite. Then u satisfies $L_u u = 0$ in $\widetilde{\Omega}$, $M_u u = -\phi$ on $S\widetilde{\Omega}$ and $u = u_0$ on $B\widetilde{\Omega}$. Setting $\delta = 1+\alpha$, it follows that the estimate $|u|_\delta \leq C_\delta$ implies that there exists $\alpha \in [0, 1)$ such that $|a^{ij}(X, \nabla u)|_\alpha$ and $|a(X, \nabla u)|_\alpha$ are bounded by a constant depending on C_δ. It follows by Theorem 8 that we have the uniform estimate

$$|u|_{2+\alpha} \leq C_1(C_\delta)|u_0|_{2+\alpha} = C_2, \text{ for } t \in [0, T). \tag{2.18}$$

Now, take a sequence of times $t_i \to T$, and define $\tilde{u}_i(\cdot) = u(\cdot, t_i)$. Then the bound $|u|_\delta$ implies that there exists a subsequence, which by abuse of notation we also write $\tilde{u}_i$, such that

$$\tilde{u}_i \to \tilde{u} \text{ uniformly as } i \to \infty. \tag{2.19}$$

Moreover, by (2.18) we have equicontinuity of $\nabla_j \tilde{u}_i$, $\nabla^2_{jk}\tilde{u}_i$ and $\tilde{u}_{i,t}$ and taking subsequences we have

$$\nabla_j \tilde{u}_i \to \nabla_j \tilde{u}, \quad \nabla^2_{jk}\tilde{u}_i \to \nabla^2_{jk}\tilde{u} \quad \text{and} \quad \tilde{u}_{i,t} \to \tilde{u}_t, \tag{2.20}$$

uniformly, where we define $\tilde{u}_t$ here to be $a^{ij}(x, \nabla\tilde{u})u_{i;j}\tilde{+}a(x, \nabla\tilde{u})$.

Then we extend u to the interval $[0, T]$ by using $\tilde{u}$. The bound $|u|_\delta$ still holds by the C^2 convergence of $\tilde{u}_i$ to $\tilde{u}$, and so by the continuity of P and M we have that u is a solution of (2.8) on $[0, T]$.

Now we will prove that $\tilde{u} \in C^{2+\alpha}$. Let $x, y \in \Omega$, for simplicity we denote by $\nabla^2 u$ an arbitrary component $\nabla^2_{jk}u$. Using the uniform convergence of the second derivatives we choose t sufficiently close to T that

$$|\nabla^2\tilde{u}(\cdot) - \nabla^2 u(\cdot, t)| < \epsilon < d(x, y).$$

Then

$$\frac{|\nabla^2 u(x) - \nabla^2 u(y)|}{d(x, y)^\alpha} \leq \frac{|\nabla^2 u(x, t) - \nabla^2 u(y, t)|}{\max\{d(x, y), |T - t|^{\frac{1}{2}}\}^\alpha} \leq 2 + C_2$$

due to the bound on $[\nabla^2 u]_\alpha$ for $t < T$. It follows that $|\tilde{u}|_{2+\alpha}$.

Now we apply the short time existence theorem to (2.8) but with $u_0 = \tilde{u}$ and get a solution $\hat{u}$ in Ω_ϵ. Then we define

$$w(x, t) = \begin{cases} u(x, t) \text{ for } (x, t) \in \Omega \times [0, T] \\ \hat{u}(x, t - T) \text{ for } (x, t) \in \Omega \times [T, T + \epsilon]. \end{cases}$$

We have that $u_t(\cdot, s) \to \hat{u}_t(0)$ as $s \to T$, then w is twice differentiable in space and once differentiable in time and satisfies $Pw = 0$ and $Mw = -\phi$. Moreover, by the strong maximum principle it is the unique solution $Lw = 0$ and by Theorem 8 it follows that $w \in H^{2+\alpha}(\Omega_{T+\epsilon})$. And this contradicts the definition of T. This completes the proof. □

Capítulo 3

Capillary Problem

Consider the capillary equation

$$\operatorname{div}\Big(\frac{\nabla u}{W}\Big) - \langle \frac{\nabla \gamma}{2\gamma}, \frac{\nabla u}{W}\rangle = \Psi. \tag{3.1}$$

Given a $C^{2,\alpha}$ function $\Phi : K \to (-1,1)$, we will impose the following prescribed angle condition

$$\langle N, \nu\rangle = \Phi \tag{3.2}$$

along $\partial\Sigma$, where

$$N = \frac{1}{W}\big(\gamma Y - \vartheta_* \nabla u\big) \tag{3.3}$$

is the unit normal vector field along Σ satisfying $\langle N, Y\rangle > 0$ and ν is the unit normal vector field along K pointing inwards the Killing cylinder over Ω.

Equation (3.1) is the prescribed mean curvature equation for Killing graphs.

We suppose that the data Ψ and Φ satisfy

i. $|\Psi| + |\bar\nabla\Psi| \leq C_\Psi$ in $\bar\Omega \times \mathbb{R}$,

ii. $\langle \bar\nabla\Psi, Y\rangle \geq \beta > 0$ in $\bar\Omega \times \mathbb{R}$,

iii. $\langle \bar\nabla\Phi, Y\rangle \leq 0$,

iv. $(1-\Phi^2) \geq \beta'$,

v. $|\Phi|_2 \leq C_\Phi$ in K,

for some positive constants C_Ψ, C_Φ, β and β', where $\bar\nabla$ denotes the Riemannian connection in M.

The main result in this chapter is the following one

Theorem 13 *Let Ω be a bounded $C^{3,\alpha}$ domain in P. Suppose that the $\Psi \in C^{1,\alpha}(\bar{\Omega}\times\mathbb{R})$ and $\Phi \in C^{2,\alpha}(K)$ with $|\Phi| \leq 1$ satisfy conditions* (i)-(v) *above. Then there exists a unique solution $u \in C^{3,\alpha}(\bar{\Omega})$ of the capillary problem* (3.1)-(3.2).

We will use the classical Continuity Method to (3.1)-(3.2) for proving the existence of result. So we need *a priori* height estimates and a interior and boundary gradient estimates.

3.1 Height estimates

In this section, we use a technique developed by N. Uraltseva [41] (see also [40] and [45] for classical references on the subject) in order to obtain a height estimate for solutions of the capillary problem (3.1)-(3.2). This estimate requires the *positive gravity* assumption (*ii*) stated in the Introduction.

Proposition 14 *Denote*

$$\beta = \inf_{\Omega\times\mathbb{R}} \langle \bar{\nabla}\Psi, Y\rangle \tag{3.4}$$

and

$$\mu = \sup_{\Omega} \Psi(x, 0). \tag{3.5}$$

Suppose that $\beta > 0$. Then any solution u of (3.1)-(3.2) satisfies

$$|u(x)| \leq \frac{\sup_\Omega |Y|}{\inf_\Omega |Y|}\frac{\mu}{\beta} \tag{3.6}$$

for all $x \in \bar{\Omega}$.

Proof. Fix an arbitrary real number k with

$$k > \frac{\sup_\Omega |Y|}{\inf_\Omega |Y|}\frac{\mu}{\beta}.$$

Suppose that the superlevel set

$$\Omega_k = \{x \in \Omega : u(x) > k\}$$

has a nonzero Lebesgue measure. Define $u_k : \Omega \to \mathbb{R}$ as

$$u_k(x) = \max\{u(x) - k, 0\}.$$

From the variational formulation we have

$$\begin{aligned} 0 &= \int_{\Omega_k} \left(\frac{1}{\sqrt{\gamma}} \frac{\langle \nabla u, \nabla u_k \rangle}{\sqrt{\gamma + |\nabla u^2|}} + \frac{1}{\sqrt{\gamma}} \Psi(x, u(x)) u_k \right) \sqrt{\sigma} \mathrm{d}x \\ &= \int_{\Omega_k} \left(\frac{1}{\sqrt{\gamma}} \frac{|\nabla u|^2}{W} + \frac{1}{\sqrt{\gamma}} \Psi(x, u(x))(u-k) \right) \sqrt{\sigma} \mathrm{d}x \\ &= \int_{\Omega_k} \left(\frac{1}{\sqrt{\gamma}} \frac{W^2 - \gamma}{W} + \frac{1}{\sqrt{\gamma}} \Psi(x, u(x))(u-k) \right) \sqrt{\sigma} \mathrm{d}x \\ &= \int_{\Omega_k} \left(\frac{W}{\sqrt{\gamma}} - \frac{\sqrt{\gamma}}{W} + \frac{1}{\sqrt{\gamma}} \Psi(x, u(x))(u-k) \right) \sqrt{\sigma} \mathrm{d}x. \end{aligned}$$

However

$$\Psi(x, u(x)) = \Psi(x, 0) + \int_0^{u(x)} \frac{\partial \Psi}{\partial s} \mathrm{d}s \geq -\mu + \beta u(x).$$

Since $\frac{\sqrt{\gamma}}{W} \leq 1$ we conclude that

$$|\Omega_k| - |\Omega_k| - \mu \int_{\Omega_k} \frac{1}{\sqrt{\gamma}}(u-k) + \beta \int_{\Omega_k} \frac{1}{\sqrt{\gamma}} u(u-k) \leq 0.$$

Hence we have

$$\beta \int_{\Omega_k} \frac{1}{\sqrt{\gamma}} u(u-k) \leq \mu \int_{\Omega_k} \frac{1}{\sqrt{\gamma}}(u-k).$$

It follows that

$$\beta k \inf_{\Omega} |Y| \int_{\Omega_k} (u-k) \leq \mu \sup_{\Omega} |Y| \int_{\Omega_k} (u-k)$$

Since $|\Omega_k| \neq 0$ we have

$$k \leq \frac{\sup_\Omega |Y|}{\inf_\Omega |Y|} \frac{\mu}{\beta},$$

what contradicts the choice of k. We conclude that $|\Omega_k| = 0$ for all $k \geq \frac{\sup_\Omega |Y|}{\inf_\Omega |Y|} \frac{\mu}{\beta}$. This implies that

$$u(x) \leq \frac{\sup_\Omega |Y|}{\inf_\Omega |Y|} \frac{\mu}{\beta},$$

for all $x \in \bar{\Omega}$. A lower estimate may be deduced in a similar way. This finishes the proof of the Proposition. □

Remark 15 *The construction of geometric barriers similar to those ones in [37] is also possible at least in the case where P is endowed with a rotationally invariant metric and Ω is contained in a normal neighborhood of a pole of P.*

3.2 Gradient estimates

Let Ω' be a subset of Ω and define

$$\Sigma' = \{\vartheta(u(x), x) : x \in \Omega'\} \subset \Sigma \tag{3.7}$$

be the graph of $u|_{\Omega'}$. Let $\mathcal{O}$ be an open subset in M containing Σ'. We consider a vector field $Z \in \Gamma(TM)$ with bounded C^2 norm and supported in $\mathcal{O}$. Hence there exists $\varepsilon > 0$ such that the local flow $\Xi : (-\varepsilon, \varepsilon) \times \mathcal{O} \to M$ generated by Z is well-defined. We also suppose that

$$\langle Z(y), \nu(y)\rangle = 0, \tag{3.8}$$

for any $y \in K \cap \mathcal{O}$. This implies that the flow line of Z passing through a point $y \in K \cap \mathcal{O}$ is entirely contained in K.

We define a variation of Σ by a one-parameter family of hypersurfaces Σ_τ, $\tau \in (-\varepsilon, \varepsilon)$, parameterized by $X_\tau : \bar{\Omega} \to M$ where

$$X_\tau(x) = \Xi(\tau, \vartheta(u(x), x)), \quad x \in \bar{\Omega}. \tag{3.9}$$

It follows from the Implicit Function Theorem that there exists $\Omega_\tau \subset P$ and $u_\tau : \bar{\Omega}_\tau \to \mathbb{R}$ such that Σ_τ is the graph of u_τ. Moreover, (3.8) implies that the $\Omega_\tau \subset \Omega$.

Hence given a point $y \in \Sigma$, denote $y_\tau = \Xi(\tau, y) \in \Sigma_\tau$. It follows that there exists $x_\tau \in \Omega_\tau$ such that $y_\tau = \vartheta(u_\tau(x_\tau), x_\tau)$. Then we denote by $\hat{y}_\tau = \vartheta(u(x_\tau), x_\tau)$ the point in Σ in the flow line of Y passing through y_τ. The vertical separation between y_τ and $\hat{y}_\tau$ is by definition the function $s(y, \tau) = u_\tau(x_\tau) - u(x_\tau)$.

Lemma 16 *For any $\tau \in (-\varepsilon, \varepsilon)$, let A_τ and H_τ be, respectively, the Weingarten map and the mean curvature of the hypersurface Σ_τ calculated with respect to the unit normal vector field N_τ along Σ_τ which satisfies $\langle N_\tau, Y\rangle > 0$. Denote $H = H_0$ and $A = A_0$. If $\zeta \in C^\infty(\mathcal{O})$ and $T \in \Gamma(T\mathcal{O})$ are defined by*

$$Z = \zeta N_\tau + T \tag{3.10}$$

with $\langle T, N_\tau\rangle = 0$ then

i. $\frac{\partial s}{\partial \tau}\Big|_{\tau=0} = \langle Z, N\rangle W.$

ii. $\bar{\nabla}_Z N\Big|_{\tau=0} = -AT - \nabla^\Sigma \zeta$

iii. $\frac{\partial H}{\partial \tau}\Big|_{\tau=0} = \Delta_\Sigma \zeta + (|A|^2 + \mathrm{Ric}_M(N, N))\zeta + \langle \bar{\nabla}\Psi, Z\rangle,$

where $W = \langle Y, N_\tau\rangle^{-1} = (\gamma + |\nabla u_\tau|^2)^{-1/2}$*. The operators* ∇^Σ *and* Δ_Σ *are, respectively, the intrinsic gradient operator and the Laplace-Beltrami operator in* Σ *with respect to the induced metric. Moreover,* $\bar\nabla$ *and* Ric_M *denote, respectively, the Riemannian covariant derivative and the Ricci tensor in* M*.*

Proof. (i) Let $(x^i)_{i=1}^n$ a set of local coordinates in $\Omega \subset P$. Differentiating (3.9) with respect to τ we obtain

$$X_{\tau *}\frac{\partial}{\partial \tau} = Z|_{X_\tau} = \zeta N_\tau + T$$

On the other hand differentiating both sides of

$$X_\tau(x) = \vartheta(u_\tau(x_\tau), x_\tau)$$

with respect to τ we have

$$\begin{aligned} X_{\tau *}\frac{\partial}{\partial \tau} &= \Big(\frac{\partial u_\tau}{\partial \tau} + \frac{\partial u_\tau}{\partial x^i}\frac{\partial x^i_\tau}{\partial \tau}\Big)\vartheta_* Y + \frac{\partial x^i_\tau}{\partial \tau}\vartheta_*\frac{\partial}{\partial x^i} \\ &= \frac{\partial u_\tau}{\partial \tau}\vartheta_* Y + \frac{\partial x^i_\tau}{\partial \tau}\Big(\vartheta_*\frac{\partial}{\partial x^i} + \frac{\partial u_\tau}{\partial x^i}\vartheta_* Y\Big)\end{aligned}$$

Since the term between parenthesis after the second equality is a tangent vector field in Σ_τ we conclude that

$$\frac{\partial u_\tau}{\partial \tau}\langle Y, N_\tau\rangle = \langle X_{\tau *}\frac{\partial}{\partial \tau}, N_\tau\rangle = \zeta$$

from what follows that

$$\frac{\partial u_\tau}{\partial \tau} = \zeta W$$

and

$$\frac{\partial s}{\partial \tau} = \frac{\partial}{\partial \tau}(u_\tau - u) = \frac{\partial u_\tau}{\partial \tau} = \zeta W.$$

(ii) Now we have

$$\begin{aligned}&\langle \bar\nabla_Z N_\tau, X_*\partial_i\rangle = -\langle N_\tau, \bar\nabla_Z X_*\partial_i\rangle = -\langle N_\tau, \bar\nabla_{X_*\partial_i} Z\rangle = -\langle N_\tau, \bar\nabla_{X_*\partial_i}(\zeta N + T)\rangle \\ &= -\langle N_\tau, \bar\nabla_{X_*\partial_i} T\rangle - \langle N_\tau, \bar\nabla_{X_*\partial_i}\zeta N_\tau\rangle = -\langle A_\tau T, X_*\partial_i\rangle - \langle \nabla^\Sigma \zeta, X_*\partial_i\rangle,\end{aligned}$$

for any $1 \le i \le n$. It follows that

$$\bar\nabla_Z N = -AT - \nabla^\Sigma \zeta.$$

(iii) This is a well-known formula whose proof may be found at a number of references (see, for instance, [46]). □

For further reference, we point out that the Comparison Principle [45] when applied to (3.1)-(3.2) may be stated in geometric terms as follows. Fixed τ, let $x \in \bar{\Omega}'$ be a point of maximal vertical separation $s(\cdot, \tau)$. If x is an interior point we have

$$\nabla u_\tau(x, \tau) - \nabla u(x) = \nabla s(x, \tau) = 0,$$

what implies that the graphs of the functions u_τ and $u + s(x, \tau)$ are tangent at their common point $y_\tau = \vartheta(u_\tau(x), x)$. Since the graph of $u + s(x, \tau)$ is obtained from Σ only by a translation along the flow lines of Y we conclude that the mean curvature of these two graphs are the same at corresponding points. Since the graph of $u + s(x, \tau)$ is locally above the graph of u_τ we conclude that

$$H(\hat{y}_\tau) \geq H_\tau(y_\tau). \tag{3.11}$$

If $x \in \partial\Omega \subset \partial\Omega'$ we have

$$\langle \nabla u_\tau, \nu \rangle|_x - \langle \nabla u, \nu \rangle|_x = \langle \nabla s, \nu \rangle \leq 0$$

since ν points toward Ω. This implies that

$$\langle N, \nu \rangle|_{y_\tau} \geq \langle N, \nu \rangle|_{\hat{y}_\tau} \tag{3.12}$$

3.2.1 Interior gradient estimate

Proposition 17 *Let $B_R(x_0) \subset \Omega$ where $R < \mathrm{inj}P$. Then there exists a constant $C > 0$ depending on β, C_Ψ, Ω and K such that*

$$|\nabla u(x)| \leq C \frac{R^2}{R^2 - d^2(x)}, \tag{3.13}$$

where $d = \mathrm{dist}(x_0, x)$ in P.

Proof. Fix $\Omega' = B_R(x_0) \subset \Omega$. We consider the vector field Z given by

$$Z = \zeta N, \tag{3.14}$$

where ζ is a function to be defined later. Fixed $\tau \in [0, \varepsilon)$, let $x \in B_R(x_0)$ be a point where the vertical separation $s(\cdot, \tau)$ attains a maximum value.

If $y = \vartheta(u(x), x)$ it follows that

$$H_\tau(y_\tau) - H_0(y) = \frac{\partial H_\tau}{\partial \tau}\Big|_{\tau=0} \tau + o(\tau). \tag{3.15}$$

However the Comparison Principle implies that $H_0(\hat{y}_\tau) \geq H_\tau(y_\tau)$. Using Lemma 16 (iii) we conclude that

$$H_0(\hat{y}_\tau) - H_0(y) \geq \frac{\partial H_\tau}{\partial \tau}\Big|_{\tau=0} \tau + o(\tau) = (\Delta_\Sigma \zeta + |A|^2 \zeta + \mathrm{Ric}_M(N,N)\zeta)\tau + o(\tau).$$

Since $\hat{y}_\tau = \vartheta(-s(y,\tau), y_\tau)$ we have

$$\frac{d\hat{y}_\tau}{d\tau}\Big|_{\tau=0} = -\frac{ds}{d\tau}\vartheta_* \frac{\partial}{\partial s} + \frac{\partial y^i_\tau}{\partial \tau}\vartheta_* \frac{\partial}{\partial x^i} = -\frac{ds}{d\tau} Y + \frac{dy_\tau}{d\tau}\Big|_{\tau=0} = -\frac{ds}{d\tau} Y + Z(y). \tag{3.16}$$

Hence using Lemma 16 (i) and (3.14) we have

$$\frac{d\hat{y}_\tau}{d\tau}\Big|_{\tau=0} = -\zeta W Y + \zeta N. \tag{3.17}$$

On the other hand for each $\tau \in (-\varepsilon, \varepsilon)$ there exists a smooth $\xi : (-\varepsilon, \varepsilon) \to TM$ such that

$$\hat{y}_\tau = \exp_y \xi(\tau).$$

Hence we have

$$\frac{d\hat{y}_\tau}{d\tau}\Big|_{\tau=0} = \xi'(0).$$

With a slight abuse of notation we denote $\Psi(s,x)$ by $\Psi(y)$ where $y = \vartheta(s,x)$. It results that

$$H_0(\hat{y}_\tau) - H_0(y) = \Psi(x_\tau, u(x_\tau)) - \Psi(x, u(x)) = \Psi(\exp_y \xi_\tau) - \Psi(y) = \langle \bar{\nabla}\Psi|_y, \xi'(0)\rangle \tau + o(\tau).$$

However

$$\langle \bar{\nabla}\Psi, \xi'(0)\rangle = \zeta \langle \bar{\nabla}\Psi, N - WY\rangle = -\zeta W \frac{\partial \Psi}{\partial s} + \zeta \langle \bar{\nabla}\Psi, N\rangle. \tag{3.18}$$

We conclude that

$$-\zeta W \frac{\partial \Psi}{\partial s}\tau + \zeta\langle \bar{\nabla}\Psi, N\rangle \tau + o(\tau) \geq (\Delta_\Sigma \zeta + |A|^2 \zeta + \mathrm{Ric}_M(N,N)\zeta)\tau + o(\tau).$$

Suppose that

$$W(x) > \frac{C + |\bar{\nabla}\Psi|}{\beta} \tag{3.19}$$

for a constant $C > 0$ to be chosen later. Hence we have

$$(\Delta_\Sigma \zeta + \mathrm{Ric}_M(N,N)\zeta)\tau + C\zeta\tau \le o(\tau).$$

Following [18] and [17] we choose

$$\zeta = 1 - \frac{d^2}{R^2},$$

where $d = \mathrm{dist}(x_0, \cdot)$. It follows that

$$\nabla^\Sigma \zeta = -\frac{2d}{R^2}\nabla^\Sigma d$$

and

$$\Delta_\Sigma \zeta = -\frac{2d}{R^2}\Delta_\Sigma d - \frac{2}{R^2}|\nabla^\Sigma d|^2$$

However using the fact that P is totally geodesic and that $[Y, \bar\nabla d] = 0$ we have

$$\begin{aligned}\Delta_\Sigma d &= \Delta_M d - \langle \bar\nabla_N \bar\nabla d, N\rangle + nH\langle \bar\nabla d, N\rangle \\ &= \Delta_P d - \langle \nabla_{\frac{\nabla u}{W}} \nabla d, \frac{\nabla u}{W}\rangle - \gamma^2\langle Y, N\rangle^2 \langle \bar\nabla_Y \bar\nabla d, Y\rangle + nH\langle \bar\nabla d, N\rangle\end{aligned}$$

Let $\pi : M \to P$ the projection defined by $\pi(\vartheta(s,x)) = x$. Then

$$\pi_* N = -\frac{\nabla u}{W}.$$

We denote

$$\pi_* N^\perp = \pi_* N - \langle \pi_* N, \nabla d\rangle \nabla d.$$

If $\mathcal{A}_d$ and $\mathcal{H}_d$ denote, respectively, the Weingarten map and the mean curvature of the geodesic ball $B_d(x_0)$ in P we conclude that

$$\Delta_\Sigma d = n\mathcal{H}_d - \langle \mathcal{A}_d(\pi_* N^\perp), \pi_* N^\perp\rangle + \gamma\langle Y, N\rangle^2 \kappa + nH\langle \bar\nabla d, N\rangle.$$

where

$$\kappa = -\gamma\langle \bar\nabla_Y \bar\nabla d, Y\rangle$$

is the principal curvature of the Kiling cylinder over $B_d(x_0)$ relative to the principal direction Y. Therefore we have

$$|\Delta_\Sigma d| \le C_1(C_\Psi, \sup_{B_R(x_0)} (\mathcal{H}_d + \kappa), \sup_{B_R(x_0)} \gamma)$$

in $B_R(x_0)$. Hence setting

$$C_2 = \sup_{B_R(x_0)} \mathrm{Ric}_M$$

we fix

$$C = \max\{2(C_1 + C_2), \sup_{\mathbb{R}\times\Omega} |\bar{\nabla}\Psi|\}. \tag{3.20}$$

With this choice we conclude that

$$C\zeta \leq \frac{o(\tau)}{\tau},$$

a contradiction. This implies that

$$W(x) \leq \frac{C - |\bar{\nabla}\Psi|}{\beta}. \tag{3.21}$$

However

$$\zeta(z)W(z) + o(\tau) = s(X(z), \tau) \leq s(X(x), \tau) = \zeta(x)W(x) + o(\tau),$$

for any $z \in B_R(x_0)$. It follows that

$$W(z) \leq \frac{R^2 - d^2(z)}{R^2 - d^2(x)} W(x) + o(\tau) \leq \frac{R^2}{R^2 - d^2(x)} \frac{C - |\bar{\nabla}\Psi|}{\beta} + o(\tau) \leq \widetilde{C} \frac{R^2}{R^2 - d^2(x)},$$

for very small $\varepsilon > 0$. This finishes the proof of the proposition. □

Remark 18 *If Ω satisfies the interior sphere condition for a uniform radius $R > 0$ we conclude that*

$$W(x) \leq \frac{C}{d_\Gamma(x)}, \tag{3.22}$$

for $x \in \Omega$, where $d_\Gamma(x) = \mathrm{dist}(x, \Gamma)$.

3.2.2 Boundary gradient estimates

Now we establish boundary gradient estimates using other local perturbation of the graph which this time has also tangential components.

Proposition 19 *Let $x_0 \in P$ and $R > 0$ such that $3R < \mathrm{inj}P$. Denote by Ω' the subdomain $\Omega \cap B_{2R}(x_0)$. Then there exists a positive constant $C = C(R, \beta, \beta', C_\Psi, C_\Phi, \Omega, K)$ such that*

$$W(x) \leq C, \tag{3.23}$$

for all $x \in \overline{\Omega}'$.

Proof. Now we consider the subdomain $\Omega' = \Omega \cap B_R(x_0)$. We define

$$Z = \eta N + X, \tag{3.24}$$

where

$$\eta = \alpha_0 v + \alpha_1 d_\Gamma$$

and α_0 and α_1 are positive constants to be chosen and d_Γ is a smooth extension of the distance function $\mathrm{dist}(\,\cdot\,,\Gamma)$ to Ω' with $|\nabla d_\Gamma| \leq 1$ and

$$v = 4R^2 - d^2,$$

where $d = \mathrm{dist}(x_0, \cdot)$. Moreover

$$X = \alpha_0 \Phi (v\nu - d_\Gamma \nabla v).$$

In this case we have

$$\zeta = \eta + \langle X, N\rangle = \alpha_0 v + \alpha_1 d_\Gamma + \alpha_0 \Phi(v\langle N, \nu\rangle - d_\Gamma \langle N, \nabla v\rangle).$$

Fixed $\tau \in [0, \varepsilon)$, let $x \in \bar{\Omega}'$ be a point where the maximal vertical separation between Σ and Σ_τ is attained. We first suppose that $x \in \mathrm{int}(\partial\Omega' \cap \partial\Omega)$. In this case denoting $y_\tau = \vartheta(u_\tau(x), x) \in \Sigma_\tau$ and $\hat{y}_\tau = \vartheta(u(x), x) \in \Sigma$ it follows from the Comparison Principle that

$$\langle N_\tau, \nu\rangle|_{y_\tau} \geq \langle N, \nu\rangle|_{\hat{y}_\tau}. \tag{3.25}$$

Notice that $\hat{y}_\tau \in \partial\Sigma$. Moreover since $Z|_{K\cap\mathcal{O}}$ is tangent to K there exists $y \in \partial\Sigma$ such that

$$y = \Xi(-\tau, y_\tau).$$

We claim that

$$|\langle \bar{\nabla}\langle N_\tau, \nu\rangle, \frac{dy_\tau}{d\tau}|_{\tau=0}\rangle| \leq \alpha_1(1 - \Phi^2) + \widetilde{C}\alpha_0 \tag{3.26}$$

for some positive constant $\widetilde{C} = C(C_\Phi, K, \Omega, R)$.

Hence (3.2) implies that

$$\langle N, \nu\rangle|_{\hat{y}_\tau} - \langle N, \nu\rangle|_y = \Phi(\hat{y}_\tau) - \Phi(y) = \tau\langle \bar{\nabla}\Phi, \frac{d\hat{y}_\tau}{d\tau}|_{\tau=0}\rangle + o(\tau).$$

Therefore

$$\langle N, \nu\rangle|_{y_\tau} - \langle N, \nu\rangle|_y \geq \tau\langle \bar{\nabla}\Phi, \frac{d\hat{y}_\tau}{d\tau}|_{\tau=0}\rangle + o(\tau).$$

On the other hand we have

$$\langle N,\nu\rangle|_{y_\tau}-\langle N,\nu\rangle|_y=\tau\langle\bar\nabla\langle N,\nu\rangle,\frac{dy_\tau}{d\tau}\big|_{\tau=0}\rangle+o(\tau).$$

We conclude that

$$\tau\langle\bar\nabla\langle N,\nu\rangle,\frac{dy_\tau}{d\tau}\big|_{\tau=0}\rangle\geq\tau\langle\bar\nabla\Phi,\frac{d\hat y_\tau}{d\tau}\big|_{\tau=0}\rangle+o(\tau).$$

Hence we have

$$\alpha_1(1-\Phi^2)\tau+\widetilde C\alpha_0\tau\geq\tau\langle\bar\nabla\Phi,\frac{d\hat y_\tau}{d\tau}\big|_{\tau=0}\rangle+o(\tau).$$

It follows from (3.16) that

$$\alpha_1(1-\Phi^2)+\widetilde C\alpha_0\geq-\zeta W\langle\bar\nabla\Phi,Y\rangle+\zeta\langle\bar\nabla\Phi,N\rangle+o(\tau)/\tau.$$

Since

$$\langle\bar\nabla\Phi,Y\rangle=\frac{\partial\Phi}{\partial s}\leq 0$$

we conclude that

$$W(x)\leq C(C_\Phi,\beta',K,\Omega,R).\tag{3.27}$$

We now prove the claim. For that, observe that Lemma 16 (*ii*) implies that

$$\begin{aligned}\langle N,\nu\rangle|_{y_\tau}-\langle N,\nu\rangle|_y&=\tau\frac{\partial}{\partial\tau}\Big|_{\tau=0}\langle N_\tau,\nu\rangle|_{y_\tau}+o(\tau)\\&=\tau(\langle N,\bar\nabla_Z\nu\rangle|_y-\langle AT+\nabla^\Sigma\zeta,\nu\rangle|_y)+o(\tau).\end{aligned}$$

Since $Z|_y\in T_yK$ it follows that

$$\langle N,\nu\rangle|_{y_\tau}-\langle N,\nu\rangle|_y=-\tau(\langle A_KZ,N\rangle|_y+\langle AT+\nabla^\Sigma\zeta,\nu\rangle|_y)+o(\tau),$$

where A_K is the Weingarten map of K with respect to ν. We conclude that

$$-\tau(\langle A_KZ,N\rangle|_y+\langle AT+\nabla^\Sigma\zeta,\nu\rangle|_y)\geq\tau\langle\bar\nabla\Phi,\frac{d\hat y_\tau}{d\tau}\big|_{\tau=0}\rangle+o(\tau)\tag{3.28}$$

where

$$\nu^T=\nu-\langle N,\nu\rangle N.$$

We have

$$\langle\nabla^\Sigma\zeta+AT,\nu^T\rangle=\alpha_0\langle\nabla v,\nu^T\rangle+\alpha_1\langle\nabla^\Sigma d_\Gamma,\nu^T\rangle+\langle\nabla^\Sigma\langle X,N\rangle,\nu^T\rangle+\langle AT,\nu^T\rangle.$$

We compute

$$\langle\nabla^\Sigma\langle X,N\rangle,\nu^T\rangle=\alpha_0(v\langle N,\nu\rangle-d_\Gamma\langle N,\nabla v\rangle)\langle\bar\nabla\Phi,\nu^T\rangle$$
$$+\alpha_0\Phi\big(\langle\nabla v,\nu^T\rangle\langle N,\nu\rangle+v(\langle\bar\nabla_{\nu^T}N,\nu\rangle+\langle N,\bar\nabla_{\nu^T}\nu\rangle)-\langle\nabla d_\Gamma,\nu^T\rangle\langle N,\nabla v\rangle$$
$$-d_\Gamma(\langle\bar\nabla_{\nu^T}N,\nabla v\rangle+\langle N,\bar\nabla_{\nu^T}\nabla v\rangle)\big).$$

Hence we have at y that

$$\langle\nabla^\Sigma\langle X,N\rangle,\nu^T\rangle=\alpha_0(v\Phi-d_\Gamma\langle N,\nabla v\rangle)\langle\bar\nabla\Phi,\nu^T\rangle$$
$$+\alpha_0\Phi\big(\langle\nabla v,\nu^T\rangle\Phi+v(-\langle A\nu^T,\nu^T\rangle+\langle N,\bar\nabla_\nu\nu\rangle-\langle N,\nu\rangle\langle N,\bar\nabla_N\nu\rangle)$$
$$-\langle\nu,\nu^T\rangle\langle N,\nabla v\rangle-d_\Gamma(-\langle A\nu^T,\nabla v\rangle+\langle N,\bar\nabla_\nu\nabla v\rangle-\langle N,\nu\rangle\langle N,\bar\nabla_N\nabla v\rangle)\big).$$

Therefore we have

$$\langle\nabla^\Sigma\langle X,N\rangle,\nu^T\rangle=\alpha_0(v\Phi-d_\Gamma\langle N,\nabla v\rangle)\langle\bar\nabla\Phi,\nu^T\rangle$$
$$+\alpha_0\Phi\big(\langle\nabla v,\nu^T\rangle\Phi-v(\langle A\nu^T,\nu^T\rangle+\langle N,\nu\rangle\langle N,\bar\nabla_N\nu\rangle)$$
$$-\langle\nu,\nu^T\rangle\langle N,\nabla v\rangle+d_\Gamma(\langle A\nu^T,\nabla v\rangle-\langle N,\bar\nabla_\nu\nabla v\rangle+\langle N,\nu\rangle\langle N,\bar\nabla_N\nabla v\rangle)\big).$$

It follows that

$$\langle\nabla^\Sigma\zeta+AT,\nu^T\rangle=\langle AT,\nu^T\rangle+\alpha_0\langle\nabla v,\nu^T\rangle+\alpha_1\langle\nu,\nu^T\rangle$$
$$+\alpha_0(v\Phi-d_\Gamma\langle N,\nabla v\rangle)\langle\bar\nabla\Phi,\nu^T\rangle$$
$$+\alpha_0\Phi\big(\langle\nabla v,\nu^T\rangle\Phi-v(\langle A\nu^T,\nu^T\rangle+\langle N,\nu\rangle\langle N,\bar\nabla_N\nu\rangle)$$
$$-\langle\nu,\nu^T\rangle\langle N,\nabla v\rangle+d_\Gamma(\langle A\nu^T,\nabla v\rangle-\langle N,\bar\nabla_\nu\nabla v\rangle+\langle N,\nu\rangle\langle N,\bar\nabla_N\nabla v\rangle)\big).$$

However

$$\langle AT,\nu^T\rangle=\langle A\nu^T,X\rangle=\alpha_0\Phi v\langle A\nu^T,\nu^T\rangle-\alpha_0\Phi d_\Gamma\langle A\nu^T,\nabla v\rangle.$$

Hence we have

$$\langle\nabla^\Sigma\zeta+AT,\nu^T\rangle=\alpha_0\langle\nabla v,\nu^T\rangle+\alpha_1\langle\nu,\nu^T\rangle+\alpha_0(v\Phi-d_\Gamma\langle N,\nabla v\rangle)\langle\bar\nabla\Phi,\nu^T\rangle$$
$$+\alpha_0\Phi\big(\langle\nabla v,\nu^T\rangle\Phi-v\Phi\langle N,\bar\nabla_N\nu\rangle-\langle\nu,\nu^T\rangle\langle N,\nabla v\rangle$$
$$-d_\Gamma(\langle N,\bar\nabla_\nu\nabla v\rangle-\langle N,\nu\rangle\langle N,\bar\nabla_N\nabla v\rangle)\big).$$

Since $d_\Gamma(y)=0$ we have

$$\langle\nabla^\Sigma\zeta+AT,\nu^T\rangle=\alpha_0\langle\nabla v,\nu^T\rangle+\alpha_1\langle\nu,\nu^T\rangle+\alpha_0v\Phi\langle\bar\nabla\Phi,\nu^T\rangle$$
$$+\alpha_0\Phi\big(\langle\nabla v,\nu^T\rangle\Phi-v\Phi\langle N,\bar\nabla_N\nu\rangle-\langle\nu,\nu^T\rangle\langle N,\nabla v\rangle\big).$$

Rearranging terms we obtain

$$\langle \nabla^\Sigma \zeta + AT, \nu^T\rangle = \alpha_1(1-\langle N,\nu\rangle^2) + \alpha_0\langle \nabla v, \nu^T\rangle(1+\Phi^2) + \alpha_0 v\Phi\langle \bar{\nabla}\Phi, \nu^T\rangle$$
$$-\alpha_0\Phi\big(v\Phi\langle N, \bar{\nabla}_N \nu\rangle + (1-\langle N,\nu\rangle^2)\langle N, \nabla v\rangle\big).$$

Therefore there exists a constant $C = C(\Phi, K, \Omega, R)$ such that

$$|\langle \nabla^\Sigma \zeta + AT, \nu^T\rangle| \le \alpha_1(1-\Phi^2) + C\alpha_0. \tag{3.29}$$

Since $d_\Gamma(y) = 0$ it holds that

$$|\langle A_K Z, N\rangle| = |A_K||Z| \le |A_K|(\eta + |X|) \le 4R^2\alpha_0|A_K|(1+\Phi).$$

from what we conclude that

$$|\langle \bar{\nabla}\langle N_\tau, \nu\rangle, \frac{dy_\tau}{d\tau}\big|_{\tau=0}\rangle| \le \alpha_1(1-\Phi^2) + \widetilde{C}\alpha_0 \tag{3.30}$$

for some constant $\widetilde{C}(C_\Phi, K, \Omega, R) > 0$.

Now we suppose that $x \in \overline{\partial\Omega' \cap \Omega}$. In this case, we have $v(x) = 0$. Then $\eta = \alpha_1 d_\Gamma$ and

$$X = -\alpha_0 \Phi d_\Gamma \nabla v$$

at x. Thus

$$\zeta = \eta + \langle X, N\rangle = \alpha_1 d_\Gamma + 2\alpha_0\Phi d d_\Gamma\langle \nabla d, N\rangle.$$

Moreover we have

$$W(x) \le \frac{C}{d_\Gamma(x)}$$

(see Remark 18). It follows that

$$\zeta W \le C(\alpha_1 + 2\alpha_0\Phi d\langle \nabla d, N\rangle) \le C(\alpha_1 + 4R\alpha_0\Phi). \tag{3.31}$$

We conclude that

$$W(x) \le C(C_\Phi, K, \Omega, R). \tag{3.32}$$

Now we consider the case when $x \in \Omega \cap \Omega'$. In this case we have

$$\Delta_\Sigma \zeta = \alpha_0\Delta_\Sigma v + \alpha_1 \Delta_\Sigma d_\Gamma + \alpha_0 \Delta_\Sigma \Phi(v\langle N,\nu\rangle - d_\Gamma\langle N, \nabla v\rangle)$$
$$+\alpha_0\Phi(\Delta_\Sigma v\langle N,\nu\rangle + v\Delta_\Sigma\langle N,\nu\rangle + 2\langle \nabla^\Sigma v, \nabla^\Sigma\langle N,\nu\rangle\rangle - \Delta_\Sigma d_\Gamma\langle N,\nabla v\rangle - d_\Gamma \Delta_\Sigma\langle N, \nabla v\rangle$$
$$-2\langle \nabla^\Sigma d_\Gamma, \nabla^\Sigma\langle N, \nabla v\rangle\rangle)$$
$$+2\alpha_0\langle \nabla^\Sigma\Phi, \nabla^\Sigma v\langle N,\nu\rangle + v\nabla^\Sigma\langle N,\nu\rangle - \nabla^\Sigma d_\Gamma\langle N,\nabla v\rangle - d_\Gamma\nabla^\Sigma\langle N, \nabla v\rangle\rangle$$

Notice that given an arbitrary vector field U along Σ we have

$$\langle \nabla^\Sigma \langle N, U\rangle, V\rangle = -\langle AU^T, V\rangle + \langle N, \bar{\nabla}_V U\rangle,$$

for any $V \in \Gamma(T\Sigma)$. Here, U^T denotes the tangential component of U. Hence using Codazzi's equation we obtain

$$\Delta_\Sigma \langle N, U\rangle \leq \langle \bar{\nabla}(nH), U^T\rangle + \mathrm{Ric}_M(U^T, N) + C|A|$$

for a constant C depending on $\bar{\nabla} U$ and $\bar{\nabla}^2 U$. Hence using (3.1) we conclude that

$$\Delta_\Sigma \langle N, U\rangle \leq \langle \bar{\nabla}\Psi, U^T\rangle + \widetilde{C}|A| \tag{3.33}$$

where $\widetilde{C}$ is a positive constant depending on $\bar{\nabla} U, \bar{\nabla}^2 U$ and Ric_M.

We also have

$$\begin{aligned} \Delta_\Sigma d_\Gamma &= \Delta_P d_\Gamma + \gamma \langle \bar{\nabla}_Y \bar{\nabla} d, Y\rangle - \langle \bar{\nabla}_N \bar{\nabla} d_\Gamma, N\rangle + nH\langle \bar{\nabla} d_\Gamma, N\rangle \\ &\leq C_0 \Psi + C_1, \end{aligned}$$

where C_0 and C_1 are positive constants depending on the second fundamental form of the Killing cylinders over the equidistant sets $d_\Gamma = \delta$ for small values of δ. Similar estimates also hold for $\Delta_\Sigma d$ and then for $\Delta_\Sigma v$.

We conclude that

$$\Delta_\Sigma \zeta \geq -\widetilde{C}_0 - \widetilde{C}_1 |A|, \tag{3.34}$$

where $\widetilde{C}_0$ and $\widetilde{C}_1$ are positive constants depending on Ω, K, Ric_M, $|\Phi|_2$.

Now proceeding similarly as in the proof of Proposition 17, we observe that Lemma 16 (iii) and the Comparison Principle yield

$$H_0(\hat{y}_\tau) - H_0(y) \geq \frac{\partial H_\tau}{\partial \tau}\Big|_{\tau=0} \tau + o(\tau) = (\Delta_\Sigma \zeta + |A|^2 \zeta + \mathrm{Ric}_M(N,N)\zeta)\tau + \tau\langle \bar{\nabla}\Psi, T\rangle + o(\tau).$$

However

$$H_0(\hat{y}_\tau) - H_0(y) = \langle \bar{\nabla}\Psi|_y, \xi'(0)\rangle \tau + o(\tau).$$

Using (3.16) we have

$$\langle \bar{\nabla}\Psi, \xi'(0)\rangle = \langle \bar{\nabla}\Psi, Z - \zeta W Y\rangle = \langle \bar{\nabla}\Psi, Z\rangle - \zeta W \frac{\partial \Psi}{\partial s}.$$

We conclude that

$$-\zeta W \frac{\partial \Psi}{\partial s}\tau + \zeta \langle \bar{\nabla}\Psi, N\rangle \tau + o(\tau) \geq (\Delta_\Sigma \zeta + |A|^2 \zeta + \mathrm{Ric}_M(N,N)\zeta)\tau + o(\tau).$$

Suppose that

$$W > \frac{C + |\bar{\nabla}\Psi|}{\beta} \tag{3.35}$$

for a constant $C > 0$ as in (3.20). Hence we have

$$(\Delta_\Sigma \zeta + |A|^2\zeta + \mathrm{Ric}_M(N,N)\zeta)\tau + C\zeta\tau \leq o(\tau)$$

We conclude that

$$-C_0 - C_1|A| + C_2|A|^2 + C \leq \frac{o(\tau)}{\tau},$$

a contradiction. It follows from this contradiction that

$$W(x) \leq \frac{C + |\bar{\nabla}\Psi|}{\beta}. \tag{3.36}$$

Now, proceeding as in the end of the proof of Proposition 17, we use the estimate for $W(x)$ in each one of the three cases for obtaining a estimate for W in Ω'. This finishes the proof of the Proposition. □

3.3 Proof of the Theorem 13

We use the classical Continuity Method for proving Theorem 13. For details, we refer the reader to [47] and [40]. For any $\tau \in [0,1]$ we consider the Neumann boundary problem $\mathcal{N}_\tau$ of finding $u \in C^{3,\alpha}(\bar{\Omega})$ such that

$$\mathcal{F}[\tau, x, u, \nabla u, \nabla^2 u] = 0, \tag{3.37}$$

$$\langle \frac{\nabla u}{W}, \nu \rangle + \tau\Phi = 0, \tag{3.38}$$

where $\mathcal{F}$ is the quasilinear elliptic operator defined by

$$\mathcal{F}[x, u, \nabla u, \nabla^2 u] = \mathrm{div}\left(\frac{\nabla u}{W}\right) - \langle \frac{\nabla\gamma}{2\gamma}, \frac{\nabla u}{W} \rangle - \tau\Psi. \tag{3.39}$$

Since the coefficients of the first and second order terms do not depend on u it follows that

$$\frac{\partial \mathcal{F}}{\partial u} = -\tau \frac{\partial \Psi}{\partial u} \leq -\tau\beta < 0. \tag{3.40}$$

We define $\mathcal{I} \subset [0,1]$ as the subset of values of $\tau \in [0,1]$ for which the Neumann boundary problem $\mathcal{N}_\tau$ has a solution. Since $u = 0$ is a solution for $\mathcal{N}_0$, it follows that $\mathcal{I} \neq \emptyset$. Moroever, the Implicit Function Theorem (see [45], Chapter 17) implies that

$\mathcal{I}$ is open in view of (3.40). Finally, the height and gradient *a priori* estimates we obtained in Sections 3.1 and 3.2 are independent of $\tau \in [0,1]$. This implies that (3.1) is uniformly elliptic. Moreover, we may assure the existence of some $\alpha_0 \in (0,1)$ for which there there exists a constant $C > 0$ independent of τ such that

$$|u_\tau|_{1,\alpha_0,\bar{\Omega}} \leq C.$$

Redefine $\alpha = \alpha_0$. Thus, combining this fact, Schauder elliptic estimates and the compactness of $C^{3,\alpha_0}(\bar{\Omega})$ into $C^3(\bar{\Omega})$ imply that $\mathcal{I}$ is closed. It follows that $\mathcal{I} = [0,1]$.

The uniqueness follows from the Comparison Principle for elliptic PDEs. We point out that a more general uniqueness statement - comparing a nonparametric solution with a general hypersurface with the same mean curvature and contact angle at corresponding points - is also valid. It is a consequence of a flux formula coming from the existence of a Killing vector field in M. We refer the reader to [16] for further details.

This finishes the proof of the Theorem 13.

Capítulo 4

Mean Curvature Flow of Killing Graphs

In this chapter we prove the following result

Theorem 20 *There exists a unique solution* $u : \bar{\Omega} \times [0, \infty) \to \mathbb{I}$ *to the problem*

$$\frac{\partial X}{\partial t} = (nH - \mathcal{H})N, \tag{4.1}$$

$$\tag{4.2}$$

with boundary condition

$$\langle N, \nu \rangle|_{\partial \Sigma_t} = \phi, \tag{4.3}$$

Moreover, if $\phi = 0$ *and* $\mathcal{H} = 0$ *the graphs* Σ_t *converge to a minimal graph which contacts the cylinder* K *orthogonally along its boundary.*

Remember that (1.6), (1.7) may be written nonparametrically as

$$u_t = \Big(\sigma^{ij} - \frac{u^i}{W}\frac{u^j}{W}\Big)u_{i;j} - \Big(\frac{1}{2\gamma} + \frac{1}{2W^2}\Big)\gamma^i u_i - W\mathcal{H} \quad \text{in} \quad \Omega \times [0, T) \tag{4.4}$$

$$u(\cdot, 0) = u_0(\cdot) \qquad \text{in} \quad \Omega \times \{0\} \tag{4.5}$$

with boundary condition

$$\langle N, \nu \rangle = \phi \quad \text{on} \quad \partial\Omega \times [0, T). \tag{4.6}$$

In what follows we prove height and boundary gradient *a priori* estimates for (1.6)-(1.8).

4.1 Height estimates

In this section we obtain an a priori height estimates.

From now on, we consider the parabolic linear operator given by

$$\mathcal{L}v = g^{ij}v_{i;j} - \Big(\frac{1}{2\gamma} + \frac{1}{2W^2}\Big)\gamma^i v_i - \mathcal{H}\frac{u^i}{W}v_i - v_t, \tag{4.7}$$

where $v \in C^\infty(\Omega \times [0,T))$.

Proposition 21 *For a solution $u \in C^\infty(\bar\Omega \times [0,T^*])$, $T^* < T$, of (4.4)-(4.6), it holds that*

$$\max_{\bar\Omega\times[0,T^*]} |u_t| = \max_{\bar\Omega} |u_t(0,\cdot)|.$$

Then it follows that

$$\max_{\bar\Omega\times[0,T^*]} |u| \leq CT^*$$

for a given constant $C > 0$ which depends on T^*.

Proof: First of all we verify that u_t is a solution for a linear parabolic equation. Indeed one has

$$\begin{aligned}
\mathcal{L}u_t &= g^{ij}u_{ti;j} - \Big(\frac{1}{2\gamma} + \frac{1}{2W^2}\Big)\langle\nabla\gamma, \nabla u_t\rangle - u_{tt}\\
&= (g^{ij}u_{i;j})_t - g^{ij}_{;t}u_{i;j} - \Big(\frac{1}{2\gamma} + \frac{1}{2W^2}\Big)\langle\nabla\gamma, \nabla u_t\rangle - u_{tt}\\
&= -g^{ij}_{;t}u_{i;j} + \Big(\frac{1}{2\gamma} + \frac{1}{2W^2}\Big)_t\langle\nabla\gamma, \nabla u\rangle + \Big(\frac{1}{2\gamma} + \frac{1}{2W^2}\Big)\langle\nabla\gamma_t, \nabla u\rangle + W_t\mathcal{H}.
\end{aligned}$$

However since $\gamma = \gamma(x)$ in (1.24) and x is independent of t it follows that

$$\Big(\frac{1}{2\gamma} + \frac{1}{2W^2}\Big)_t = \Big(\frac{1}{2\gamma}\Big)_t - \frac{1}{W^3}W_t = -\frac{1}{2W^4}(\gamma_t + 2u^k u_{k;t}) = -\frac{1}{W^4}u^k u_{t;k}.$$

In the same way we have

$$W_t = \frac{1}{2W}(\gamma_t + 2u^k u_{k;t}) = \frac{1}{W}u^k u_{t;k}. \tag{4.8}$$

We conclude that

$$\mathcal{L}u_t = -g^{ij}_{;t}u_{i;j} - \frac{1}{W^4}\langle\nabla\gamma, \nabla u\rangle u^k(u_t)_k + \frac{1}{W}\mathcal{H}u^k(u_t)_k.$$

Now using the fact that $\sigma^{ij}_{;t} = 0$ and $\gamma_t = 0$ we have

$$\begin{aligned}
\mathcal{L}u_t &= \frac{2}{W}\Big(\frac{u^i_{;t}u^j}{W} - \frac{u^i}{W}\frac{u^j}{W}W_t\Big)u_{i;j} - \frac{1}{W^4}\langle\nabla\gamma, \nabla u\rangle u^k(u_t)_k + \frac{1}{W}\mathcal{H}u^k(u_t)_k\\
&= \frac{2}{W}\Big((W_i - \frac{\gamma_i}{2W})u^i_{;t} - (W_i - \frac{\gamma_i}{2W})\frac{u^i}{W}\frac{u^k}{W}u_{t;k}\Big) - \frac{1}{W^4}\langle\nabla\gamma, \nabla u\rangle u^k(u_t)_k + \frac{1}{W}\mathcal{H}u^k(u_t)_k\\
&= \frac{2}{W}(W_i - \frac{\gamma_i}{2W})(\sigma^{ik} - \frac{u^i}{W}\frac{u^k}{W})u_{t;k} - \frac{1}{W^4}\langle\nabla\gamma, \nabla u\rangle u^k(u_t)_k + \frac{1}{W}\mathcal{H}u^k(u_t)_k.
\end{aligned}$$

Hence it follows that

$$\mathcal{L}u_t - \frac{2}{W}g^{ik}(W_i - \frac{\gamma_i}{2W})(u_t)_k + \frac{1}{W^4}\langle\nabla\gamma, \nabla u\rangle u^k(u_t)_k - \frac{1}{W}\mathcal{H}u^k(u_t)_k = 0. \tag{4.9}$$

Thus fixed $T^* \in [0, T)$ let (x_0, t_0) be a point in $\bar{\Omega} \times [0, T^*]$ such that

$$u_t(x_0, t_0) = \max_{\bar{\Omega}\times[0,T^*]} |u_t|.$$

Hence we choose a coordinate system adapted to the boundary Γ in such a way that $\frac{\partial}{\partial x^n} = \nu$ at x_0. Then, at the point (x_0, t_0) we have

$$u_{i;t} = u_{t;i} = 0$$

for $1 \leq i < n$ what implies that

$$W_t = \frac{1}{W}u^n u_{n;t} = -\phi(x_0)u_{n;t},$$

where we used (4.6) and (4.8). On the other hand, (4.6) implies that

$$u_{t;n} = u_{n;t} = -(\phi W)_t = -\phi(x_0)W_t. \tag{4.10}$$

at (x_0, t_0). We conclude that

$$(1 - \phi^2(x_0))u_{n;t} = 0.$$

However since $| \phi |< 1$, it follows that $u_{t;n} = 0$ what contradicts the parabolic Hopf Lemma [48].

From this contradiction we conclude that $t_0 = 0$. Since T^* is arbitrary, the conclusion follows. □

4.2 Boundary gradient estimates

Now we will prove a gradient bound for a solution of (4.4)-(4.6) by applying a modification of the Korevaar's technique [17] which appeared formerly in [33].

From now on, we consider a non-negative extension $d : \bar{\Omega} \to \mathbb{R}$ of the distance function $\text{dist}_P(\cdot, \Gamma)$ satisfying $|\nabla d| \leq 1$ in $\bar{\Omega}$. In the same way, we consider a C^∞ extension of the boundary data ϕ to the domain $\bar{\Omega}$ which we denote also by ϕ. Then we define

$$\eta = e^{Ku}h \tag{4.11}$$

where

$$h = 1 + \alpha d - \phi\langle \nabla d, N\rangle, \tag{4.12}$$

where K and α are positive numbers to be fixed later.

Proposition 22 *For $\alpha > 0$ sufficiently large independent of K and t, if for some $t \geq 0$ fixed, $\eta W(\cdot, t)$ attains a local maximum value at a point $x_0 \in \partial\Omega$, then $W(x_0, t) \leq K$.*

Proof: Let $t \geq 0$ be such that

$$\max_{\bar{\Omega}} \eta W(t, \cdot) = \eta W(t, x_0)$$

for a point $x_0 \in \Gamma$. Hence we choose a coordinate system adapted to Γ such that $\frac{\partial}{\partial x^n} = \nu$ at x_0 and

$$u_1(x_0) \geq 0 \quad \text{and} \quad u_i(x_0) = 0, \quad \text{for} \quad 2 \leq i \leq n-1. \tag{4.13}$$

We have at x_0

$$0 = (\eta W)_1 = \eta_1 W + \eta W_1 = e^{Ku}\big(WKu_1(1-\phi^2) - 2W\phi\phi_1 + W_1(1-\phi^2)\big) \tag{4.14}$$

from what follows that

$$W_1 = -Ku_1 W + \frac{2\phi\phi_1}{(1-\phi^2)}W. \tag{4.15}$$

On the other hand at x_0 we have

$$\begin{aligned}
\eta_n &= e^{Ku}\big(Ku_n(1-\phi^2) + \alpha - \phi\phi_n - \phi(\langle \nabla_{\nabla d} N, \nabla d\rangle + \langle N, \nabla_{\nabla d}\nabla d\rangle)\big) \\
&= e^{Ku}\big(Ku_n(1-\phi^2) + \alpha - \phi\phi_n - \phi(\langle \partial_n \frac{1}{W}(\gamma Y - \nabla u), \partial_n\rangle + \langle \frac{1}{W}\nabla_{\partial_n}(\gamma Y - \nabla u), \partial_n\rangle)\big) \\
&= e^{Ku}\big(Ku_n(1-\phi^2) + \alpha - \phi\phi_n - \frac{1}{W^2}\phi u_n W_n + \frac{1}{W}\phi u_{n;n}\big).
\end{aligned}$$

Since $(\eta W)_n \leq 0$ at x_0 it holds that

$$\begin{aligned}
0 &\geq WKu_n(1-\phi^2) + \alpha W - W\phi\phi_n - \frac{1}{W}\phi u_n W_n + \phi u_{n;n} + (1-\phi^2)W_n \\
&= WKu_n(1-\phi^2) + \alpha W + W_n + \phi u_{n;n} + u_n\phi_n \\
&= WKu_n(1-\phi^2) + \alpha W + W_n + \phi u_{n;n} - W\phi\phi_n.
\end{aligned}$$

On the other hand

$$W_n = \frac{\gamma_n}{2W} + \frac{1}{W}(u_1 u_{1;n} + u_n u_{n;n}) = \frac{\gamma_n}{2W} - \frac{1}{W}\phi u_1 W_1 - \phi_1 u_1 - \phi u_{n;n} \tag{4.16}$$

what implies that

$$\begin{aligned} W_n &= \frac{\gamma_n}{2W} - \frac{1}{W}\phi u_1\Big(\frac{2\phi\phi_1 W}{1-\phi^2} - Ku_1W\Big) - \phi_1 u_1 - \phi u_{n;n} \\ &= \frac{\gamma_n}{2W} - \frac{1+\phi^2}{1-\phi^2}u_1\phi_1 + K\phi u_1^2 - \phi u_{n;n}. \end{aligned}$$

Therefore since

$$u_1^2 = |\nabla u|^2 - u_n^2 = W^2 - \gamma - \phi^2 W^2 = W^2(1-\phi^2) - \gamma$$

we conclude that

$$\begin{aligned} 0 &\geq \alpha + \frac{\gamma_n}{2W^2} - \frac{1+\phi^2}{1-\phi^2}\frac{u_1}{W}\phi_1 + \frac{K\phi u_1^2}{W} - \phi\phi_n + Ku_n(1-\phi^2) \\ &= \alpha + \frac{\gamma_n}{2W^2} + \frac{1+\phi^2}{1-\phi^2}N_1\phi_1 + K\phi\Big(W(1-\phi^2) - \frac{\gamma}{W}\Big) - \phi\phi_n - K\phi W(1-\phi^2) \\ &= \alpha + \frac{\gamma_n}{2W^2} + \frac{1+\phi^2}{1-\phi^2}N_1\phi_1 - \frac{K\phi\gamma}{W} - \phi\phi_n \\ &\geq \alpha + C - \frac{K\gamma}{W}, \end{aligned}$$

for a given constant C depending solely on γ and ϕ. It follows that $W(x_0,t) \leq K$ if α is chosen large enough and independent of K and t.

□

4.3 Interior gradient estimates

In this section we deduce a global gradient bound using the techniques in [18] and [33]. However the more general context of warped product gives rise to a long list of additional terms which require a careful tracking along the calculations.

In the sequel, we consider the parabolic linear operator given by

$$Lv = g^{ij}v_{i;j} - \Big(\frac{1}{2\gamma} + \frac{1}{2W^2}\Big)\gamma^i v_i - v_t, \tag{4.17}$$

where $v \in C^\infty(\Omega \times [0,T))$.

Some lemmata will be needed in the sequel. Their content could be also of independent interest for other applications.

Lemma 23 *Denote $\theta = \langle \nabla d, N\rangle$. The differentials of the functions θ and h have components given by*

$$\theta_i = -a_i^j d_j + (d_{i;j} - \kappa\sigma_{ij})N^j \tag{4.18}$$

and

$$h_i = (\alpha\delta_i^j + \phi a_i^j)d_j - (\phi(d_{i;j} - \kappa\sigma_{ij}) + \phi_i d_j)N^j \tag{4.19}$$

respectively, where $\kappa = \langle\gamma\bar\nabla_Y Y, \nabla d\rangle$.

Proof: We have

$$\begin{aligned}\frac{\partial\theta}{\partial x^i} &= X_*\frac{\partial}{\partial x^i}\langle N, \bar\nabla d\rangle = \langle\bar\nabla_{X_*\frac{\partial}{\partial x^i}} N, \bar\nabla d\rangle + \langle N, \bar\nabla_{X_*\frac{\partial}{\partial x^i}}\bar\nabla d\rangle \\ &= -\langle AX_*\frac{\partial}{\partial x^i}, \bar\nabla d\rangle + \langle N, \bar\nabla_{\frac{\partial}{\partial x^i}+u_i\frac{\partial}{\partial x^0}}\bar\nabla d\rangle \\ &= -\langle AX_*\frac{\partial}{\partial x^i}, \bar\nabla d\rangle + \frac{\gamma}{W}\langle\frac{\partial}{\partial x^0}, \bar\nabla_{\frac{\partial}{\partial x^i}}\bar\nabla d\rangle - \langle\frac{\nabla u}{W}, \bar\nabla_{\frac{\partial}{\partial x^i}}\bar\nabla d\rangle \\ &\quad + u_i\frac{\gamma}{W}\langle\frac{\partial}{\partial x^0}, \bar\nabla_{\frac{\partial}{\partial x^0}}\bar\nabla d\rangle - u_i\langle\frac{\nabla u}{W}, \bar\nabla_{\frac{\partial}{\partial x^0}}\bar\nabla d\rangle\end{aligned}$$

Since P is totally geodesic we have

$$\langle\frac{\partial}{\partial x^0}, \bar\nabla_{\frac{\partial}{\partial x^i}}\bar\nabla d\rangle = \langle\frac{\partial}{\partial x^0}, \bar\nabla_{\frac{\partial}{\partial x^i}}\nabla d\rangle = 0.$$

Moreover we compute

$$\langle\frac{\partial}{\partial x^0}, \bar\nabla_{\frac{\partial}{\partial x^0}}\bar\nabla d\rangle = |Y|^2\langle\frac{Y}{|Y|}, \bar\nabla_{\frac{Y}{|Y|}}\bar\nabla d\rangle = |Y|^2\kappa = \frac{1}{\gamma}\kappa$$

and

$$\langle\frac{\nabla u}{W}, \bar\nabla_{\frac{\partial}{\partial x^0}}\bar\nabla d\rangle = \langle\frac{\nabla u}{W}, \bar\nabla_{\bar\nabla d}\frac{\partial}{\partial x^0}\rangle + \langle\frac{\nabla u}{W}, [\frac{\partial}{\partial x^0}, \bar\nabla d]\rangle = 0,$$

where we used the fact that $[\frac{\partial}{\partial x^0}, \bar\nabla d] = 0$ and that P is totally geodesic.

Thus we conclude that

$$\frac{\partial\theta}{\partial x^i} = -\langle AX_*\frac{\partial}{\partial x^i}, \bar\nabla d\rangle - \langle\frac{\nabla u}{W}, \nabla_{\frac{\partial}{\partial x^i}}\nabla d\rangle + \kappa\frac{u_i}{W}.$$

However

$$\langle AX_*\frac{\partial}{\partial x^i}, \bar\nabla d\rangle = a_i^j\langle X_*\frac{\partial}{\partial x^j}, \bar\nabla d\rangle = a_i^j\langle\frac{\partial}{\partial x^j} + u_jY, \bar\nabla d\rangle = a_i^j d_j = g^{jk}a_{ik}d_j$$

Therefore we write

$$\theta_i = -g^{jk}a_{ik}d_j + (d_{i;j} - \kappa\sigma_{ij})N^j. \tag{4.20}$$

This finishes the proof of the proposition.

We denote the components of the tensor X^*II in P by

$$b_{ij} = X^*II(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) := \langle AX_*\frac{\partial}{\partial x^i}, X_*\frac{\partial}{\partial x^j}\rangle \tag{4.21}$$

Notice that the covariant derivatives of X^*II and II are related by

$$\nabla_k b_{ij} = \langle (\nabla^{\Sigma}_{X_*\frac{\partial}{\partial x^k}} A) X_* \frac{\partial}{\partial x^i}, X_* \frac{\partial}{\partial x^j}\rangle + \langle AX_* \frac{\partial}{\partial x^j}, \bar{\nabla}_{X_*\frac{\partial}{\partial x^k}} X_* \frac{\partial}{\partial x^i} - X_* \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i}\rangle$$
$$+\langle AX_* \frac{\partial}{\partial x^i}, \bar{\nabla}_{X_*\frac{\partial}{\partial x^k}} X_* \frac{\partial}{\partial x^j} - X_* \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j}\rangle.$$

However since $X_*\frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i} + u_i Y$ we compute

$$\bar{\nabla}_{X_*\frac{\partial}{\partial x^k}} X_* \frac{\partial}{\partial x^i} - X_* \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i} = \bar{\nabla}_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i} + u_{i,k} Y + u_i \bar{\nabla}_{\frac{\partial}{\partial x^k}} Y + u_k \bar{\nabla}_Y \frac{\partial}{\partial x^i} + u_i u_k \bar{\nabla}_Y Y$$
$$-\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i} - \langle \nabla u, \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i}\rangle Y.$$

Therefore

$$\bar{\nabla}_{X_*\frac{\partial}{\partial x^k}} X_* \frac{\partial}{\partial x^i} - X_* \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i} = u_{i;k} Y + u_i \bar{\nabla}_{\frac{\partial}{\partial x^k}} Y + u_k \bar{\nabla}_{\frac{\partial}{\partial x^i}} Y + u_i u_k \bar{\nabla}_Y Y.$$

Hence using (1.14), (1.18) and (1.19) we obtain

$$\bar{\nabla}_{X_*\frac{\partial}{\partial x^k}} X_* \frac{\partial}{\partial x^i} - X_* \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i} = (W a_{ik} + u_i u_k u^l \frac{\gamma_l}{2\gamma^2}) Y + \frac{1}{2} u_i u_k \frac{\nabla\gamma}{\gamma^2}$$
$$= W a_{ik} Y + \frac{1}{2\gamma^2} u_i u_k (\langle \nabla u, \nabla\gamma\rangle Y + \nabla\gamma) = W a_{ik} Y + \frac{1}{2\gamma^2} u_i u_k X_* \nabla\gamma.$$

Hence it follows that

$$\langle AX_* \frac{\partial}{\partial x^j}, \bar{\nabla}_{X_*\frac{\partial}{\partial x^k}} X_* \frac{\partial}{\partial x^i} - X_* \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i}\rangle = \langle AX_* \frac{\partial}{\partial x^j}, \frac{u_i u_k}{2\gamma^2} X_* \nabla\gamma + W a_{ik} Y\rangle$$
$$= \frac{1}{\gamma} W a_{ik} a^l_j u_l + \frac{u_i u_k}{2\gamma^2} a_{jl} \gamma^l.$$

We conclude that

$$\nabla_k b_{ij} = \langle (\nabla^{\Sigma}_{X_*\frac{\partial}{\partial x^k}} A) X_* \frac{\partial}{\partial x^i}, X_* \frac{\partial}{\partial x^j}\rangle + \frac{1}{\gamma} W a_{ik} a^l_j u_l + \frac{u_i u_k}{2\gamma^2} a_{jl} \gamma^l$$
$$+\frac{1}{\gamma} W a_{jk} a^l_i u_l + \frac{u_j u_k}{2\gamma^2} a_{il} \gamma^l,$$

that is,

$$\nabla_k b_{ij} = \nabla^{\Sigma}_k a_{ij} + \frac{1}{\gamma} W a_{ik} a^l_j u_l + \frac{1}{\gamma} W a_{jk} a^l_i u_l + \frac{u_i u_k}{2\gamma^2} a_{jl} \gamma^l + \frac{u_j u_k}{2\gamma^2} a_{il} \gamma^l. \tag{4.22}$$

Now we use (4.22) for computing the Hessian of the function θ.

Lemma 24 *The trace of the Hessian of θ in Ω calculated with respect to the metric in Σ is given by*

$$g^{ik}\theta_{i;k} = -|A|^2\theta - 2\langle \nabla^2 d, X^* II\rangle_{\Sigma} - n\langle \nabla^{\Sigma} H, \nabla^{\Sigma} d\rangle - nHW\langle AY^T, \nabla^{\Sigma} d\rangle - \mathrm{Ric}(\nabla d, \frac{\nabla u}{W})$$
$$-\mathrm{tr}_{\Sigma} \nabla_{\frac{\nabla u}{W}} \nabla^2 d - \frac{|\nabla u|^2}{W^2} \langle A\nabla^{\Sigma} d, X_* \frac{\nabla\gamma}{2\gamma}\rangle + \frac{1}{2}\langle AY^T, Y^T\rangle \langle \nabla d, \nabla\gamma\rangle - \frac{1}{2W^2} \nabla^2 d(\frac{\nabla u}{W}, \nabla\gamma)$$
$$-\frac{\gamma}{W^2}\langle N, \nabla\kappa\rangle + \kappa(nH - \gamma\langle AY^T, Y^T\rangle) - \kappa \frac{1}{2W^2}\langle N, \nabla\gamma\rangle.$$

Proof: Notice that we may write (4.20) as

$$\theta_i = -g^{jl}b_{il}d_j + (d_{i;j} - \kappa\sigma_{ij})N^j. \tag{4.23}$$

Hence we have

$$\begin{aligned}
g^{ik}\theta_{i;k} &= -g^{ik}(g^{jl}b_{il}d_j)_{;k} + g^{ik}(d_{i;jk} - \kappa_k\sigma_{ij})N^j + g^{ik}(d_{i;j} - \kappa\sigma_{ij})N^j_{;k} \\
&= -g^{ik}(g^{jl}b_{il}d_j)_{;k} + g^{ik}(d_{i;kj} + R^l_{jki}d_l - \kappa_k\sigma_{ij})N^j - g^{ik}(d_{i;j} - \kappa\sigma_{ij})(a^j_k - N_k\frac{\gamma^j}{2\gamma}).
\end{aligned}$$

However

$$\begin{aligned}
&g^{ik}(g^{jl}b_{il}d_j)_{;k} = g^{jl}g^{ik}b_{il;k}d_j + g^{ik}g^{jl}_{;k}b_{il}d_j + g^{ik}g^{jl}b_{il}d_{j;k} \\
&= g^{jl}g^{ik}(\nabla^\Sigma_k a_{il} + \frac{1}{\gamma}Wa_{ik}a^m_l u_m + \frac{1}{\gamma}Wa_{lk}a^m_i u_m + u_iu_ka_{lm}\frac{\gamma^m}{2\gamma^2} + u_lu_ka_{im}\frac{\gamma^m}{2\gamma^2})d_j \\
&+g^{ik}g^{jl}_{;k}b_{il}d_j + g^{ik}g^{jl}b_{il}d_{j;k}
\end{aligned}$$

Hence using Codazzi's equation we obtain

$$\begin{aligned}
&g^{ik}(g^{jl}b_{il}d_j)_{;k} = g^{jl}(nH_l + n\frac{1}{\gamma}WHa^m_l u_m + \frac{1}{\gamma}Wa^i_la^m_iu_m + \frac{|\nabla u|^2}{W^2}a_{lm}\frac{\gamma^m}{2\gamma} + u_lu_ka^k_m\frac{\gamma^m}{2\gamma^2})d_j \\
&+g^{jl}g^{ik}\langle\bar{R}(X_*\frac{\partial}{\partial x^i}, X_*\frac{\partial}{\partial x^k})N, X_*\frac{\partial}{\partial x^l}\rangle d_j + g^{ik}g^{jl}_{;k}b_{il}d_j + g^{ik}g^{jl}b_{il}d_{j;k}
\end{aligned}$$

Using that $g^{jl}u_l = \frac{\gamma}{W^2}u^j$ we conclude that

$$\begin{aligned}
&g^{ik}(g^{jl}b_{il}d_j)_{;k} = ng^{jl}H_ld_j - n\frac{1}{\gamma}W^2Hg^{jl}a^m_lN_md_j - \frac{1}{\gamma}W^2g^{jl}a^i_la^m_iN_md_j \\
&+\frac{|\nabla u|^2}{W^2}a^j_m\frac{\gamma^m}{2\gamma}d_j + N^jN_ka^k_m\frac{\gamma^m}{2\gamma}d_j + g^{ik}g^{jl}_{;k}b_{il}d_j + g^{ik}g^{jl}b_{il}d_{j;k}
\end{aligned}$$

However we have

$$g^{jl}_{;k} = (\sigma^{jl} - N^jN^l)_{;k} = -N^j_{;k}N^l - N^jN^l_{;k} = (a^j_k - N_k\frac{\gamma^j}{2\gamma})N^l + N^j(a^l_k - N_k\frac{\gamma^l}{2\gamma}).$$

and

$$\begin{aligned}
&\bar{\nabla}_{\frac{\partial}{\partial x^k}}N = \bar{\nabla}_{X_*\frac{\partial}{\partial x^k}}N - \bar{\nabla}_{u_kY}N = -AX_*\frac{\partial}{\partial x^k} - u_k\bar{\nabla}_Y(\frac{\gamma}{W}Y - \frac{\nabla u}{W}) \\
&= -AX_*\frac{\partial}{\partial x^k} - \frac{u_k}{2W}\big(\frac{\nabla\gamma}{\gamma} + \langle\nabla u, \frac{\nabla\gamma}{\gamma}\rangle Y)
\end{aligned}$$

from what follows that

$$\begin{aligned}
&g^{ik}(g^{jl}b_{il}d_j)_{;k} = ng^{jl}H_ld_j - n\frac{1}{\gamma}W^2Ha^m_lN_mg^{jl}d_j - \frac{1}{\gamma}W^2a^i_la^m_iN_mg^{jl}d_j \\
&+\frac{|\nabla u|^2}{W^2}a^j_m\frac{\gamma^m}{2\gamma}d_j + N^jN_ka^k_m\frac{\gamma^m}{2\gamma}d_j + a^k_la^j_kN^ld_j \\
&-a^k_lN_kN^l\frac{\gamma^j}{2\gamma}d_j + a^k_la^l_kN^jd_j - a^k_lN_k\frac{\gamma^l}{2\gamma}N^jd_j + g^{jl}a^k_ld_{j;k}
\end{aligned}$$

Therefore

$$g^{ik}\theta_{i;k} = -ng^{jl}H_l d_j + n\frac{1}{\gamma}W^2 H a_l^m N_m g^{jl} d_j + \frac{1}{\gamma}W^2 a_l^i a_i^m N_m g^{jl} d_j - \frac{|\nabla u|^2}{W^2} a_m^j \frac{\gamma^m}{2\gamma} d_j$$
$$-a_l^k a_k^j N^l d_j + a_l^k N_k N^l \langle \nabla d, \frac{\nabla\gamma}{2\gamma}\rangle - a_l^k a_k^l \theta - g^{jl} a_l^k d_{j;k}$$
$$+g^{ik}(d_{i;kj} + R^l_{jki} d_l - \kappa_k \sigma_{ij})N^j - g^{ik}(d_{i;j} - \kappa\sigma_{ij})(a_k^j - N_k \frac{\gamma^j}{2\gamma})$$

Now using the fact that $g^{ij}u_j = \frac{\gamma}{W^2}u^i$ and therefore $g^{ij}N_j = \frac{\gamma}{W^2}N^i$ we obtain

$$a_i^m N_m = g^{km} a_{ik} N_m = \frac{\gamma}{W^2} a_{ik} N^k = \frac{\gamma}{W^2}\langle AX_*\frac{\partial}{\partial x^i}, N^k X_* \frac{\partial}{\partial x^k}\rangle$$
$$= \frac{\gamma}{W^2}\langle AX_*\frac{\partial}{\partial x^i}, N^k \frac{\partial}{\partial x^k} + \langle N^k \frac{\partial}{\partial x^k}, \nabla u\rangle Y\rangle$$
$$= \frac{\gamma}{W^2}\langle AX_*\frac{\partial}{\partial x^i}, N - \frac{\gamma}{W}Y + \langle N, \nabla u\rangle Y\rangle$$
$$= -\frac{\gamma}{W^2}\langle AX_*\frac{\partial}{\partial x^i}, Y\rangle (\frac{\gamma}{W} + \frac{|\nabla u|^2}{W}) = -\frac{\gamma}{W}\langle AX_*\frac{\partial}{\partial x^i}, Y\rangle = -\frac{\gamma}{W}\langle AY^T, X_*\frac{\partial}{\partial x^i}\rangle.$$

Therefore

$$a_l^m N_m g^{jl} d_j = -\frac{\gamma}{W}\langle AY^T, g^{jl} d_j X_* \frac{\partial}{\partial x^l}\rangle = -\frac{\gamma}{W}\langle AY^T, \nabla^\Sigma d\rangle$$

Moreover notice that

$$a_l^k N^l = g^{km} a_{ml} N^l = -g^{km} W \langle AY^T, X_* \frac{\partial}{\partial x^m}\rangle$$

and

$$a_{ik} N^k = -W\langle AY^T, X_*\frac{\partial}{\partial x^i}\rangle.$$

Similarly we have

$$a_k^j d_j = g^{jm} d_j \langle AX_* \frac{\partial}{\partial x^k}, X_* \frac{\partial}{\partial x^m}\rangle = \langle AX_*\frac{\partial}{\partial x^k}, \nabla^\Sigma d\rangle = \langle A\nabla^\Sigma d, X_* \frac{\partial}{\partial x^k}\rangle.$$

Replacing this above we obtain

$$g^{ik}\theta_{i;k} = -n\langle \nabla^\Sigma H, \nabla^\Sigma d\rangle - nHW\langle AY^T, \nabla^\Sigma d\rangle - W\langle AY^T, A\nabla^\Sigma d\rangle - \frac{|\nabla u|^2}{W^2}\langle A\nabla^\Sigma d, X_* \frac{\nabla\gamma}{2\gamma}\rangle$$
$$+W\langle AY^T, A\nabla^\Sigma d\rangle + \gamma\langle AY^T, Y^T\rangle\langle \nabla d, \frac{\nabla\gamma}{2\gamma}\rangle - |A|^2\theta - g^{jl} a_l^k d_{j;k}$$
$$+g^{ik}(d_{i;kj} + R^l_{jki} d_l - \kappa_k \sigma_{ij})N^j - g^{ik}(d_{i;j} - \kappa\sigma_{ij})(a_k^j - N_k \frac{\gamma^j}{2\gamma})$$

Therefore

$$g^{ik}\theta_{i;k} = -n\langle\nabla^\Sigma H, \nabla^\Sigma d\rangle - nHW\langle AY^T, \nabla^\Sigma d\rangle - \frac{|\nabla u|^2}{W^2}\langle A\nabla^\Sigma d, X_*\frac{\nabla\gamma}{2\gamma}\rangle$$
$$+\gamma\langle AY^T, Y^T\rangle\langle\nabla d, \frac{\nabla\gamma}{2\gamma}\rangle - |A|^2\theta - g^{jl}a_l^k d_{j;k}$$
$$+g^{ik}(d_{i;kj} + R^l_{jki}d_l - \kappa_k\sigma_{ij})N^j - g^{ik}(d_{i;j} - \kappa\sigma_{ij})(a_k^j - N_k\frac{\gamma^j}{2\gamma})$$

However

$$g^{ik}\sigma_{ij} = g^{ik}(g_{ij} - \frac{u_i u_j}{\gamma}) = \delta_j^k - \frac{1}{W^2}u^k u_j = \delta_j^k - N^k N_j.$$

Hence we have

$$g^{ik}\theta_{i;k} = -n\langle\nabla^\Sigma H, \nabla^\Sigma d\rangle - nHW\langle AY^T, \nabla^\Sigma d\rangle - \frac{|\nabla u|^2}{W^2}\langle A\nabla^\Sigma d, X_*\frac{\nabla\gamma}{2\gamma}\rangle$$
$$+\frac{1}{2}\langle AY^T, Y^T\rangle\langle\nabla d, \nabla\gamma\rangle - |A|^2\theta - 2g^{ik}g^{jl}d_{i;j}a_{kl} + \frac{1}{2W^2}d_{i;j}N^i\gamma^j$$
$$+g^{ik}d_{i;kj}N^j - \mathrm{Ric}(\nabla d, \frac{\nabla u}{W}) - \frac{\gamma}{W^2}\langle N, \nabla\kappa\rangle + \kappa(nH - \gamma\langle AY^T, Y^T\rangle) - \kappa\frac{1}{2W^2}\langle N, \nabla\gamma\rangle$$

This finishes the proof of the Lemma.

Using Lemma 24 we will obtain an expression for Lh. Notice that

$$h_{i;k} = \alpha d_{i;k} - \phi_i\theta_k - \phi_k\theta_i - \phi_{i;k}\theta - \phi\theta_{i;k}.$$

Moreover it holds that

$$2g^{ik}\phi_i\theta_k = 2g^{ik}\phi_i\langle A\nabla^\Sigma d, X_*\frac{\partial}{\partial x^k}\rangle - 2g^{ik}d_{k;l}\phi_i N^l + 2\kappa g^{ik}\sigma_{kl}\phi_i N^l$$
$$= 2\langle A\nabla^\Sigma d, \nabla^\Sigma\phi\rangle - 2g^{ik}d_{k;l}\phi_i N^l + 2\kappa\frac{\gamma}{W^2}\langle\nabla\phi, N\rangle.$$

We conclude that

$$g^{ik}h_{i;k} = \alpha g^{ik}d_{i;k} + 2\langle A\nabla^\Sigma d, \nabla^\Sigma\phi\rangle - 2g^{ik}d_{k;l}\phi_i N^l + 2\kappa\frac{\gamma}{W^2}\langle\nabla\phi, N\rangle - g^{ik}\phi_{i;k}\theta$$
$$+n\phi\langle\nabla^\Sigma H, \nabla^\Sigma d\rangle + n\phi HW\langle AY^T, \nabla^\Sigma d\rangle + \frac{|\nabla u|^2}{W^2}\phi\langle A\nabla^\Sigma d, X_*\frac{\nabla\gamma}{2\gamma}\rangle$$
$$-\frac{1}{2}\phi\langle AY^T, Y^T\rangle\langle\nabla d, \nabla\gamma\rangle + |A|^2\phi\theta + 2g^{ik}g^{jl}d_{i;j}a_{kl}\phi - \frac{1}{2W^2}\phi d_{i;j}N^i\gamma^j$$
$$-g^{ik}d_{i;kj}N^j\phi + \mathrm{Ric}(\nabla d, \frac{\nabla u}{W})\phi + \frac{\gamma}{W^2}\langle N, \nabla\kappa\rangle\phi - \kappa(nH - \gamma\langle AY^T, Y^T\rangle)\phi$$
$$+\kappa\frac{1}{2W^2}\langle N, \nabla\gamma\rangle\phi.$$

Now we compute the derivatives with respect to t. We have

$$\begin{aligned}\theta_t = X_*\frac{\partial}{\partial t}\langle N, \bar\nabla d\rangle = \langle \bar\nabla_{X_*\frac{\partial}{\partial t}} N, \bar\nabla d\rangle + \langle N, \bar\nabla_{X_*\frac{\partial}{\partial t}}\bar\nabla d\rangle \\ = -\langle \nabla^\Sigma(nH-\mathcal{H}), \bar\nabla d\rangle + (nH-\mathcal{H})\langle N, \bar\nabla_N\bar\nabla d\rangle.\end{aligned}$$

However

$$\langle N, \bar\nabla_N\bar\nabla d\rangle = -\frac{1}{2W^2}\langle\bar\nabla\gamma, \bar\nabla d\rangle + \langle\frac{\nabla u}{W}, \bar\nabla_{\frac{\nabla u}{W}}\bar\nabla d\rangle.$$

Hence we have

$$\theta_t = -\langle \nabla^\Sigma(nH-\mathcal{H}), \bar\nabla d\rangle + (nH-\mathcal{H})(-\frac{1}{2W^2}\langle\nabla\gamma, \nabla d\rangle + \langle\frac{\nabla u}{W}, \bar\nabla_{\frac{\nabla u}{W}}\bar\nabla d\rangle).$$

Moreover we have

$$d_t = \langle X_*\frac{\partial}{\partial t}, \bar\nabla d\rangle = (nH-\mathcal{H})\langle N, \bar\nabla d\rangle = (nH-\mathcal{H})\theta. \tag{4.24}$$

Therefore

$$\begin{aligned}h_t = \alpha(nH-\mathcal{H})\theta - (nH-\mathcal{H})\langle N, \bar\nabla\phi\rangle\theta + \phi\langle\nabla^\Sigma(nH-\mathcal{H}), \bar\nabla d\rangle \\ -\phi(nH-\mathcal{H})(-\frac{1}{2W^2}\langle\nabla\gamma, \nabla d\rangle + \langle\frac{\nabla u}{W}, \bar\nabla_{\frac{\nabla u}{W}}\bar\nabla d\rangle)\end{aligned}$$

We also compute

$$\langle\nabla\gamma, \nabla h\rangle = \alpha\langle\nabla d, \nabla\gamma\rangle + \phi\langle A\nabla^\Sigma d, X_*\nabla\gamma\rangle - \langle\nabla\phi, \nabla\gamma\rangle\theta - \phi d_{i;j}\gamma^i N^j + \kappa\phi\langle N, \nabla\gamma\rangle.$$

Now we obtain

$$g^{ik}d_{i;k} = \Delta d - \langle\nabla_{\frac{\nabla u}{W}}\nabla d, \frac{\nabla u}{W}\rangle = -nH_d - \langle\nabla_{\frac{\nabla u}{W}}\nabla d, \frac{\nabla u}{W}\rangle$$

and

$$g^{ik}d_{k;l}\phi_i N^l = d_{k;l}\phi^k N^l - d_{k;l}N^k N^l N^i\phi_i = -\langle\nabla_{\frac{\nabla u}{W}}\nabla d, \nabla\phi\rangle - \langle\nabla_{\frac{\nabla u}{W}}\nabla d, \frac{\nabla u}{W}\rangle\langle N, \nabla\phi\rangle.$$

Moreover we have

$$g^{ij}\phi_{i;j} = \Delta\phi - \langle\nabla_{\frac{\nabla u}{W}}\nabla\phi, \frac{\nabla u}{W}\rangle$$

and

$$g^{ik}d_{i;kj}N^j = (\sigma^{ik}d_{i;k})_{;j}N^j - d_{i;kj}N^i N^k N^j = -n(H_d)_j N^j + \nabla^3 d(\frac{\nabla u}{W}, \frac{\nabla u}{W}, \frac{\nabla u}{W}).$$

Therefore grouping and rearranging these expressions we obtain

$$
\begin{aligned}
Lh &= |A|^2\phi\theta + n\phi HW\langle AY^T, \nabla^\Sigma d\rangle + \big(\kappa\gamma - \frac{1}{2}\langle\nabla d, \nabla\gamma\rangle\big)\phi\langle AY^T, Y^T\rangle \\
&+2\langle A\nabla^\Sigma d, \nabla^\Sigma\phi\rangle + 2\langle A, \nabla^2 d\rangle_\Sigma\phi - \frac{1}{W^2}\phi\langle A\nabla^\Sigma d, X_*\nabla\gamma\rangle \\
&\big(nH - \mathcal{H}\big)\big(\langle N, \nabla\phi\rangle\theta - \alpha\theta - \frac{1}{2W^2}\langle\nabla\gamma, \nabla d\rangle\phi + \langle\nabla_{\frac{\nabla u}{W}}\nabla d, \frac{\nabla u}{W}\rangle\phi\big) - n\kappa H\phi \\
&-n\alpha H_d + \big(2\langle N, \nabla\phi\rangle - \alpha\big)\langle\nabla_{\frac{\nabla u}{W}}\nabla d, \frac{\nabla u}{W}\rangle + 2\langle\nabla_{\frac{\nabla u}{W}}\nabla d, \nabla\phi\rangle \\
&-\phi\langle\nabla_{\frac{\nabla u}{W}}\nabla d, \frac{\nabla\gamma}{2\gamma}\rangle + \phi\langle\nabla^\Sigma\mathcal{H}, \bar\nabla d\rangle + n\langle\nabla H_d, N\rangle\phi - \phi\nabla^3 d(\frac{\nabla u}{W}, \frac{\nabla u}{W}, \frac{\nabla u}{W}) + \mathrm{Ric}(\nabla d, \frac{\nabla u}{W})\phi \\
&+\frac{\gamma}{W^2}\langle N, \nabla\kappa\rangle\phi - \big(\frac{1}{2\gamma} + \frac{1}{2W^2}\big)\alpha\langle\nabla d, \nabla\gamma\rangle + \big(\frac{1}{2\gamma} + \frac{1}{2W^2}\big)\langle\nabla\phi, \nabla\gamma\rangle\theta \\
&-\kappa\phi\langle N, \frac{\nabla\gamma}{2\gamma}\rangle + 2\kappa\frac{\gamma}{W^2}\langle\nabla\phi, N\rangle - \big(\Delta\phi - \langle\nabla_{\frac{\nabla u}{W}}\nabla\phi, \frac{\nabla u}{W}\rangle\big)\theta.
\end{aligned}
$$

Lemma 25 *We have*

$$
\begin{aligned}
LW - \frac{2}{W}g^{ij}W_iW_j &= |A|^2 W + nHW^3\langle AY^T, Y^T\rangle - nHW^3\langle\frac{\nabla\gamma}{2\gamma^2}, N\rangle - 3\gamma\langle AY^T, X_*\frac{\nabla\gamma}{2\gamma}\rangle \\
&+g^{ij}\frac{\gamma_{i;j}}{2\gamma}W - \frac{3}{4}\frac{|\nabla\gamma|^2}{4\gamma^2}W - \frac{1}{4}\langle\frac{\nabla\gamma}{2\gamma}, N\rangle^2 W + \gamma W\langle\bar\nabla_N\frac{\bar\nabla\gamma}{2\gamma^2}, N\rangle - W\langle\nabla^\Sigma\mathcal{H}, N\rangle \\
&-\frac{|\nabla\gamma|^2}{4\gamma}\frac{1}{W} - W_t.
\end{aligned}
$$

Proof: Notice that

$$
\begin{aligned}
W_i &= -W^2\big(\langle\bar\nabla_{X_*\frac{\partial}{\partial x^i}}Y, N\rangle + \langle Y, \bar\nabla_{X_*\frac{\partial}{\partial x^i}}N\rangle\big) \\
&= -W^2\big(\langle\bar\nabla_{\frac{\partial}{\partial x^i}}Y, N\rangle + u_i\langle\bar\nabla_Y Y, N\rangle - \langle Y, AX_*\frac{\partial}{\partial x^i}\rangle\big) \\
&= -W^2\big(-\frac{\gamma_i}{2\gamma}\langle Y, N\rangle + u_i\langle\frac{\nabla\gamma}{2\gamma^2}, N\rangle - \langle Y, AX_*\frac{\partial}{\partial x^i}\rangle\big).
\end{aligned}
$$

Therefore

$$
W_i = \frac{\gamma_i}{2\gamma}W + N_iW^3\langle\frac{\nabla\gamma}{2\gamma^2}, N\rangle + W^2\langle AY^T, X_*\frac{\partial}{\partial x^i}\rangle.
$$

However

$$
\langle AY^T, X_*\frac{\partial}{\partial x^i}\rangle = g^{kl}\langle Y, X_*\frac{\partial}{\partial x^k}\rangle\langle X_*\frac{\partial}{\partial x^l}, AX_*\frac{\partial}{\partial x^i}\rangle = g^{kl}\langle Y, u_kY\rangle b_{il} = \frac{1}{W^2}u^l b_{il}.
$$

Hence it follows that

$$
W_i = \frac{\gamma_i}{2\gamma}W + N_iW^3\langle\frac{\nabla\gamma}{2\gamma^2}, N\rangle - WN^l b_{il}.
$$

Hence we obtain

$$\frac{1}{W}g^{ij}W_iW_j = \frac{|\nabla^\Sigma\gamma|^2}{4\gamma^2}W + W\langle\nabla\gamma, N\rangle\langle\frac{\nabla\gamma}{2\gamma^2}, N\rangle + \langle AY^T, \frac{\nabla^\Sigma\gamma}{\gamma}\rangle W^2$$
$$+\gamma|\nabla u|^2\langle\frac{\nabla\gamma}{2\gamma^2}, N\rangle^2 W - \langle AY^T, Y^T\rangle\langle\frac{\nabla\gamma}{\gamma}, N\rangle W^3 + \langle AY^T, AY^T\rangle W^3$$

Now we compute

$$W_{i;j} = \big(\frac{\gamma_{i;j}}{2\gamma} - \frac{\gamma_i\gamma_j}{2\gamma^2}\big)W + \frac{\gamma_i}{2\gamma}W_j + N_{i;j}W^3\langle\frac{\nabla\gamma}{2\gamma^2}, N\rangle + 3N_iW^2W_j\langle\frac{\nabla\gamma}{2\gamma^2}, N\rangle$$
$$+N_iW^3\big(\langle\bar\nabla_{X_*\frac{\partial}{\partial x^j}}\frac{\bar\nabla\gamma}{2\gamma^2}, N\rangle - \langle\frac{\nabla\gamma}{2\gamma^2}, AX_*\frac{\partial}{\partial x^j}\rangle\big) - W_jN^lb_{il} - WN^l_{;j}b_{il} - WN^lb_{il;j}.$$

However we have

$$g^{ij}\frac{\gamma_i}{2\gamma}W_j = \frac{|\nabla^\Sigma\gamma|^2}{4\gamma^2}W + \langle\frac{\nabla\gamma}{2\gamma}, N\rangle^2 W + W^2\langle AY^T, \frac{\nabla^\Sigma\gamma}{2\gamma}\rangle$$

and

$$g^{ij}N_{i;j} = g^{ij}\sigma_{ik}N^k_{;j} = -(\delta^j_k - N^jN_k)(a^k_j - N_j\frac{\gamma^k}{2\gamma})$$
$$= -nH + \frac{\gamma}{W^2}\langle N, \frac{\nabla\gamma}{2\gamma}\rangle + \gamma\langle AY^T, Y^T\rangle.$$

Moreover we compute

$$g^{ij}N_iW_j = \frac{\gamma}{W}\langle N, \frac{\nabla\gamma}{2\gamma}\rangle + \frac{\gamma|\nabla u|^2}{W}\langle\frac{\nabla\gamma}{2\gamma^2}, N\rangle - \gamma W\langle AY^T, Y^T\rangle$$

and

$$g^{ij}N_iW^3\big(\langle\bar\nabla_{X_*\frac{\partial}{\partial x^j}}\frac{\bar\nabla\gamma}{2\gamma^2}, N\rangle - \langle\frac{\bar\nabla\gamma}{2\gamma^2}, AX_*\frac{\partial}{\partial x^j}\rangle\big)$$
$$= \gamma W\big(\langle\bar\nabla_{N-WY}\frac{\bar\nabla\gamma}{2\gamma^2}, N\rangle - \langle A\frac{\nabla^\Sigma\gamma}{2\gamma^2}, -WY\rangle\big)$$
$$= \gamma W\langle\bar\nabla_N\frac{\bar\nabla\gamma}{2\gamma^2}, N\rangle + W\frac{|\nabla\gamma|^2}{4\gamma^2} + \gamma W^2\langle A\frac{\nabla^\Sigma\gamma}{2\gamma^2}, Y^T\rangle.$$

We also have

$$2Wg^{ij}W_j\langle AY^T, X_*\frac{\partial}{\partial x^i}\rangle = 2W^2\langle AY^T, \frac{\nabla^\Sigma\gamma}{2\gamma}\rangle - W^3\langle\frac{\nabla\gamma}{\gamma}, N\rangle\langle AY^T, Y^T\rangle + 2W^3\langle AY^T, AY^T\rangle.$$

Now we compute

$$g^{ij}WN^lb_{il;j} = WN^lg^{ij}\nabla^\Sigma_j a_{il} + \frac{1}{\gamma}W^2g^{ij}a_{ij}a^m_lN^lu_m + \frac{1}{\gamma}W^2g^{ij}a_{lj}N^la^m_iu_m$$
$$+Wg^{ij}\frac{u_iu_j}{2\gamma^2}a_{lm}N^l\gamma^m + Wg^{ij}N^l\frac{u_lu_j}{2\gamma^2}a_{im}\gamma^m.$$

Hence we have

$$g^{ij}WN^l b_{il;j} = WN^l(n\nabla^\Sigma_l H + g^{ij}\langle \bar{R}(X_*\frac{\partial}{\partial x^i}, X_*\frac{\partial}{\partial x^j})N, X_*\frac{\partial}{\partial x^l}\rangle)$$
$$+nHW^2\langle AY^T, N^k X_*\frac{\partial}{\partial x^k}\rangle + W^2 g^{ij}\langle AY^T, X_*\frac{\partial}{\partial x^j}\rangle\big(-W\langle AY^T, X_*\frac{\partial}{\partial x^i}\rangle\big)$$
$$-|\nabla u|^2\langle AY^T, X_*\frac{\nabla\gamma}{2\gamma}\rangle + \frac{|\nabla u|^2}{2\gamma W}(-W\langle AY^T, X_*\frac{\partial}{\partial x^m}\rangle)\gamma^m.$$

Therefore

$$g^{ij}WN^l b_{il;j} = nWN^l\nabla^\Sigma_l H - nHW^3\langle AY^T, Y^T\rangle - W^3\langle AY^T, AY^T\rangle - |\nabla u|^2\langle AY^T, X_*\frac{\nabla\gamma}{\gamma}\rangle.$$

Moreover

$$g^{ij}W_j N^l b_{il} = -W^2\langle AY^T, \frac{\nabla^\Sigma\gamma}{2\gamma}\rangle + W^3\langle\frac{\nabla\gamma}{2\gamma}, N\rangle\langle AY^T, Y^T\rangle - W^3\langle AY^T, AY^T\rangle$$

and

$$Wg^{ij}N^l_{;j}b_{il} = -Wg^{ij}(a^l_j - N_j\frac{\gamma^l}{2\gamma})a_{il} = -|A|^2W - \frac{1}{2}\langle AY^T, X_*\nabla\gamma\rangle.$$

We conclude that

$$g^{ij}W_{i;j} = |A|^2W + 2W^3\langle AY^T, AY^T\rangle + \big(nH - 3\langle\frac{\nabla\gamma}{2\gamma}, N\rangle\big)W^3\langle AY^T, Y^T\rangle$$
$$+3W^2\langle AY^T, \frac{\nabla^\Sigma\gamma}{2\gamma}\rangle + |\nabla u|^2\langle AY^T, X_*\frac{\nabla\gamma}{\gamma}\rangle + \frac{1}{2}\langle AY^T, X_*\nabla\gamma\rangle$$
$$+g^{ij}\frac{\gamma_{i;j}}{2\gamma}W - \frac{|\nabla^\Sigma\gamma|^2}{4\gamma^2}W + \frac{|\nabla\gamma|^2}{4\gamma^2}W + \big(5W + 3\frac{W}{\gamma}|\nabla u|^2\big)\langle\frac{\nabla\gamma}{2\gamma}, N\rangle^2$$
$$-nHW^3\langle\frac{\nabla\gamma}{2\gamma^2}, N\rangle + \gamma W\langle\bar{\nabla}_N\frac{\bar{\nabla}\gamma}{2\gamma^2}, N\rangle - nWN^l\nabla^\Sigma_l H.$$

Now

$$\langle\nabla\gamma, \nabla W\rangle = \frac{|\nabla\gamma|^2}{2\gamma}W + \frac{1}{2\gamma^2}\langle\nabla\gamma, N\rangle^2W^3 + W^2\langle AY^T, X_*\nabla\gamma\rangle.$$

Hence

$$
\begin{aligned}
LW - \frac{2}{W} g^{ij} W_i W_j &= |A|^2 W + \big(nH + \langle \frac{\nabla\gamma}{2\gamma}, N\rangle\big) W^3 \langle AY^T, Y^T\rangle \\
&- W^2 \langle AY^T, \frac{\nabla^\Sigma \gamma}{2\gamma}\rangle + |\nabla u|^2 \langle AY^T, X_* \frac{\nabla\gamma}{\gamma}\rangle + \frac{1}{2}\langle AY^T, X_* \nabla\gamma\rangle \\
&- \big(\frac{1}{2\gamma} + \frac{1}{2W^2}\big) W^2 \langle AY^T, X_* \nabla\gamma\rangle \\
&+ g^{ij} \frac{\gamma_{i;j}}{2\gamma} W - \frac{3}{4}\frac{|\nabla^\Sigma \gamma|^2}{4\gamma^2} W + \frac{|\nabla\gamma|^2}{4\gamma^2} W + \big(5W + 3\frac{W}{\gamma}|\nabla u|^2\big)\langle \frac{\nabla\gamma}{2\gamma}, N\rangle^2 \\
&- nHW^3 \langle \frac{\nabla\gamma}{2\gamma^2}, N\rangle + \gamma W \langle \bar{\nabla}_N \frac{\bar{\nabla}\gamma}{2\gamma^2}, N\rangle - nWN^l \nabla^\Sigma_l H \\
&- \big(\frac{1}{2\gamma} + \frac{1}{2W^2}\big)\big(\frac{|\nabla\gamma|^2}{2\gamma} W + \frac{1}{2\gamma^2}\langle \nabla\gamma, N\rangle^2 W^3\big) \\
&- \frac{1}{\gamma^2}\langle \nabla\gamma, N\rangle^2 W - 2\gamma |\nabla u|^2 \langle \frac{\nabla\gamma}{2\gamma^2}, N\rangle^2 W - W_t.
\end{aligned}
$$

However

$$
3\frac{W}{\gamma}|\nabla u|^2 \langle \frac{\nabla\gamma}{2\gamma}, N\rangle^2 - 2\gamma |\nabla u|^2 \langle \frac{\nabla\gamma}{2\gamma^2}, N\rangle^2 W = \frac{W}{\gamma}|\nabla u|^2 \langle \frac{\nabla\gamma}{2\gamma}, N\rangle^2
$$

and

$$
5W \langle \frac{\nabla\gamma}{2\gamma}, N\rangle^2 - \frac{1}{\gamma^2}\langle \nabla\gamma, N\rangle^2 W = \langle \frac{\nabla\gamma}{2\gamma}, N\rangle^2 W
$$

and

$$
\begin{aligned}
&- W^2 \langle AY^T, \frac{\nabla^\Sigma \gamma}{2\gamma}\rangle + |\nabla u|^2 \langle AY^T, X_* \frac{\nabla\gamma}{\gamma}\rangle + \frac{1}{2}\langle AY^T, X_* \nabla\gamma\rangle - \big(\frac{1}{2\gamma} + \frac{1}{2W^2}\big) W^2 \langle AY^T, X_* \nabla\gamma\rangle \\
&= -3\gamma \langle AY^T, X_* \frac{\nabla\gamma}{2\gamma}\rangle - W^3 \langle \frac{\nabla\gamma}{2\gamma}, N\rangle \langle AY^T, Y^T\rangle.
\end{aligned}
$$

Moreover we compute

$$
\begin{aligned}
&\big(\frac{1}{2\gamma} + \frac{1}{2W^2}\big)\big(\frac{|\nabla\gamma|^2}{2\gamma} W + \frac{1}{2\gamma^2}\langle \nabla\gamma, N\rangle^2 W^3\big) \\
&= \frac{|\nabla\gamma|^2}{4\gamma^2} W + \frac{1}{\gamma} W^3 \langle \frac{\nabla\gamma}{2\gamma}, N\rangle^2 + \frac{|\nabla\gamma|^2}{4\gamma}\frac{1}{W} + \langle \frac{\nabla\gamma}{2\gamma}, N\rangle^2 W
\end{aligned}
$$

and

$$
-nWN^l \nabla^\Sigma_l H = -nW\langle \nabla^\Sigma H, N\rangle = -W\langle \nabla^\Sigma \mathcal{H}, N\rangle.
$$

We conclude that

$$
\begin{aligned}
LW - \frac{2}{W} g^{ij} W_i W_j &= |A|^2 W + nHW^3 \langle AY^T, Y^T \rangle - 3\gamma \langle AY^T, X_* \frac{\nabla\gamma}{2\gamma} \rangle \\
&+ g^{ij} \frac{\gamma_{i;j}}{2\gamma} W - \frac{3}{4} \frac{|\nabla^\Sigma \gamma|^2}{4\gamma^2} W + \frac{|\nabla u|^2}{\gamma} \langle \frac{\nabla\gamma}{2\gamma}, N \rangle^2 W \\
&- nHW^3 \langle \frac{\nabla\gamma}{2\gamma^2}, N \rangle + \gamma W \langle \bar{\nabla}_N \frac{\bar{\nabla}\gamma}{2\gamma^2}, N \rangle - W \langle \nabla^\Sigma \mathcal{H}, N \rangle \\
&- \frac{1}{\gamma} W^3 \langle \frac{\nabla\gamma}{2\gamma}, N \rangle^2 - \frac{|\nabla\gamma|^2}{4\gamma} \frac{1}{W} - W_t.
\end{aligned}
$$

However

$$
\frac{|\nabla u|^2}{\gamma} \langle \frac{\nabla\gamma}{2\gamma}, N \rangle^2 W - \frac{1}{\gamma} W^3 \langle \frac{\nabla\gamma}{2\gamma}, N \rangle^2 = -\langle \frac{\nabla\gamma}{2\gamma}, N \rangle^2 W
$$

and

$$
\begin{aligned}
-\frac{3}{4} \frac{|\nabla^\Sigma \gamma|^2}{4\gamma^2} W - \langle \frac{\nabla\gamma}{2\gamma}, N \rangle^2 W &= -\frac{3}{4} \frac{|\nabla\gamma|^2}{4\gamma^2} W + \frac{3}{4} \langle \frac{\nabla\gamma}{2\gamma}, N \rangle^2 W - \langle \frac{\nabla\gamma}{2\gamma}, N \rangle^2 W \\
= -\frac{3}{4} \frac{|\nabla\gamma|^2}{4\gamma^2} W - \frac{1}{4} \langle \frac{\nabla\gamma}{2\gamma}, N \rangle^2 W.
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
LW - \frac{2}{W} g^{ij} W_i W_j &= |A|^2 W + nHW^3 \langle AY^T, Y^T \rangle - nHW^3 \langle \frac{\nabla\gamma}{2\gamma^2}, N \rangle - 3\gamma \langle AY^T, X_* \frac{\nabla\gamma}{2\gamma} \rangle \\
+ g^{ij} \frac{\gamma_{i;j}}{2\gamma} W &- \frac{3}{4} \frac{|\nabla\gamma|^2}{4\gamma^2} W - \frac{1}{4} \langle \frac{\nabla\gamma}{2\gamma}, N \rangle^2 W + \gamma W \langle \bar{\nabla}_N \frac{\bar{\nabla}\gamma}{2\gamma^2}, N \rangle - W \langle \nabla^\Sigma \mathcal{H}, N \rangle \\
- \frac{|\nabla\gamma|^2}{4\gamma} \frac{1}{W} &- W_t.
\end{aligned}
$$

This finishes the proof of the lemma.

Now we are able to prove the following result

Proposition 26 *For fixed $T^* < T$ there exists $K > 0$ sufficiently large so that if*

$$
\eta W(x_0, t_0) = \max_{\bar{\Omega} \times [0, T^*]} \eta W
$$

for some $(x_0, t_0) \in \bar{\Omega} \times [0, T^*]$, then $W(x_0, t_0) \leq C$, for some constant C.

Proof: We can assume $x_0 \in \Omega$ and $t_0 > 0$. At a point (x_0, t_0) where ηW attains maximum value we have

$$
\eta_i W + \eta W_i = 0 \tag{4.25}
$$

and

$$\frac{1}{\eta}L\eta + \frac{1}{W}\Big(LW - \frac{2}{W}g^{ij}W_iW_j\Big) \leq 0. \tag{4.26}$$

We conclude that

$$\begin{aligned}\frac{1}{\eta}L\eta &= KLu + \frac{1}{h}Lh + K^2 g^{ij}u_iu_j + 2Kg^{ij}u_i\frac{h_j}{h}\\ &= K\mathcal{H}W + \frac{1}{h}Lh + K^2\frac{\gamma|\nabla u|^2}{W^2} + 2Kg^{ij}u_i\frac{h_j}{h}.\end{aligned}$$

Now we have

$$g^{ij}u_ih_j = \frac{\gamma}{W^2}u^jh_j = -\frac{\gamma}{W}(\alpha\langle N, \nabla d\rangle - \langle N, \nabla\phi\rangle\theta - \phi\langle N, \nabla\theta\rangle).$$

However

$$\langle N, \nabla\theta\rangle = W\langle AY^T, \nabla^\Sigma d\rangle + \langle\nabla_{\frac{\nabla u}{W}}\nabla d, \frac{\nabla u}{W}\rangle - \kappa\frac{|\nabla u|^2}{W^2}.$$

Therefore

$$g^{ij}u_ih_j = -\alpha\frac{\gamma}{W}\langle N, \nabla d\rangle + \frac{\gamma}{W}\langle N, \nabla\phi\rangle\theta + \gamma\phi\langle AY^T, \nabla^\Sigma d\rangle + \frac{\gamma}{W}\phi\langle\nabla_{\frac{\nabla u}{W}}\nabla d, \frac{\nabla u}{W}\rangle - \gamma\phi\kappa\frac{|\nabla u|^2}{W^3}.$$

Thus the expression for Lh in Appendix allows us to conclude that

$$\begin{aligned}\frac{1}{\eta}L\eta &= K\mathcal{H}W + K^2\frac{\gamma|\nabla u|^2}{W^2}\\ &+\frac{2K}{h}\Big(-\alpha\frac{\gamma}{W}\langle N, \nabla d\rangle + \frac{\gamma}{W}\langle N, \nabla\phi\rangle\theta + \gamma\phi\langle AY^T, \nabla^\Sigma d\rangle + \frac{\gamma}{W}\phi\langle\nabla_{\frac{\nabla u}{W}}\nabla d, \frac{\nabla u}{W}\rangle - \gamma\phi\kappa\frac{|\nabla u|^2}{W^3}\Big)\\ &+\frac{1}{h}|A|^2\phi\theta + n\frac{1}{h}\phi HW\langle AY^T, \nabla^\Sigma d\rangle + \frac{1}{h}\Big(\kappa\gamma - \frac{1}{2}\langle\nabla d, \nabla\gamma\rangle\Big)\phi\langle AY^T, Y^T\rangle\\ &+\frac{2}{h}\langle A\nabla^\Sigma d, \nabla^\Sigma\phi\rangle + \frac{2}{h}\langle A, \nabla^2 d\rangle_\Sigma\phi - \frac{1}{hW^2}\phi\langle A\nabla^\Sigma d, X_*\nabla\gamma\rangle\\ &+\frac{1}{h}\big(nH - \mathcal{H}\big)\Big(\langle N, \nabla\phi\rangle\theta - \alpha\theta - \frac{1}{2W^2}\langle\nabla\gamma, \nabla d\rangle\phi + \langle\nabla_{\frac{\nabla u}{W}}\nabla d, \frac{\nabla u}{W}\rangle\phi\Big) - n\frac{1}{h}\kappa H\phi\\ &-n\frac{\alpha}{h}H_d + \frac{1}{h}\big(2\langle N, \nabla\phi\rangle - \alpha\big)\langle\nabla_{\frac{\nabla u}{W}}\nabla d, \frac{\nabla u}{W}\rangle + \frac{2}{h}\langle\nabla_{\frac{\nabla u}{W}}\nabla d, \nabla\phi\rangle\\ &-\frac{1}{h}\phi\langle\nabla_{\frac{\nabla u}{W}}\nabla d, \frac{\nabla\gamma}{2\gamma}\rangle + \frac{1}{h}\phi\langle\nabla^\Sigma\mathcal{H}, \bar{\nabla} d\rangle + n\frac{1}{h}\langle\nabla H_d, N\rangle\phi - \frac{1}{h}\phi\nabla^3 d\Big(\frac{\nabla u}{W}, \frac{\nabla u}{W}, \frac{\nabla u}{W}\Big)\\ &+\frac{1}{h}\mathrm{Ric}\Big(\nabla d, \frac{\nabla u}{W}\Big)\phi + \frac{\gamma}{hW^2}\langle N, \nabla\kappa\rangle\phi - \frac{1}{h}\Big(\frac{1}{2\gamma} + \frac{1}{2W^2}\Big)\alpha\langle\nabla d, \nabla\gamma\rangle + \frac{1}{h}\Big(\frac{1}{2\gamma} + \frac{1}{2W^2}\Big)\langle\nabla\phi, \nabla\gamma\rangle\theta\\ &-\kappa\frac{1}{h}\phi\langle N, \frac{\nabla\gamma}{2\gamma}\rangle + \frac{2}{h}\kappa\frac{\gamma}{W^2}\langle\nabla\phi, N\rangle - \frac{1}{h}\Big(\Delta\phi - \langle\nabla_{\frac{\nabla u}{W}}\nabla\phi, \frac{\nabla u}{W}\rangle\Big)\theta.\end{aligned}$$

On the other hand Lemma 25 yields

$$\frac{1}{W}\Big(LW-\frac{2}{W}g^{ij}W_iW_j\Big)=|A|^2+nHW^2\langle AY^T,Y^T\rangle-nHW^2\langle\frac{\nabla\gamma}{2\gamma^2},N\rangle$$
$$-3\frac{1}{W}\gamma\langle AY^T,X_*\frac{\nabla\gamma}{2\gamma}\rangle+g^{ij}\frac{\gamma_{i;j}}{2\gamma}-\frac{3}{4}\frac{|\nabla\gamma|^2}{4\gamma^2}-\frac{1}{4}\langle\frac{\nabla\gamma}{2\gamma},N\rangle^2+\gamma\langle\bar\nabla_N\frac{\bar\nabla\gamma}{2\gamma^2},N\rangle$$
$$-\langle\nabla^\Sigma\mathcal{H},N\rangle-\frac{|\nabla\gamma|^2}{4\gamma}\frac{1}{W^2}-\frac{W_t}{W}.$$

Now we use the fact that x_0 is a critical point to ηW. We have

$$e^{Ku}(Ku_ih+h_i)W=-e^{Ku}hW_i.$$

what implies that

$$-KW^2hN_iN^i+Wh_iN^i=-hW_iN^i$$

and then

$$-Kh|\nabla u|^2+Wh_iN^i=-hW_iN^i.$$

However

$$W_iN^i=\frac{\gamma_i}{2\gamma}N^iW+N_iN^iW^3\langle\frac{\nabla\gamma}{2\gamma^2},N\rangle+W^2\langle AY^T,N^iX_*\frac{\partial}{\partial x^i}\rangle$$
$$=\frac{1}{2\gamma}\langle\nabla\gamma,N\rangle W+|\nabla u|^2W\langle\frac{\nabla\gamma}{2\gamma^2},N\rangle-W^3\langle AY^T,Y^T\rangle$$

and

$$h_iN^i=\alpha\theta-\langle\nabla\phi,N\rangle\theta+\phi a_i^jN^id_j-\phi(d_{i;j}N^iN^j-\kappa\sigma_{ij})N^iN^j$$
$$=\alpha\theta-\langle\nabla\phi,N\rangle\theta-\phi W\langle AY^T,\nabla^\Sigma d\rangle-\phi\langle\nabla_{\frac{\nabla u}{W}}\nabla d,\frac{\nabla u}{W}\rangle+\phi\kappa\frac{|\nabla u|^2}{W^2}.$$

We then conclude that

$$-K\frac{|\nabla u|^2}{W}+\frac{\alpha\theta}{h}-\frac{1}{h}\langle\nabla\phi,N\rangle\theta-\frac{\phi}{h}W\langle AY^T,\nabla^\Sigma d\rangle-\frac{\phi}{h}\langle\nabla_{\frac{\nabla u}{W}}\nabla d,\frac{\nabla u}{W}\rangle+\frac{\phi}{h}\kappa\frac{|\nabla u|^2}{W^2}$$
$$=-\frac{1}{2\gamma}\langle\nabla\gamma,N\rangle-|\nabla u|^2\langle\frac{\nabla\gamma}{2\gamma^2},N\rangle+W^2\langle AY^T,Y^T\rangle$$

Moreover

$$\begin{aligned}-\frac{W_t}{W}&=\frac{\eta_t}{\eta}=Ku_t+\frac{h_t}{h}=WK(nH-\mathcal{H})+\frac{h_t}{h}\\&=nHKW-KW\mathcal{H}-\frac{1}{h}(nH-\mathcal{H})\Big(\langle\nabla\phi,N\rangle\theta-\alpha\theta-\frac{\phi}{2W^2}\langle\nabla\gamma,\nabla d\rangle\Big)\\&\quad+\frac{\phi}{h}\langle\frac{\nabla u}{W},\nabla_{\frac{\nabla u}{W}}\nabla d\rangle.\end{aligned}$$

Then we have

$$\frac{1}{W}\Big(LW-\frac{2}{W}g^{ij}W_iW_j\Big)=|A|^2+KnH\frac{\gamma}{W}+\frac{\alpha\theta}{h}nH-\frac{1}{h}nH\langle\nabla\phi,N\rangle\theta-\frac{\phi}{h}nHW\langle AY^T,\nabla^\Sigma d\rangle$$
$$-\frac{1}{h}(nH-\mathcal{H})\Big(\langle\nabla\phi,N\rangle\theta-\alpha\theta-\frac{\phi}{2W^2}\langle\nabla\gamma,\nabla d\rangle\Big)-\frac{\phi}{h}nH\langle\nabla_{\frac{\nabla u}{W}}\nabla d,\frac{\nabla u}{W}\rangle+\frac{\phi}{h}nH\kappa\frac{|\nabla u|^2}{W^2}$$
$$-3\frac{1}{W}\gamma\langle AY^T,X_*\frac{\nabla\gamma}{2\gamma}\rangle+g^{ij}\frac{\gamma_{i;j}}{2\gamma}-\frac{3}{4}\frac{|\nabla\gamma|^2}{4\gamma^2}-\frac{1}{4}\langle\frac{\nabla\gamma}{2\gamma},N\rangle^2+\gamma\langle\bar\nabla_N\frac{\bar\nabla\gamma}{2\gamma^2},N\rangle-\langle\nabla^\Sigma\mathcal{H},N\rangle$$
$$-\frac{|\nabla\gamma|^2}{4\gamma}\frac{1}{W^2}-KW\mathcal{H}+\frac{\phi}{h}\langle\frac{\nabla u}{W},\nabla_{\frac{\nabla u}{W}}\nabla d\rangle.$$

We conclude that

$$\frac{1}{\eta}L\eta+\frac{1}{W}\Big(LW-\frac{2}{W}g^{ij}W_iW_j\Big)=K^2\frac{\gamma|\nabla u|^2}{W^2}+\mathcal{A}+\mathcal{B},$$

where

$$\mathcal{A}=\Big(1+\frac{\phi\theta}{h}\Big)|A|^2+\frac{2K}{h}\gamma\phi\langle AY^T,\nabla^\Sigma d\rangle+\frac{\phi}{h}\big(\kappa\gamma-\frac{1}{2}\langle\nabla d,\nabla\gamma\rangle\big)\langle AY^T,Y^T\rangle$$
$$+\frac{2}{h}\langle A\nabla^\Sigma d,\nabla^\Sigma\phi\rangle+\frac{2}{h}\langle A,\nabla^2 d\rangle_\Sigma\phi-\frac{1}{hW^2}\phi\langle A\nabla^\Sigma d,X_*\nabla\gamma\rangle$$
$$+KnH\frac{\gamma}{W}+\frac{\alpha\theta}{h}nH-\frac{1}{h}nH\langle\nabla\phi,N\rangle\theta-\frac{\phi}{h}nH\kappa\frac{\gamma}{W^2}-3\frac{1}{W}\gamma\langle AY^T,X_*\frac{\nabla\gamma}{2\gamma}\rangle$$

and

$$\mathcal{B}=\frac{2K}{h}\big(-\alpha\frac{\gamma}{W}\langle N,\nabla d\rangle+\frac{\gamma}{W}\langle N,\nabla\phi\rangle\theta+\frac{\gamma}{W}\phi\langle\nabla_{\frac{\nabla u}{W}}\nabla d,\frac{\nabla u}{W}\rangle-\gamma\phi\kappa\frac{|\nabla u|^2}{W^3}\big)$$
$$-\mathcal{H}\langle\nabla_{\frac{\nabla u}{W}}\nabla d,\frac{\nabla u}{W}\rangle\phi-n\frac{\alpha}{h}H_d+\frac{1}{h}\big(2\langle N,\nabla\phi\rangle-\alpha\big)\langle\nabla_{\frac{\nabla u}{W}}\nabla d,\frac{\nabla u}{W}\rangle+\frac{2}{h}\langle\nabla_{\frac{\nabla u}{W}}\nabla d,\nabla\phi\rangle$$
$$-\frac{1}{h}\phi\langle\nabla_{\frac{\nabla u}{W}}\nabla d,\frac{\nabla\gamma}{2\gamma}\rangle+\frac{1}{h}\phi\langle\nabla^\Sigma\mathcal{H},\bar\nabla d\rangle+n\frac{1}{h}\langle\nabla H_d,N\rangle\phi-\frac{1}{h}\phi\nabla^3 d(\frac{\nabla u}{W},\frac{\nabla u}{W},\frac{\nabla u}{W})$$
$$+\frac{1}{h}\mathrm{Ric}(\nabla d,\frac{\nabla u}{W})\phi+\frac{\gamma}{hW^2}\langle N,\nabla\kappa\rangle\phi-\frac{1}{h}(\frac{1}{2\gamma}+\frac{1}{2W^2})\alpha\langle\nabla d,\nabla\gamma\rangle$$
$$+\frac{1}{h}(\frac{1}{2\gamma}+\frac{1}{2W^2})\langle\nabla\phi,\nabla\gamma\rangle\theta-\kappa\frac{1}{h}\phi\langle N,\frac{\nabla\gamma}{2\gamma}\rangle+\frac{2}{h}\kappa\frac{\gamma}{W^2}\langle\nabla\phi,N\rangle$$
$$-\frac{1}{h}\big(\Delta\phi-\langle\nabla_{\frac{\nabla u}{W}}\nabla\phi,\frac{\nabla u}{W}\rangle\big)\theta+g^{ij}\frac{\gamma_{i;j}}{2\gamma}-\frac{3}{4}\frac{|\nabla\gamma|^2}{4\gamma^2}-\frac{1}{4}\langle\frac{\nabla\gamma}{2\gamma},N\rangle^2+\gamma\langle\bar\nabla_N\frac{\bar\nabla\gamma}{2\gamma^2},N\rangle$$
$$-\langle\nabla^\Sigma\mathcal{H},N\rangle-\frac{|\nabla\gamma|^2}{4\gamma}\frac{1}{W^2}+\frac{\phi}{h}\langle\frac{\nabla u}{W},\nabla_{\frac{\nabla u}{W}}\nabla d\rangle.$$

However using some standard inequalites we obtain

$$\mathcal{A}\geq\Big(1+\frac{\phi\theta}{h}\Big)|A|^2-\Big(\frac{2K\gamma}{h\sqrt{\gamma}}+\frac{\kappa}{h}+\frac{1}{2h\gamma}|\nabla\gamma|+\frac{2}{h}|\nabla\phi|+\frac{2}{h}|\nabla^2 d|_\Sigma$$
$$+\frac{1}{hW^2}|X_*\nabla\gamma|+\frac{K\gamma\sqrt{n}}{W}+\frac{\alpha\theta\sqrt{n}}{h}+\frac{\theta\sqrt{n}}{h}|\nabla\phi|+\frac{\gamma\sqrt{n}\kappa}{hW^2}+\frac{3\gamma}{\sqrt{\gamma}W}|X_*\frac{\nabla\gamma}{2\gamma}|\Big)|A|$$

Using that $W^2 \geq \gamma$ and choosing α sufficiently large and depending only on n, γ, ϕ and κ we have

$$\mathcal{A} \geq \frac{1}{2}|A|^2 - \Big(\epsilon + 2\sqrt{\gamma}\frac{K}{h} + \frac{K\gamma\sqrt{n}}{W} + \frac{3\sqrt{\gamma}}{W}|X_*\frac{\nabla\gamma}{2\gamma}|\Big)|A|$$
$$\geq -\Big(\epsilon + 2\sqrt{\gamma}\frac{K}{h} + \frac{K\gamma\sqrt{n}}{W} + \frac{3\sqrt{\gamma}}{W}|X_*\frac{\nabla\gamma}{2\gamma}|\Big)^2.$$

Moreover

$$\mathcal{B} \geq -C\bigg(1 + \frac{\alpha}{h} + \frac{\alpha}{hW^2} + \frac{1}{h} + \frac{1}{W^2} + \frac{1}{hW^2} + K\frac{\alpha}{h} + \frac{K}{h}\bigg),$$

where C is a constant depending on $n, \gamma, \phi, d, \kappa$ and $\mathcal{H}$.

Hence we obtain

$$\frac{1}{\eta}L\eta + \frac{1}{W}\Big(\mathcal{L}W - \frac{2}{W}g^{ij}W_iW_j\Big) \geq K^2\frac{\gamma|\nabla u|^2}{W^2} - C(\epsilon) - \frac{K}{W}C(\epsilon,\gamma,n) - \frac{K^2}{W^2}C(\gamma,n)$$
$$-\frac{1}{W}C(\gamma,\epsilon) - \frac{K}{W^2}C(\gamma,n) - \frac{K^2}{h^2}C(\gamma) - \frac{K^2}{hW}C(\gamma,n) - \frac{1}{W^2}C(\gamma) - \frac{K}{hW}C(\gamma)$$
$$-K\frac{\alpha}{h}C - \frac{K}{h}C(\epsilon,\gamma) - C - \frac{\alpha}{h}C - \frac{\alpha}{hW^2}C - \frac{1}{h}C - \frac{1}{W^2}C - \frac{1}{hW^2}C.$$

Then

$$-K^2\frac{\gamma|\nabla u|^2}{W^2} \geq -C\bigg(\frac{K^2}{W^2} + \frac{K}{W^2} + \frac{1}{W} + \frac{K}{hW} + \frac{K^2}{h^2} + \frac{\alpha}{hW^2} + \frac{1}{W^2} + \frac{1}{hW^2} + \frac{K}{W} + \frac{1}{W}$$
$$+K\frac{\alpha}{h} + \frac{K}{h} + \frac{\alpha}{h} + \frac{1}{h} + 1\bigg).$$

It follows that

$$\bigg(K^2\gamma - \Big(\frac{K^2}{h^2} + K\frac{\alpha}{h} + \frac{K}{h} + \frac{1+\alpha}{h} + 1\Big)C\bigg)W^2 \leq \Big(K^2 + K + \frac{1+\alpha}{h} + 1\Big)C$$
$$+(K + \frac{K}{h} + 1)CW.$$

Now suppose that $W(x_0,t_0) \geq 1$. Otherwise we are done. In this case we have $W \leq W^2$ and absorbing the terms with W into that one with W^2 transforms the inequality above into

$$\Big(K^2\gamma - \frac{K^2}{h^2}C - \frac{K}{h}C - KC - C - \frac{1}{h}(\alpha+1)(K+1)C\Big)W^2 \leq \Big(K^2 + K + 1 + \frac{1}{h}(\alpha+1)\Big)C.$$

If $d_0 = d(x_0)$ then choosing $\alpha \geq 1/(C(d_0)d_0 - 1)$ for some constant $C(d_0) > 1/d_0$ we obtain $(1+\alpha)/h \leq C(d_0)$ what implies that

$$\Big(K^2\gamma - \frac{K^2}{h^2}C - \frac{K}{h}C - KC(d_0) - C(d_0)\Big)W^2 \leq (K^2 + K + C(d_0))C.$$

Then for $\alpha > \frac{1}{d_0}\max\{1, \sqrt{2C/\gamma}\}$ we have

$$\Big(K^2\frac{\gamma}{2} - KC(d_0) - C(d_0)\Big)W^2 \leq (K^2 + K + C(d_0))C.$$

It follows that for $K > \frac{C(d_0)+\sqrt{C(d_0)^2+2\gamma C(d_0)}}{\gamma}$ we have $K^2\frac{\gamma}{2} - KC(d_0) - C(d_0) > 0$ and

$$W^2 \leq \frac{C(K^2 + K + C(d_0))}{K^2\frac{\gamma}{2} - KC(d_0) - C(d_0)}. \tag{4.27}$$

This finishes the proof of the proposition.

Theorem 27 *There exists a unique solution* $u : \bar{\Omega} \times [0, \infty) \to \mathbb{I}$ *to the problem* (1.6)-(1.8).

Proof: Propositions 21, 22 and 26 yield the following global gradient bound

$$W(x,t) \leq W(x_0,t_0)\frac{\eta(x,t)}{\eta(x_0,t_0)} \leq C_1 e^{C_2 M T^*}, \tag{4.28}$$

for $(x,t) \in \bar{\Omega} \times [0, T^*]$, where C_1 and C_2 are positive constants and

$$M = \max_{\bar{\Omega}\times[0,T^*]} |u - u_0|.$$

It results that (4.4) is uniformly parabolic and then the standard theory of quasilinear parabolic PDEs may be applied for assuring the existence of a unique smooth solution to (4.4)-(4.6).

4.4 Asymptotic behavior

Suppose from now on that $\mathcal{H} = 0$ and $\phi = 0$. In the particular case when the evolving functions have the form $u(x,t) = v(x) + Ct$, $(x,t) \in \bar{\Omega} \times [0, T)$, the initial value problem (4.4)-(4.6) becomes

$$\mathrm{div}\frac{\nabla v}{W} - \gamma\langle \bar{\nabla}_Y Y, \frac{\nabla v}{W}\rangle = \frac{C}{W} \quad \text{in} \quad \Omega \tag{4.29}$$

$$\langle \nu, N\rangle = 0 \qquad \text{on} \quad \partial\Omega \tag{4.30}$$

Conversely, notice that if $v(x)$ is a solution of (4.29)-(4.30) then $u = v + Ct$ is a solution of (4.4) which is translating along the flow lines of Y with speed C.

Now observe that

$$\mathrm{div}\frac{\nabla v}{W} - \gamma\langle \bar{\nabla}_Y Y, \frac{\nabla v}{W}\rangle = \mathrm{div}\frac{\nabla v}{W} + \gamma\langle \bar{\nabla}_{\frac{\nabla v}{W}} Y, Y\rangle = \mathrm{div}\frac{\nabla v}{W} + \gamma\langle \bar{\nabla}_Y \frac{\nabla v}{W} Y, Y\rangle = \mathrm{div}_M \frac{\nabla v}{W}.$$

Therefore it follows from divergence theorem that

$$\int_{\vartheta([0,s]\times\bar{\Omega})} \frac{C}{W} + \mathcal{H} = -\int_{\vartheta([0,s]\times\Gamma)} \langle \frac{\nabla v}{W}, \nu\rangle = \int_{\vartheta([0,s]\times\Gamma)} \langle N, \nu\rangle = \int_{\vartheta([0,s]\times\Gamma)} \phi. \tag{4.31}$$

Since the integrands do not depend on s we have

$$\int_\Omega C\frac{1}{\sqrt{\gamma}W} = \int_\Gamma \frac{1}{\sqrt{\gamma}}\phi. \tag{4.32}$$

from what results that

$$C = 0. \tag{4.33}$$

We then obtain the following height estimate

Proposition 28 *Given a solution $u(x,t)$ of (4.4) there exists a constant M such that*

$$|u(x,t)| \leq M \tag{4.34}$$

for $(x,t) \in \bar{\Omega}\times[0,+\infty)$.

Proof: We observe that since C is necessarily zero, $v=$ cte. is a solution to (4.29). In particular the constant functions $v_1 = \inf_\Omega u_0$ and $v_2 = \sup_\Omega u_0$ are solutions of (4.29) with $v_1 \leq u_0 \leq v_2$. Hence the parabolic maximum principle implies that

$$v_1 \leq u(\cdot,t) \leq v_2,$$

for $t\in[0,T)$ from what we obtain (4.34).

Now, proceeding as in [31], we prove the following convergence result

Theorem 29 *Suppose that $\mathcal{H}=0$ and $\phi=0$. Then $\lim_{t\to\infty} u_t = 0$. In particular the mean curvature flow converges to a slice of the form $\vartheta(\{s\}\times\bar{\Omega})$ for some $s\in\mathbb{I}$.*

Proof: It is immediate that $v=s$ is a trivial solution to (4.29) with (necessarily) $C=0$. We also have

$$\frac{d}{dt}\int_\Omega W = \int_\Omega \frac{u^i u_{i;t}}{W} = -\int_\Omega \frac{u_t^2}{W} - \int_\Omega \frac{1}{2W^3}\langle\nabla u,\nabla\gamma\rangle - \int_\Omega \frac{|\nabla u|^2}{2\gamma W^2}\langle\nabla u,\nabla\gamma\rangle.$$

Therefore

$$-\int_\Omega \frac{u_t^2}{W} = \frac{d}{dt}\Big(\int_\Omega W\Big) + \int_\Omega \frac{1}{2W^3}\langle\nabla u,\nabla\gamma\rangle + \int_\Omega \frac{|\nabla u|^2}{2\gamma W^2}\langle\nabla u,\nabla\gamma\rangle. \tag{4.35}$$

It follows that

$$\int_0^T \int_\Omega \frac{u_t^2}{W} = -\int_\Omega W(x,T) + \int_\Omega W(x,0)$$
$$+ \int_0^T \int_\Omega \frac{1}{2W^3} \langle \nabla u, \nabla \gamma \rangle + \int_0^T \int_\Omega \frac{|\nabla u|^2}{2\gamma W^2} \langle \nabla u, \nabla \gamma \rangle \leq \widetilde{C}$$

for some positive constant $\widetilde{C}$. It follows that $\lim_{t\to\infty} \frac{u_t^2}{W} = 0$. Since W is bounded then $\lim_{t\to\infty} u_t = 0$. This finishes the proof of the theorem.

$\square$

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