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# On a Class of Nonlinear Schrödinger Equations in $\mathbb{R}^2$ Involving Critical Growth

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In this paper we deal with semilinear elliptic problem of the form

$$\begin{aligned} -\varepsilon^2 \Delta u + V(z)\, u = f(u), \qquad & \text{in} \quad \mathbb{R}^2 \\ u \in C^2(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)\,,\, u > 0, \qquad & \text{in} \quad \mathbb{R}^2, \end{aligned}$$

where $\varepsilon$ is a small positive parameter, $V\colon \mathbb{R}^2 \to \mathbb{R}$ is a positive potential bounded away from zero, and $f(u)$ behaves like $\exp(\alpha s^2)$ when $s \to +\infty$. We prove the existence of solutions concentrating around a local minima not necessarily non-degenerate of $V(x)$, when $\varepsilon$ tends to 0. 

*Key Words:* Schrödinger equation; elliptic equations; critical growth; mountain-pass theorem; Trudinger–Moser inequality.

## 1. INTRODUCTION

This paper has been motivated by recent works concerning standing wave solutions of the nonlinear Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m}\, \Delta \psi + V(z)\, \psi - \gamma\, |\psi|^{p-1}\, \psi, \qquad \text{in} \quad \mathbb{R}^N \tag{1}$$

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i.e., solutions of the form

$$\psi(z, t) = \exp(-iEt/\hbar)\, v(z),$$

where $\hbar$, $m$, $\gamma$ are positive constants, $p > 1$, $E \in \mathbb{R}$, $V \in C^1(\mathbb{R}^N, \mathbb{R})$ and $v$ is a real function. It is well known that $\psi$ satisfies (1) if and only if the function $v(z)$ solves the elliptic equation

$$-\frac{\hbar^2}{2m}\, \Delta v + (V(z) - E)\, v = \gamma\, |v|^{p-1}\, v, \qquad \text{in} \quad \mathbb{R}^N.$$

In [10], Floer and Weinstein studied the case $N = 1$ and $p = 3$. They used a Lyapunov–Schmidt type reduction to prove the existence of standing wave solutions concentrating at each given nondegenerate critical point of the potential $V(z)$, when $\hbar$ tends to 0, under the assumption that $V$ is bounded. This method and result was extended by Oh in [16] to prove a similar result to higher dimensional cases with $1 < p < (N+2)/(N-2)$. The Lyapunov–Schmidt reduction method requires basically local conditions for the potential $V(z)$ and a nondegeneracy condition is essential. On the other hand, the calculus of variations based on variants of the mountain-pass theorem has been used by Rabinowitz in [18] to prove the existence of a positive "least-energy" solution when $\hbar$ is small, $1 < p < (N+2)/(N-2)$, and $V(z)$ satisfies the following global condition:

$$\inf_{z \in \mathbb{R}^N} V(z) < \liminf_{|z| \to \infty} V(z). \tag{2}$$

Moreover, Wang, in [19], has obtained the concentration behavior around the global minimum of $V(z)$ for these solutions, when $\hbar$ tends to 0. In [7], Felmer and del Pino have used the variational method based on local mountain-pass to prove the existence of standing wave solutions concentrating around local minima not necessarily nondegenerate of $V(z)$, when $\hbar$ tends to 0. It is natural to ask if this result is true, under a similar local condition for the potential $V(z)$, when we consider nonlinearities in the critical growth range. In [3], a positive answer to this question was given in the case $N \geqslant 3$ and here we consider the two-dimensional case. To be more precise, we deal with a semilinear elliptic problem of the form

$$\begin{aligned} &-\varepsilon^2\, \Delta u + V(z)\, u = f(u), && \text{in} \quad \mathbb{R}^2, \\ &u \in C^2(\mathbb{R}^2) \cap H^1(\mathbb{R}^2), && u > 0, \text{ in } \mathbb{R}^2, \end{aligned} \tag{$P_\varepsilon$}$$

where $\varepsilon$ is a small positive parameter and the potential $V\colon \mathbb{R}^2 \to \mathbb{R}$ satisfies the following conditions:

($V_1$) $V$ is locally Hölder continuous in $\mathbb{R}^2$ and there exists a positive constant $V_0$ such that

$$V(z) \geqslant V_0, \qquad \forall z \in \mathbb{R}^2;$$

($V_2$) there exists a bounded domain $\Omega \subset \mathbb{R}^2$ such that

$$V_1 \doteq \inf_{\Omega} V(z) < \min_{\partial\Omega} V(z).$$

We also assume that the nonlinearity $f(s)$ satisfies the following conditions:

($f_1$) $f \in C^1(\mathbb{R})$ and $f(s) \equiv 0$ for $s \leqslant 0$;

($f_2$) $f(s) = o_1(s)$ near origin;

($f_3$) $f$ has critical growth at $+\infty$; namely, there exists $\alpha_0 > 0$ such that

$$\lim_{s \to +\infty} \frac{f(s)}{\exp(\alpha s^2)} = 0, \ \forall \alpha > \alpha_0; \qquad \lim_{s \to +\infty} \frac{f(s)}{\exp(\alpha s^2)} = +\infty, \ \forall \alpha < \alpha_0;$$

($f_4$) there is a constant $\mu > 2$ such that, for all $s > 0$,

$$0 \leqslant \mu F(s) = \mu \int_0^s f(t)\, dt < s f(s);$$

($f_5$) the function $s \to f(s)/s$ is increasing;

($f_6$) there is $p > 2$ and $\delta > 0$ such that for all $s > 0$,

$$f(s) \geqslant \left( \frac{p}{2} (S_p + \delta)^p \left( \frac{4\pi}{\alpha_0} \right)^{1 - p/2} \right) s^{p-1},$$

where

$$S_p = \inf_{u \in H^1(\mathbb{R}^2) \setminus \{0\}} \frac{\left(\int_{\mathbb{R}^2} (|\nabla u|^2 + V_1 u^2)\, dz\right)^{1/2}}{\left(\int_{\mathbb{R}^2} |u|^p\, dz\right)^{1/p}}. \tag{3}$$

The main result of this paper is stated as follows.

THEOREM 1. *Suppose that the potential $V$ satisfies* ($V_1$)–($V_2$) *and that the nonlinearity $f$ satisfies* ($f_1$)–($f_6$). *Then there is $\varepsilon_0 > 0$ such that when $0 < \varepsilon < \varepsilon_0$, problem ($P_\varepsilon$) possesses a positive bound state solution $u_\varepsilon(z)$ with the following properties*:

(i) *$u_\varepsilon$ has at most one local (hence global) maximum $z_\varepsilon$ in $\mathbb{R}^2$ and $z_\varepsilon \in \Omega$;*

(ii) $\lim_{\varepsilon \to 0^+} V(z_\varepsilon) = V_1 = \inf_\Omega V$;

(iii) *there are $C$ and $\zeta$ positive constants such that for all $z \in \mathbb{R}^2$,*

$$u_\varepsilon(z) \leqslant C \exp\left(-\zeta \left|\frac{z - z_\varepsilon}{\varepsilon}\right|\right).$$

In order to treat variationally this class of problems, with $f$ behaving like $\exp(\alpha s^2)$ when $s \to +\infty$, we use the so-called Trudinger–Moser inequality which says that if $u$ is a $H^1(\mathbb{R}^2)$ function then for all $\alpha > 0$ the integral $\int_{\mathbb{R}^2} [\exp(\alpha u^2) - 1]\, dz$ is finite (see Lemma 2 in the next section). Indeed, this inequality motivates the notion of criticality given in ($f_3$). There is an extensive bibliography on this subject. See, for example, [5, 6] for the semilinear elliptic equations and [2, 8, 9] for quasilinear equations. We adapt some of their ideas to overcome the difficulties arising from the critical growth and unboundedness of the domain. Furthermore, as in [3] and [7], we make a suitable modification on the nonlinearity $f(s)$ outside the domain $\Omega$ such that the associated energy functional satisfies the Palais–Smale condition, and then using some elliptic estimates we can prove that, for sufficiently small $\varepsilon$, the associated minimax critical point is indeed a solution to the original equation. This elementary idea allows us to use the variational methods to deal with local conditions for the potential $V(z)$.

This paper is composed of three sections; taking preliminaries in the following section, we shall prove the existence and concentration behavior in the last section.

## 2. AUXILIARY PROBLEM

We make a suitable modification on the nonlinearity $f(s)$ outside the domain $\Omega$ such that the associated energy functional satisfies the Palais–Smale condition and to which we can apply the mountain-pass theorem. Namely, we consider the following Carathéodory function

$$g(z, s) = \chi_\Omega(z)\, f(s) + (1 - \chi_\Omega(z))\, \tilde{f}(s)$$

where $\chi_\Omega$ is the characteristic of $\Omega$ and

$$\tilde{f}(s) = \begin{cases} f(s), & \text{if} \quad s \leqslant a, \\ \dfrac{V_0}{k}\, s, & \text{if} \quad s > a, \end{cases}$$

with $k > \mu/(\mu - 2) > 1$ and $a > 0$ such that $f(a) = aV_0/k$.

Using assumptions ($f_1$)–($f_5$) it is easy to check that $g(z, s)$ satisfies the following properties:

($g_1$) $g(z, s)$ is piecewise $C^1$ in $s$ for any fixed $z$ and $g(z, s) \equiv 0$ for $s \leqslant 0$;

($g_2$) for each $\delta > 0$ and $\beta > \alpha_0$ there is a constant $c = c(\delta, \beta) > 0$ such that

$$g(z, s) \leqslant \delta s + c \exp(\beta s^2), \qquad \forall s \geqslant 0;$$

($g_3$)

$$0 < \mu G(z, s) \leqslant g(z, s)\, s, \qquad (z, s) \in [\Omega \times (0, +\infty)] \cup [(\mathbb{R}^2 - \Omega) \times (0, a]]$$

and

$$0 \leqslant 2G(z, s) \leqslant g(z, s)\, s \leqslant \frac{1}{k} V(z)\, s^2, \qquad (z, s) \in (\mathbb{R}^2 - \Omega) \times [0, +\infty),$$

where $G(z, s) = \int_0^s g(z, t)\, dt$;

($g_4$) the function $s \to g(z, s)/s$ is increasing,

Now, we consider an energy functional given by

$$J(u) = \tfrac{1}{2} \int_{\mathbb{R}^2} [|\nabla u|^2 + V(z)\, u^2]\, dz - \int_{\mathbb{R}^2} G(z, u)\, dz,$$

defined on the Hilbert space

$$H = \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} V(z)\, u^2\, dz < \infty \right\},$$

endowed with the inner product given by $\langle u, v \rangle = \int_{\mathbb{R}^2} [\nabla u\, \nabla v + V(z)\, uv]\, dz$ and the induced norm $\|u\| \doteq \sqrt{\langle u, u \rangle}$. $J$ is well defined and it is a $C^1$ functional with Fréchet derivative given by

These statements are standard (see [17]) and they follow from the conditions ($g_1$)–($g_2$) taking into account the following Trudinger–Moser inequality, which was proved in [9] (see also [5] for a slightly different version).

LEMMA 2. *If $u \in H^1(\mathbb{R}^2)$ and $\alpha > 0$, then*

$$\int_{\mathbb{R}^2} [\exp(\alpha u^2) - 1]\, dz < \infty.$$

*Moreover, if* $\|\nabla u\|_{L^2} \leqslant 1$, $\|u\|_{L^2} \leqslant M$ *and* $\alpha < 4\pi$, *then there exists a constant* $C$, *which depends only on* $\alpha$ *and* $M$, *such that*

$$\int_{\mathbb{R}^2} [\exp(\alpha u^2) - 1] \, dz \leqslant C.$$

The next result concerns the mountain-pass geometry of $J$. Its proof is a consequence of our assumptions $(f_2)$, $(f_3)$, and $(g_3)$ and can be found in [5, 9].

LEMMA 3. *The functional* $J$ *satisfies the following conditions*:

(i) *there exist* $\rho, \sigma > 0$, *such that* $J(u) \geqslant \sigma$ *if* $\|u\| = \rho$,

(ii) *for any nonnegative function* $u \in C_0^\infty(\Omega) \setminus \{0\}$, *we have* $J(tu) \to -\infty$ *as* $t \to +\infty$.

LEMMA 4. *J satisfies the Palais–Smale condition.*

*Proof.* Let $(u_n) \subset H$ be a Palais–Smale sequence of the functional $J$. For $n$ big enough, using $(g_3)$ we have

$$\begin{aligned} c\mu + 1 + \|u_n\| &\geqslant \mu J(u_n) - J'(u_n)\, u_n \\ &= \left(\frac{\mu - 2}{2}\right) \|u_n\|^2 + \int_{\mathbb{R}^2} [u_n g(z, u_n) - \mu G(z, u_n)] \, dz \\ &\geqslant \left(\frac{\mu - 2}{2}\right) \|u_n\|^2 - \mu \int_{\mathbb{R}^2 - \Omega} G(z, u_n) \, dz \\ &\geqslant \left(\frac{\mu - 2}{2}\right) \|u_n\|^2 - \frac{\mu}{2k} \int_{\mathbb{R}^2 - \Omega} V(z)\, u_n^2 \, dz \\ &\geqslant \left[\frac{(\mu - 2)\, k - \mu}{2k}\right] \|u_n\|^2; \end{aligned}$$

thus $\|u_n\|$ is bounded, since $(\mu - 2)\, k > \mu$. Now we can take a subsequence, denote again by $(u_n)$, weakly convergent to some $u \in H$. We are going to prove that this convergence is actually strong. For that matter it suffices to show that, given $\delta > 0$, there is an $R > 0$ such that

$$\limsup_{n \to \infty} \int_{\{|z| \geqslant R\}} [|\nabla u_n|^2 + V(z)\, u_n^2] \, dz < \delta. \tag{4}$$

Consider the test function $\psi_R(z)\,u_n$, where $\psi_R \in C_0^\infty(\mathbb{R}^2, [0,1])$, $\psi_R(z)=0$ if $|z| \leqslant R/2$, $\psi_R(z)=1$ if $|z| \geqslant R$ and $|\nabla\psi_R(z)| \leqslant C/R$ for all $z \in \mathbb{R}^2$. Since $(u_n)$ is bounded, from (??) we obtain

$$\int_{\mathbb{R}^2} [|\nabla u_n|^2 + V(z)\,u_n^2]\,\psi_R\,dz + \int_{\mathbb{R}^2} u_n\,\nabla u_n\,\nabla\psi_R\,dz$$

$$= \int_{\mathbb{R}^2} g(z, u_n)\,\psi_R u_n\,dz + o_n(1).$$

Thus, from property $(g_3)$ for $R>0$ suitably large,

$$\int_{\mathbb{R}^2} [|\nabla u_n|^2 + V(z)\,u_n^2]\,\psi_R\,dz + \int_{\mathbb{R}^2} u_n\nabla u_n\,\nabla\psi_R\,dz$$

$$\leqslant \frac{1}{k}\int_{\mathbb{R}^2} V(z)\,u_n^2\,\psi_R\,dz + o_n(1),$$

which implies that

$$\int_{\{|z| \geqslant R\}} [|\nabla u_n|^2 + V(z)\,u_n^2]\,dz \leqslant \frac{C}{R}\,\|u_n\|_{L^2} \cdot \|\nabla u_n\|_{L^2} + o_n(1)$$

and (4) follows. Thus, the proof of Lemma 4 is complete. ▮

In view of the previous lemmas, applying the mountain-pass theorem (see [17]) we obtain the main result of this section.

THEOREM 5. *For all $\varepsilon > 0$, the functional*

$$J_\varepsilon(u) = \tfrac{1}{2}\int_{\mathbb{R}^2} [\varepsilon^2\,|\nabla u|^2 + V(z)\,u^2]\,dz - \int_{\mathbb{R}^2} G(z,u)\,dz$$

*possesses a nonnegative critical point $u_\varepsilon \in H\backslash\{0\}$ at the level*

$$c_\varepsilon = \inf_{u \in H\backslash\{0\}} \max_{t \geqslant 0} J_\varepsilon(tu). \tag{5}$$

*Remark* 1. (i) This characterization of the mountain-pass level $c_\varepsilon$ given in (5) has been established in [7] and [18] as a consequence of properties of $g(z,s)$.

(ii) Since $g(z,s)=0$ for $s \leqslant 0$ and $J'_\varepsilon(u_\varepsilon)\,\phi = 0$ for all $\phi \in H$, choosing the test function $\phi = u_\varepsilon^- = \max\{-u_\varepsilon, 0\} \in H$, we have that $\|u_\varepsilon^-\| = 0$. Thus, we conclude that $u_\varepsilon$ is a nonnegative function.

## 3. PROOF OF THEOREM 1

First, we scale the spatial variable by setting $z=\varepsilon x$ and let $I_\varepsilon$ denote the energy functional

$$I_\varepsilon(u)=\tfrac{1}{2}\int_{\mathbb{R}^2}[|\nabla u|^2+V(\varepsilon x)\,u^2]\,dx-\int_{\mathbb{R}^2}G(\varepsilon x,u)\,dx$$

defined on the Hilbert space

$$H_\varepsilon=\{u\in H^1(\mathbb{R}^2):\int_{\mathbb{R}^2}V(\varepsilon x)\,u^2<\infty\},$$

$$\langle u,v\rangle\doteq\int_{\mathbb{R}^2}[\nabla u\nabla v+V(\varepsilon x)\,uv]\,dx,$$

associated to the problem

$$-\Delta u+V(\varepsilon x)\,u=g(\varepsilon x,u),\qquad \mathbb{R}^2. \tag{6}$$

Thus, from Theorem 5, $v_\varepsilon(x)=u_\varepsilon(z)$ is a critical point of $I_\varepsilon$ at the level

$$b_\varepsilon=I_\varepsilon(v_\varepsilon)=\inf_{v\in H_\varepsilon\setminus\{0\}}\max_{t\geqslant 0}I_\varepsilon(tv). \tag{7}$$

We suppose, without loss of generality, that $\partial\Omega$ is smooth, $0\in\Omega$, and $V(0)=V_1$.

In order to derive some estimates on the mountain-pass level $b_\varepsilon$ we consider the following autonomous problem

$$-\Delta u+V_1u=f(u),\qquad \mathbb{R}^2. \tag{8}$$

The energy functional corresponding to Eq. (8) is

$$I_1(u)=\tfrac{1}{2}\int_{\mathbb{R}^2}[|\nabla u|^2+V_1u^2]\,dx-\int_{\mathbb{R}^2}F(u)\,dx,\qquad \forall u\in H^1(\mathbb{R}^2).$$

Now, we state the following basic result.

THEOREM 6. *Suppose that the nonlinearity $f$ satisfies* $(f_1)$–$(f_6)$. *Then, problem* (8) *possesses a positive ground state solution $\omega$ at the level*

$$c_1=I_1(\omega)=\inf_{v\in H^1\setminus\{0\}}\max_{t\geqslant 0}I_1(tv)<\frac{4\pi}{\alpha_0}. \tag{9}$$

*Furthermore, $\omega$ is spherically symmetric about some point in $\mathbb{R}^2$ and $\partial\omega/\partial r$ is negative for all $r>0$, where $r$ is the radial coordinate about that point.*

*Proof.* First we observe that the radial symmetry for any solution of problem (8) follows from a result due to Gidas *et al.* (see [11, Theorem 2]).

By the definition of $S_p$ given in (3), there exists $u_\delta \in H^1(\mathbb{R}^2)\backslash\{0\}$ such that

$$S_p + \frac{\delta}{2} > \frac{(\int_{\mathbb{R}^2} (|\nabla u_\delta|^2 + V_1 u_\delta^2)\, dx)^{1/2}}{(\int |u_\delta|^p\, dx)^{1/p}}.$$

Let

$$v_\delta = \left(\frac{4\pi}{\alpha_0}\right)^{1/2} \frac{|u_\delta|}{(\int_{\mathbb{R}^2} (|\nabla u_\delta|^2 + V_1 u_\delta^2)\, dx)^{1/2}},$$

We have

$$\int_{\mathbb{R}^2} (|\nabla v_\delta|^2 + V_1 v_\delta^2)\, dx = \frac{4\pi}{\alpha_0}$$

and

$$S_p + \frac{\delta}{2} > \frac{\left(\int_{\mathbb{R}^2} (|\nabla v_\delta|^2 + V_1 v_\delta^2)\, dx\right)^{1/2}}{\left(\int v_\delta^p\, dx\right)^{1/p}} = \frac{\left(\frac{4\pi}{\alpha_0}\right)^{1/2}}{\left(\int v_\delta^p\, dx\right)^{1/p}},$$

which implies that

$$\int v_\delta^p\, dx > \frac{\left(\frac{4\pi}{\alpha_0}\right)^{p/2}}{\left(S_p + \frac{\delta}{2}\right)^p}.$$

Hence, by $(f_6)$ we have

$$\begin{aligned}\int_{\mathbb{R}^2} F(v_\delta)\, dx &= \int_{\mathbb{R}^2} \int_0^{v_\delta} f(s)\, ds\, dx \\ &\geqslant \left(\frac{1}{2}(S_p+\delta)^p \left(\frac{4\pi}{\alpha_0}\right)^{1-p/2}\right) \int_{\mathbb{R}^2} v_\delta^p\, dx \\ &> \frac{1}{2}\left(\frac{4\pi}{\alpha_0}\right) = \frac{1}{2}\int_{\mathbb{R}^2} (|\nabla v_\delta|^2 + V_1 v_\delta^2)\, dx.\end{aligned}$$

Therefore, we have proved that

$$\int_{\mathbb{R}^2} F(v_\delta)\, dx > \tfrac{1}{2} \int_{\mathbb{R}^2} (|\nabla v_\delta|^2 + V_1 v_\delta^2)\, dx. \tag{10}$$

In order to proceed further we introduce the following manifold:

$$M = \left\{ u \in H^1(\mathbb{R}^2) \backslash \{0\} : \int_{\mathbb{R}^2} F(u)\, dx = \frac{V_1}{2} \int_{\mathbb{R}^2} u^2\, dx \right\}$$

and we consider the constrained problem

$$I_1^0 = \inf \left\{ \int_{\mathbb{R}^2} |\nabla u|^2\, dx : u \in M \right\}. \tag{11}$$

CLAIM 1. *$M \neq \emptyset$ and $0 < I_1^0 \leqslant \int_{\mathbb{R}^2} |\nabla v_\delta|^2 < 4\pi/\alpha_0$.*

*Verification of Claim* 1. From (10) and $(f_2)$ it is easy to see that there exists $\bar{t} \in (0, 1]$ such that $\bar{t} v_\delta \in M$. Thus, $I_1^0 \leqslant \int_{\mathbb{R}^2} |\nabla v_\delta|^2$. Assume for the sake of contradiction that $I_1^0 = 0$; thus there exists a sequence $(u_n) \subset H^1(\mathbb{R}^2) \backslash \{0\}$ such that

$$\int_{\mathbb{R}^2} F(u_n)\, dx = \frac{V_1}{2} \int_{\mathbb{R}^2} u_n^2\, dx \qquad \text{and} \qquad \lim_{n \to \infty} \int_{\mathbb{R}^2} |\nabla u_n|^2\, dx = 0.$$

From our assumptions we have that

$$F(s) \leqslant \frac{V_1}{4} s^2 + cs[\exp(\beta s^2) - 1], \qquad \forall s \in \mathbb{R}.$$

Also, it holds that

$$\int_{\mathbb{R}^2} |u_n| [\exp(\beta u_n^2) - 1]\, dx \leqslant c\, \|\nabla u_n\|_{L^2}\, \|u_n\|_{L^2}^2, \tag{12}$$

since from the Gagliardo–Nirenberg inequality (see [1])

$$\int_{\mathbb{R}^2} |u_n|^{2k+1}\, dx \leqslant C k^k\, \|\nabla u_n\|_{L^2}^{2k-1}\, \|u_n\|_{L^2}^2,$$

which implies

$$\int_{\mathbb{R}^2} |u_n|[\exp(\beta u_n^2)-1]\,dx = \sum_{k=1}^{\infty} \frac{\beta^k}{k!} \int_{\mathbb{R}^2} |u_n|^{2k+1}\,dx$$
$$\leqslant C\,\|\nabla u_n\|_{L^2}\,\|u_n\|_{L^2}^2,$$

because $\sum_{k=1}^{\infty} \frac{(\beta k)^k}{k!}\,\|\nabla u_n\|_{L^2}^{2k-2}$ converges.

Now, using (12), one obtains

$$\frac{V_1}{2}\int_{\mathbb{R}^2} u_n^2\,dx \leqslant \frac{V_1}{4}\int_{\mathbb{R}^2} u_n^2\,dx + c\,\|\nabla u_n\|_{L^2}\,\|u_n\|_{L^2}^2.$$

Thus,

$$\|\nabla u_n\|_{L^2} \geqslant \frac{V_1}{4c} > 0,$$

which is a contradiction and Claim 1 is proved.

CLAIM 2. *$I_1^0$ is achieved.*

*Verification of Claim* 2. Let $(u_n) \subset H^1(\mathbb{R}^2)\backslash\{0\}$ be a minimizing sequence of $I_1^0$; thus

$$\int_{\mathbb{R}^2} F(u_n)\,dx = \frac{V_1}{2}\int_{\mathbb{R}^2} u_n^2\,dx \qquad \text{and} \qquad \lim_{n\to\infty}\int_{\mathbb{R}^2} |\nabla u_n|^2\,dx = I_1^0.$$

Furthermore, taking $\tilde{u}_n(x) = u_n(\sigma_n x)$ where $\sigma_n = \|u_n\|_{L^2}$, we have that

$$\|\tilde{u}_n\|_{L^2} = 1, \qquad \|\nabla \tilde{u}_n\|_{L^2} = \|\nabla u_n\|_{L^2} \qquad \text{and} \qquad \int_{\mathbb{R}^2} F(\tilde{u}_n)\,dx = \frac{V_1}{2}.$$

Thus, up to subsequence, we may assume that

$$\tilde{u}_n \rightharpoonup \tilde{u} \text{ in } H^1(\mathbb{R}^2), \qquad \tilde{u}_n(x) \to \tilde{u}(x) \text{ a.e. in } \mathbb{R}^2, \qquad \text{and} \qquad \|\nabla \tilde{u}_n\|_{L^2}^2 < \frac{4\pi}{\alpha_0}.$$

In what follows, we make use of the limit

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} F(\tilde{u}_n)\,dx = \int_{\mathbb{R}^2} F(\tilde{u})\,dx, \tag{13}$$

which we prove later. From (13), one sees easily that $\tilde{u}$ is nontrivial. Also, we have

$$\frac{V_1}{2}\|\tilde{u}\|_{L^2}^2 \leqslant \liminf_{n\to\infty}\frac{V_1}{2}\|\tilde{u}_n\|_{L^2}^2 = \liminf_{n\to\infty}\int_{\mathbb{R}^2} F(\tilde{u}_n)\,dx = \int_{\mathbb{R}^2} F(\tilde{u})\,dx.$$

Indeed, we have that

$$\frac{V_1}{2}\|\tilde{u}\|_{L^2}^2 = \int_{\mathbb{R}^2} F(\tilde{u})\,dx$$

and therefore $I_1^0$ is achieved by $\tilde{u}$. To prove this fact we argue by contradiction. Thus, assume that

$$\tfrac{1}{2}\|\tilde{u}\|_{L^2}^2 < \int_{\mathbb{R}^2} F(\tilde{u})\,dx.$$

Hence, there exists $t \in (0, 1)$ that such that $t\tilde{u} \in M$. Hence,

$$I_1^0 \leqslant t^2\int_{\mathbb{R}^2} |\nabla\tilde{u}|^2\,dx < \int_{\mathbb{R}^2} |\nabla\tilde{u}|^2\,dx \leqslant \liminf_{n\to\infty}\int_{\mathbb{R}^2} |\nabla\tilde{u}_n|^2\,dx = I_1^0,$$

which is a contradiction.

Now we give the proof of limit in (13). We use the Schwarz symmetrization method. Notice that we may assume that $\tilde{u}_n(x) \geqslant 0$ for all $x \in \mathbb{R}^2$. Since $F(s)$ is an increasing function and $F(0)=0$ then

$$\int_{\{|x|\geqslant R\}} F(\tilde{u}_n)\,dx = \int_{\{|x|\geqslant R\}} F(\tilde{u}_n^*)\,dx,$$

where $\tilde{u}_n^*$ denotes the Schwarz symmetrization of $\tilde{u}_n$.

By our assumptions $(f_2)$–$(f_3)$, given $\eta_1 > 0$, there exist $\rho_1, \rho_2 > 0$ such that

$$F(s) \leqslant \eta_1 s^2, \qquad \forall\ |s| \leqslant \rho_1. \tag{14}$$

and

$$F(s) \leqslant \eta_1[\exp(\beta s^2) - 1], \qquad \forall\ |s| \geqslant \rho_2, \tag{15}$$

where $\beta > \alpha_0$ is a number to be determined. Radial lemma (see [4, Lemma A.II]) leads to

$$|\tilde{u}_n^*(x)| \leqslant C\frac{\|\tilde{u}_n\|_{H^1}}{\sqrt{|x|}} \leqslant C\frac{1}{\sqrt{|x|}}. \tag{16}$$

From (14) and (16) we see easily that for all $\eta > 0$ there exists $R > 0$ such that

$$\int_{\{|x| \geqslant R\}} F(\tilde{u}_n^*)\, dx < \eta. \tag{17}$$

Let $A \subset \{x : |x| < R_1\}$ be a Lebesgue mensurable set; using (14) and (15) we have

$$\int_A F(\tilde{u}_n)\, dx \leqslant \eta_1 \int_{\mathbb{R}^2} |\tilde{u}_n|^2\, dx + \eta_1 \int_{\mathbb{R}^2} [\exp(\beta \tilde{u}_n^2) - 1]\, dx + |A| \sup_{\rho_1 \leqslant |s| \leqslant \rho_2} F(s).$$

We use $|A|$ to denote the Lebesgue measure of a mensurable subset $A$. By Lemma 2, if we choose $\beta > \alpha_0$ sufficiently close to $\alpha_0$, we see that $\int_{\mathbb{R}^2} [\exp(\beta \tilde{u}_n^2) - 1]\, dx$ is bounded, independent of $n$, since $\|\nabla \tilde{u}_n\|_{L^2}^2 < 4\pi/\alpha_0$. So, by this estimate we have

$$\int_A F(\tilde{u}_n)\, dx < \eta, \tag{18}$$

if $|A|$ is suitably small. In view of (17) and (18), applying Vitali's theorem we obtain (13).

Since $I_1^0 > 0$ is achieved, according to the Lagrange multiplier method we have

$$\int_{\mathbb{R}^2} \nabla \tilde{u}\, \nabla \phi\, dx = \lambda \int_{\mathbb{R}^2} [f(\tilde{u}) - V_1 \tilde{u}]\, \phi\, dx, \qquad \forall \phi \in H^1(\mathbb{R}^2).$$

Choosing the test function $\phi = \tilde{u}$ we have that

$$\lambda \int_{\mathbb{R}^2} [f(\tilde{u}) - V_1 \tilde{u}]\, \tilde{u}\, dx = \int_{\mathbb{R}^2} |\nabla \tilde{u}|^2\, dx > 0.$$

Also, it holds that

$$\int_{\mathbb{R}^2} [f(\tilde{u}) - V_1 \tilde{u}]\, \tilde{u}\, dx \geqslant (\mu - 2) \int_{\mathbb{R}^2} F(\tilde{u})\, dx.$$

Thus, $\lambda$ is a positive number. Choosing the test function $\phi = \max\{-\tilde{u}, 0\}$ we conclude that $\tilde{u} \geqslant 0$. By the standard regularity theory of the elliptic equations (see Proposition 8 below), we conclude that $\tilde{u}$ is a classical solution and $\tilde{u}(x) \to 0$ as $|x| \to \infty$. Hence, applying the maximum principle $\tilde{u} > 0$ in $\mathbb{R}^2$.

Let $\omega(x)=\tilde{u}(\lambda^{-1/2}x)$; we get

$$-\Delta\omega+V_1\omega=f(\omega), \qquad \text{in} \quad \mathbb{R}^2.$$

By Pohozaevs identity (see [13]),

$$\int_{\mathbb{R}^2} F(\omega)\,dx=\frac{V_1}{2}\int_{\mathbb{R}^2}\omega^2\,dx.$$

Thus,

$$I_1(\omega)=\tfrac{1}{2}\int_{\mathbb{R}^2}|\nabla\omega|^2\,dx=\tfrac{1}{2}\int_{\mathbb{R}^2}|\nabla\tilde{u}|^2\,dx<\frac{4\pi}{\alpha_0}.$$

Since $I_1'(\omega)\,\omega=0$, we have that $\max_{t\geqslant 0} I_1(t\omega)=I_1(\omega)$. Therefore

$$c_1=\inf_{v\in H^1\backslash\{0\}}\max_{t\geqslant 0} I_1(tv)\leqslant I_1(\omega)=\tfrac{1}{2}\int_{\mathbb{R}^2}|\nabla\omega|^2\,dx=\tfrac{1}{2}\,I_1^0.$$

Indeed, we have that $c_1=I_1^0/2$. Notice that, given $\eta>0$, there is $v\in H^1(\mathbb{R}^2)\backslash\{0\}$ such that

$$c_1\leqslant J(v)=\max_{t\geqslant 0} J(tv)\leqslant c_1+\eta.$$

Also, there exists $t_0>0$ such that $t_0 v\in M$, that is,

$$\int_{\mathbb{R}^2} F(t_0 v)\,dx=\frac{V_1}{2}\int_{\mathbb{R}^2}(t_0 v)^2\,dx.$$

So

$$\tfrac{1}{2}\,I_1^0\leqslant\tfrac{1}{2}\int_{\mathbb{R}^2}|\nabla(t_0 v)|^2\,dx=J(t_0 v)\leqslant J(v)\leqslant c_1+\eta.$$

Thus, Theorem 6 is completely proved.

LEMMA 7. $\limsup_{\varepsilon\to 0} b_\varepsilon\leqslant c_1$.

*Proof.* Let $\omega$ be a ground state solution of problem (8). Without loss of generality we may assume that $\omega$ maximizes at zero. Now consider the test function $\varpi_\varepsilon(x)=\phi(\varepsilon x)\,\omega(x)$, where $\phi\in C_0^\infty(\mathbb{R}^2,[0,1])$, $\phi(x)=1$ if $x\in B_\rho(0)$ and $\phi(x)=0$ if $x\notin B_{2\rho}(0)$. Here we are assuming that $B_{2\rho}(0)\subset\subset\Omega$. It is easy to check that $\varpi_\varepsilon\to\omega$ in $H^1(\mathbb{R}^2)$, $I_1(\varpi_\varepsilon)\to I_1(\omega)$, as $\varepsilon\to 0$, and the support of

$\varpi_\varepsilon$ is contained in $\Omega_\varepsilon = \{x \in \mathbb{R}^2 \colon \varepsilon x \in \Omega\}$. In particular, $\varpi_\varepsilon \in H_\varepsilon$. For each $\varepsilon > 0$ consider $t_\varepsilon \in (0, +\infty)$ such that

$$\max_{t \geqslant 0} I_\varepsilon(t\varpi_\varepsilon) = I_\varepsilon(t_\varepsilon \varpi_\varepsilon).$$

Thus,

$$\begin{aligned} b_\varepsilon &= \inf\nolimits_{v \in H_\varepsilon \setminus \{0\}} \max\nolimits_{t \geqslant 0} I_\varepsilon(tv) \\ &\leqslant \max\nolimits_{t \geqslant 0} I_\varepsilon(t\varpi_\varepsilon) = I_\varepsilon(t_\varepsilon \varpi_\varepsilon) \\ &= \frac{t_\varepsilon^2}{2} \int_{\mathbb{R}^2} [|\nabla \varpi_\varepsilon|^2 + V(\varepsilon x)\, \varpi_\varepsilon^2]\, dx - \int_{\mathbb{R}^2} F(t_\varepsilon \varpi_\varepsilon)\, dx. \end{aligned}$$

Claim. *$t_\varepsilon \to 1$ as $\varepsilon \to 0$.*

*Verification of claim.* Since $I'_\varepsilon(t_\varepsilon \varpi_\varepsilon)(t_\varepsilon \varpi_\varepsilon) = 0$, using assumption $(\mathrm{f}_6)$ we have

$$\begin{aligned} t_\varepsilon^2 \int_{\mathbb{R}^2} [|\nabla \varpi_\varepsilon|^2 + V(\varepsilon x)\, \varpi_\varepsilon^2]\, dx &= \int_{\mathbb{R}^2} f(t_\varepsilon \varpi_\varepsilon)\, t_\varepsilon \varpi_\varepsilon\, dx \\ &\geqslant C t_\varepsilon^p \int_{\mathbb{R}^2} \varpi_\varepsilon^p\, dx. \end{aligned} \tag{19}$$

Since $\|\varpi_\varepsilon\|_{H_\varepsilon} \leqslant C$ and $\varpi_\varepsilon \to \omega > 0$ in $L^p$, from (19) we derive easily that $(t_\varepsilon)$ is bounded. Thus, up to subsequence, we have $t_\varepsilon \to t_1 \geqslant 0$. Indeed, $t_1 > 0$ because $t_\varepsilon^2 \|\varpi_\varepsilon\|_{H_\varepsilon} \geqslant 2b_\varepsilon \geqslant 2\bar{c} > 0$ where $\bar{c}$ is the mountain-pass level of the functional $\bar{I}$ defined as

$$\bar{I}(u) \doteq \tfrac{1}{2} \int_{\mathbb{R}^2} [|\nabla u|^2 + V_0\, |u|^2]\, dx - \int_{\mathbb{R}^2} F(u)\, dx.$$

Passing to the limit in (19), we get

$$\int_{\mathbb{R}^2} [|\nabla \omega|^2 + V_1 \omega^2]\, dx = t_1^{-2} \int_{\mathbb{R}^2} f(t_1 \omega)\, t_1 \omega\, dx. \tag{20}$$

Now, subtracting (20) from

$$\int_{\mathbb{R}^2} [|\nabla \omega|^2 + V_1 \omega^2]\, dx = \int_{\mathbb{R}^2} f(\omega)\, \omega\, dx,$$

we achieve

$$0=\int_{\mathbb{R}^2}\left[\frac{f(t_1\omega)}{(t_1\omega)}-\frac{f(\omega)}{\omega}\right]\omega^2\,dx,$$

which implies that $t_1=1$, because of our assumption ($f_5$). Thus, the proof of the claim is complete.

Notice that we also have that

$$I_\varepsilon(t_\varepsilon\varpi_\varepsilon)=I_1(t_\varepsilon\varpi_\varepsilon)+\frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^2}[V(\varepsilon x)-V_1]|\varpi_\varepsilon|^2\,dx.$$

Thus, taking the limit as $\varepsilon\to 0$ and using the fact that $V(\varepsilon x)$ is bounded on the support of $\varpi_\varepsilon$ and the Lebesgue dominated convergence theorem, we conclude the proof of the lemma. ∎

Now we have $I_\varepsilon(v_\varepsilon)\leqslant c_1+o_\varepsilon(1)$, where $o_\varepsilon(1)$ goes to zero as $\varepsilon\to 0$. Notice that

$$\|v_\varepsilon\|_{H_\varepsilon}^2=\int_{\mathbb{R}^2} g(\varepsilon x, v_\varepsilon)\,v_\varepsilon\,dx$$

and that there exists $\varepsilon_0>0$ such that

$$\frac{\mu}{2}\|v_\varepsilon\|_{H_\varepsilon}^2\leqslant\int_{\mathbb{R}^2}\mu G(\varepsilon x, v_\varepsilon)\,dx+\mu c_1+1,\qquad \forall\varepsilon\in(0,\varepsilon_0),$$

which together with assumption ($g_3$) implies that

$$\begin{aligned}\left(\frac{\mu}{2}-1\right)\|v_\varepsilon\|_{H_\varepsilon}^2 &\leqslant\int_{\mathbb{R}^2-\Omega_\varepsilon}[\mu G(\varepsilon x, v_\varepsilon)-g(\varepsilon x, v_\varepsilon)\,v_\varepsilon]\,dx+\mu c_1+1\\ &\leqslant\int_{\mathbb{R}^2-\Omega_\varepsilon}(\mu-2)\,G(\varepsilon x, v_\varepsilon)\,dx+\mu c_1+1\\ &\leqslant\int_{\mathbb{R}^2-\Omega_\varepsilon}\left(\frac{\mu-2}{2k}\right)V(\varepsilon x)\,v_\varepsilon^2\,dx+\mu c_1+1\\ &\leqslant\left(\frac{\mu-2}{2k}\right)\|v_\varepsilon\|_{H_\varepsilon}^2+\mu c_1+1.\end{aligned}$$

Thus, $\|v_\varepsilon\|_{H_\varepsilon}\leqslant C$, for all $\varepsilon\in(0,\varepsilon_0)$. Of course, we have also that $(v_\varepsilon)_{\{0<\varepsilon\leqslant\varepsilon_0\}}$ is bounded in $H^1(\mathbb{R}^2)$.

The next result is fundamental for our proof of Theorem 1 and concerns the regularity of the family $(v_\varepsilon)$.

PROPOSITION 8. *The functions* $v_\varepsilon$ *belong to* $L^\infty(\mathbb{R}^2)$. *Moreover,* $\|v_\varepsilon\|_{L^\infty} \leqslant C$ *for all* $0 < \varepsilon \leqslant \varepsilon_0$ *and the functions* $v_\varepsilon$ *decay uniformly to zero as* $|x| \to +\infty$.

*Proof.* We set $\sigma_n = s_n + 2 = 2^{n+1}$ and consider the test function $\phi = \psi^2 v_\varepsilon [T_k(v_\varepsilon)]^{s_n}$, where $T_k(v_\varepsilon) = \min\{k, v_\varepsilon\}$ and $\psi \in C_0^\infty(\mathbb{R}^2, [0,1])$. Using that $v_\varepsilon$ is a critical point of $I_\varepsilon$ and our assumptions we find that

$$\int_{\mathbb{R}^2} [\nabla v_\varepsilon \nabla\phi + V(\varepsilon x)\, v_\varepsilon\, \phi]\, dx \leqslant \int_{\mathbb{R}^2} \left[\frac{V_0}{2} v_\varepsilon + C(V_0, \beta)\, v_\varepsilon[\exp(\beta v_\varepsilon^2) - 1]\right] \phi\, dx,$$

which implies that

$$\int_{\mathbb{R}^2} \left[\nabla v_\varepsilon \nabla\phi + \frac{V(\varepsilon x)}{2} v_\varepsilon\, \phi\right] dx \leqslant C \int_{\mathbb{R}^2} v_\varepsilon[\exp(\beta v_\varepsilon^2) - 1]\, \phi\, dx. \tag{21}$$

From (21), it is easy to achieve

$$\begin{aligned} &\int_{\mathbb{R}^2} |\nabla v_\varepsilon|^2\, \psi^2 [T_k(v_\varepsilon)]^{s_n}\, dx + s_n \int_{\mathbb{R}^2} \psi^2 v_\varepsilon [T_k(v_\varepsilon)]^{s_n - 1}\, \nabla v_\varepsilon\, \nabla[T_k(v_\varepsilon)]\, dx \\ &\qquad + 2\int_{\mathbb{R}^2} v_\varepsilon \psi [T_k(v_\varepsilon)]^{s_n}\, \nabla v_\varepsilon\, \nabla\psi\, dx + \int_{\mathbb{R}^2} \frac{V(\varepsilon x)}{2}\, \psi^2\, v_\varepsilon^2 [T_k(v_\varepsilon)]^{s_n}\, dx \\ &\quad \leqslant C \int_{\mathbb{R}^2} \psi^2 v_\varepsilon^2 [T_k(v_\varepsilon)]^{s_n}\, [\exp(\beta v_\varepsilon^2) - 1]\, dx. \end{aligned}$$

Thus,

$$\begin{aligned} &\int_{\mathbb{R}^2} |\nabla v_\varepsilon|^2\, \psi^2 [T_k(v_\varepsilon)]^{s_n}\, dx + s_n \int_{\mathbb{R}^2} \psi^2 [T_k(v_\varepsilon)]^{s_n}\, |\nabla[T_k(v_\varepsilon)]|^2\, dx \\ &\qquad + 2\int_{\mathbb{R}^2} v_\varepsilon \psi [T_k(v_\varepsilon)]^{s_n}\, \nabla v_\varepsilon\, \nabla\psi\, dx + \int_{\mathbb{R}^2} \frac{V(\varepsilon x)}{2}\, \psi^2 v_\varepsilon^2 [T_k(v_\varepsilon)]^{s_n}\, dx \\ &\quad \leqslant C \int_{\mathbb{R}^2} \psi^2 v_\varepsilon^2 [T_k(v_\varepsilon)]^{s_n}\, [\exp(\beta v_\varepsilon^2) - 1]\, dx. \end{aligned}$$

By Young's inequality, it follows that

$$\begin{aligned} &\int_{\mathbb{R}^2} v_\varepsilon \psi [T_k(v_\varepsilon)]^{s_n}\, \nabla v_\varepsilon \nabla\psi\, dx \\ &\qquad \leqslant \frac{\delta_2}{2} \int_{\mathbb{R}^2} \psi^2 [T_k(v_\varepsilon)]^{s_n}\, |\nabla v_\varepsilon|^2\, dx + \frac{1}{2\delta^2} \int_{\mathbb{R}^2} v_\varepsilon^2 [T_k(v_\varepsilon)]^{s_n}\, |\nabla\psi|^2\, dx. \end{aligned}$$

Now, using the Gagliardo–Nirenberg inequality (see [13, Proposition 8.12]),

$$\|u\|_{L^4}^2 \leqslant C\,\|u\|_{L^2}\,\|\nabla u\|_{L^2} \leqslant \frac{C}{2}\,(\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2),$$

we obtain

$$\begin{aligned}
&\|\psi v_\varepsilon[T_k(v_\varepsilon)]^{s_n/2}\|_{L^4}^2\\
&\quad\leqslant \frac{C}{2}\,\{\|\psi v_\varepsilon[T_k(v_\varepsilon)]^{s_n/2}\|_{L^2}^2 + \|\nabla\{\psi v_\varepsilon[T_k(v_\varepsilon)]^{s_n/2}\}\|_{L^2}^2\}\\
&\quad\leqslant \frac{C}{2}\left\{\int_{\mathbb{R}^2}\psi^2 v_\varepsilon^2[T_k(v_\varepsilon)]^{s_n}\,dx + 4\int_{\mathbb{R}^2}|\nabla v_\varepsilon|^2\,\psi^2[T_k(v_\varepsilon)]^{s_n}\,dx\right.\\
&\qquad + 4\int_{\mathbb{R}^2}|\nabla\psi|^2\,v_\varepsilon^2[T_k(v_\varepsilon)]^{s_n}\,dx\\
&\qquad\left. + 2s_n\int_{\mathbb{R}^2}\psi^2 v_\varepsilon^2[T_k(v_\varepsilon)]^{s_n-2}\,|\nabla[T_k(v_\varepsilon)]|^2\,dx\right\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\|\psi v_\varepsilon[T_k(v_\varepsilon)]^{s_n/2}\|_{L^4}^2\\
&\quad\leqslant \frac{C}{2}\left\{\int_{\mathbb{R}^2}\psi^2 v_\varepsilon^2[T_k(v_\varepsilon)]^{s_n}\,dx + 4\int_{\mathbb{R}^2}|\nabla v_\varepsilon|^2\,\psi^2[T_k(v_\varepsilon)]^{s_n}\,dx\right.\\
&\qquad + 4\int_{\mathbb{R}^2}|\nabla\psi|^2\,v_\varepsilon^2[T_k(v_\varepsilon)]^{s_n}\,dx\\
&\qquad\left. + 2s_n\int_{\mathbb{R}^2}\psi^2[T_k(v_\varepsilon)]^{s_n}\,|\nabla[T_k(v_\varepsilon)]|^2\,dx\right\}.
\end{aligned}$$

These estimates imply that

$$\begin{aligned}
\|\psi v_\varepsilon[T_k(v_\varepsilon)]^{s_n/2}\|_{L^4}^2 \leqslant C&\left\{\int_{\mathbb{R}^2}|\nabla\psi|^2\,v_\varepsilon^2[T_k(v_\varepsilon)]^{s_n}\,dx\right.\\
&\left.+\int_{\mathbb{R}^2}\psi^2 v_\varepsilon^2[T_k(v_\varepsilon)]^{s_n}\,[\exp(\beta v_\varepsilon^2)-1]\,dx\right\}. \qquad (22)
\end{aligned}$$

Notice that, by the radial lemma, we can choose $\rho$ suitably large such that

$$\left\{\int_{|x|\geqslant\rho/2}[\exp(\beta v_\varepsilon^2)-1]^2\,dx\right\}^{1/2}\leqslant 1/2C.$$

Consider $\psi \in C_0^\infty(\mathbb{R}^2, [0,1])$ such that $\psi \equiv 1$ if $|x| \geqslant \rho \geqslant 4$, $\psi \equiv 0$ if $|x| \leqslant \rho - 2$ and $|\nabla\psi| \leqslant 1$ and hence, by Hölders inequality,

$$\int_{\mathbb{R}^2} \psi^2 v_\varepsilon^2 [T_k(v_\varepsilon)]^{s_n} [\exp(\beta v_\varepsilon^2) - 1]\, dx \leqslant \frac{1}{2C} \|\psi v_\varepsilon [T_k(v_\varepsilon)]^{s_n/2}\|_{L^4}^2. \quad (23)$$

From (22) and (23) we find

$$\begin{aligned} \|v_\varepsilon [T_k(v_\varepsilon)]^{s_n/2}\|_{L^4(|x| \geqslant \rho)}^2 &\leqslant \|\psi v_\varepsilon [T_k(v_\varepsilon)]^{s_n/2}\|_{L^4}^2 \\ &\leqslant C \int_{\mathbb{R}^2} |\nabla\psi|^2\, v_\varepsilon^2 [T_k(v_\varepsilon)]^{s_n}\, dx \\ &\leqslant C \int_{|x| \geqslant \rho/2} v_\varepsilon^{s_n+2}\, dx. \end{aligned}$$

Thus, letting $k \to +\infty$, by the dominated convergence theorem,

$$\|v_\varepsilon\|_{L^{\sigma_{n+1}}(|x| \geqslant \rho)} \leqslant C^{1/\sigma_n} \|v_\varepsilon\|_{L^{\sigma_n}(|x| \geqslant \rho/2)}. \quad (24)$$

We can use the same argument taking $\psi \in C_0^\infty(B_{2\rho'}(x_0), [0,1])$ such that $\psi \equiv 1$ if $|x_0 - x| \leqslant \rho'$ and $|\nabla\psi| \leqslant 2/\rho'$ to prove that

$$\|v_\varepsilon\|_{L^{\sigma_{n+1}}(B_{\rho'}(x_0))} \leqslant C^{1/\sigma_n} \|v_\varepsilon\|_{L^{\sigma_n}(B_{2\rho'}(x_0))}. \quad (25)$$

Therefore, from (24) and (25), by a standard covering argument, we can show that

$$\|v_\varepsilon\|_{L^{\sigma_{n+1}}} \leqslant C^{1/\sigma_n} \|v_\varepsilon\|_{L^{\sigma_n}}.$$

Iteration yields

$$\|v_\varepsilon\|_{L^{\sigma_{n+1}}} \leqslant C^{\sum 1/\sigma_n} \gamma^{\sum n - 1/\sigma_n} \|v_\varepsilon\|_{L^{\sigma_1}}, \qquad \forall \varepsilon \in (0, \varepsilon_0),$$

where $C$ is independent of $n$, since both series are convergent. Finally, letting $n \to \infty$ and observing that $\|u\|_\infty \leqslant \lim_{n\to\infty} \|u\|_{L^{\sigma_n}}$, we deduce easily that $v_\varepsilon \in L^\infty(\mathbb{R}^2)$ and in addition that

$$\|v_\varepsilon\|_\infty \leqslant C, \qquad \text{for all} \quad 0 < \varepsilon < \varepsilon_0. \quad (26)$$

From (21) and (26), it is easy to see that for all nonnegative $\phi \in C_0^\infty(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} \nabla v_\varepsilon\, \nabla\phi\, dx \leqslant C \int_{\mathbb{R}^2} v_\varepsilon \phi\, dx.$$

Also, it is known that $H^1(\mathbb{R}^2) \hookrightarrow L^s(\mathbb{R}^2)$ for all $s \geqslant 2$. By standard regularity result [12, Theorem 8.17], for any ball $B_r(x)$ of radius $r$ centered at any $x \in \mathbb{R}^2$,

$$\sup_{y \in B_r(x)} v_\varepsilon(y) \leqslant C\{ \|v_\varepsilon\|_{L^2(B_{2r}(x))} + \|v_\varepsilon\|_{L^4(B_{2r}(x))}\}, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

Thus, the uniform vanishing of the family $(v_\varepsilon)_{\{0<\varepsilon\leqslant\varepsilon_0\}}$ is implied. ∎

Lemma 9. *If the family* $(y_\varepsilon)_{\{0<\varepsilon\leqslant\varepsilon_0\}} \subset \mathbb{R}^2$ *is such that* $\varepsilon y_\varepsilon \in \Omega$ *and* $v_\varepsilon(y_\varepsilon) \geqslant \eta_0 > 0$, *for all* $\varepsilon \in (0, \varepsilon_0)$. *Then*

$$\lim_{\varepsilon \to 0} V(\varepsilon y_\varepsilon) = V_1.$$

*Furthermore,* $\omega_\varepsilon(x) \doteq v_\varepsilon(x + y_\varepsilon)$ *converges uniformly over compacts to the* $\omega$ *solution of problem* (8).

*Proof.* Let us take a sequence $\varepsilon_n \searrow 0$ and $y_n \in \mathbb{R}^2$ such that $\varepsilon_n y_n \in \Omega$ and $v_{\varepsilon_n}(y_n) = u_{\varepsilon_n}(\varepsilon_n y_n) \geqslant \eta_0 > 0$. Since $\varepsilon_n\, y_n \in \bar{\Omega}$, up to subsequence, we have $\varepsilon_n y_n \to x_0 \in \bar{\Omega}$. Set $v_n = v_{\varepsilon_n}$ and $\omega_n(x) = v_n(x + y_n)$. Thus, for all $\phi \in C_0^\infty(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} [\nabla\omega_n \nabla\phi + V(\varepsilon_n x + \varepsilon_n y_n)\, \omega_n \phi]\, dx = \int_{\mathbb{R}^2} g(\varepsilon_n x + \varepsilon_n y_n, \omega_n)\, \phi\, dx. \quad (27)$$

Since $\|\omega_n\|_{H^1} = \|v_n\|_{H^1}$ is bounded, up to subsequence, we may assume that there is $\omega \in H^1(\mathbb{R}^2)$ such that

$$\omega_n \rightharpoonup \omega \text{ in } H^1(\mathbb{R}^2) \text{ and } \omega_n(x) \to \omega(x) \qquad \text{a.e. in } \mathbb{R}^2.$$

We set

$$\tilde{g}(x, \omega) = \chi(x) f(\omega) + (1 - \chi(x))\, \tilde{f}(\omega)$$

and

$$\chi(x) = \lim_{n \to \infty} \chi_\Omega(\varepsilon_n\, x + \varepsilon_n y_n) \qquad \text{a.e. in } \mathbb{R}^2.$$

Using similar arguments as for Lemma 2.1 in [6], we can prove that

$$\lim_{n \to \infty} \int_{\mathbb{R}^2} g(\varepsilon_n x + \varepsilon_n y_n, \omega_n)\, \phi\, dx = \int_{\mathbb{R}^2} \tilde{g}(x, \omega)\, \phi\, dx, \qquad \forall \phi \in C_0^\infty(\mathbb{R}^2). \quad (28)$$

Now taking the limit in (27) we achieve that $\omega$ satisfies

$$\int_{\mathbb{R}^2} [\nabla\omega\,\nabla\phi + V(x_0)\,\omega\phi]\,dx = \int_{\mathbb{R}^2} g(x,\omega)\,\phi\,dx, \qquad \forall\phi \in C_0^\infty(\mathbb{R}^2).$$

Thus, $\omega$ is a critical point of the energy functional

$$\tilde{I}(\omega) = \tfrac{1}{2}\int_{\mathbb{R}^N} [|\nabla\omega|^2 + V(x_0)\,\omega^2]\,dx - \int_{\mathbb{R}^N} \tilde{G}(x,\omega)\,dx,$$

where $\tilde{G}$ is the primite of $\tilde{g}$. Notice that in the case that $x_0 \in \Omega$ we have $\varepsilon_n x + \varepsilon_n y_n \in \Omega$ for $n$ sufficiently large. Hence, $\chi(x) = 1$ for all $x \in \mathbb{R}^N$, and so $\omega$ is a critical point of the energy functional

$$I_{x_0}(\omega) = \tfrac{1}{2}\int_{\mathbb{R}^N} [|\nabla\omega|^2 + V(x_0)\,\omega^2]\,dx - \int_{\mathbb{R}^N} F(\omega)\,dx.$$

On the other hand, if $x_0 \in \partial\Omega$, without loss of generality we may suppose that the outer normal vector $v$ in $x_0$ is $(1,0)$. Let $P = \{x \in \mathbb{R}^N : x_1 < 0\}$. Notice that $\chi \equiv 1$ on $P$, since for each $x \in P$, we have that $\varepsilon_n x + \varepsilon_n y_n \in \Omega$, for $n$ sufficiently large, because $\varepsilon_n y_n \in \Omega$. Thus, in both cases $\tilde{g}(x,s) = f(s)$, for all $x \in P$. This implies that the mountain-pass level $\tilde{c}$ associated to the functional $\tilde{I}$ is identical to the mountain-pass level $c_{x_0}$ associated to the functional $I_{x_0}$. Indeed, from $\tilde{G}(x,s) \leqslant F(s)$, we have $I_{x_0}(u) \leqslant \tilde{I}(u)$, for all $u \in H^1(\mathbb{R}^2)$ and then $c_{x_0} \leqslant \tilde{c}$. On the other hand, $I_{x_0}(u) = \tilde{I}(u)$ for all $u$ with support contained in $P$.

Also, the dependence of the mountain-pass level $c_1$ (as defined in (9)) on the constant potential $V_1$ is continuous and increasing (for details see [18]). Hence, using Fatou's lemma and Lemma 7, we get

$$\begin{aligned}
2c_1 \leqslant 2\tilde{I}(\omega) &= \int_{\mathbb{R}^2} [\tilde{\omega} g(x,\omega) - 2\tilde{G}(x,\omega)]\,dx \\
&\leqslant \liminf_{n\to\infty} \left\{ \int_{\mathbb{R}^2} [\omega_n g(\varepsilon_n x + \varepsilon_n y_n, \omega_n) - 2G(\varepsilon_n x + \varepsilon_n y_n, \omega_n)]\,dx \right\} \\
&= \liminf_{n\to\infty} \left\{ \int_{\mathbb{R}^2} [v_n g(\varepsilon_n x, v_n) - 2G(\varepsilon_n x, v_n)]\,dx \right\} \\
&= \liminf_{n\to\infty} \left\{ 2I_{\varepsilon_n}(v_{\varepsilon_n}) - I'_{\varepsilon_n}(v_{\varepsilon_n})\,v_{\varepsilon_n} \right\} \leqslant 2c_1.
\end{aligned}$$

Thus, $\lim_{\varepsilon\to 0} b_\varepsilon = c_1$. Furthermore, if $V(x_0) > V_1$ we have $c_1 < \tilde{c} \leqslant \tilde{I}(\omega) = c_0$, which is a contradiction; then $V(x_0) = V_1$.

We conclude also from what we have proved that $\omega_\varepsilon \rightharpoonup \omega$ in $H^1(\mathbb{R}^2)$, where $\omega$ is a solution of problem (8). From this fact, together with elliptic estimates (see Proposition 8), we conclude the second part of this lemma. ∎

From Proposition 8, we conclude that there exists a $\rho > 0$ such that $\omega_\varepsilon(x) \leqslant a$ for all $|x| \geqslant \rho$. Also, we can choose $\varepsilon_0 > 0$ suitably small such that $B_\rho(0) \subset \Omega_{\varepsilon_0}$. Therefore, for all $\varepsilon \in (0, \varepsilon_0)$,

$$-\Delta\omega_\varepsilon + V(\varepsilon x + \varepsilon y_\varepsilon)\,\omega_\varepsilon = f(\omega_\varepsilon), \qquad \text{in} \quad \mathbb{R}^2.$$

Thus, there is $\varepsilon_0 > 0$ such that when $0 < \varepsilon < \varepsilon_0$, problem $(P_\varepsilon)$ possesses a positive bound state solution.

Taking translations, if necessary, we may assume that $\omega_\varepsilon$ achieved its global maximum at the origin of $\mathbb{R}^2$. Now, by the fact that $\omega_\varepsilon$ converges uniformly over compacts to $\omega$ together with Lemma 4.2 in [15], we conclude that $\omega_\varepsilon$ possesses no critical point other than the origin for all $\varepsilon \in (0, \varepsilon_0)$.

We note that the maximum value of $u_\varepsilon(\varepsilon x) = v_\varepsilon(x)$ is achieved at a point $z_\varepsilon = \varepsilon x_\varepsilon \in \Omega$ and it is away from zero. Thus, the second item in Theorem 1 is a consequence of Lemma 9.

Finally, we are going to prove the exponential decay of the solutions. Since the functions $\omega_\varepsilon$ decay uniformly to zero as $|x| \to +\infty$, we can choose $R_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$,

$$f(w_\varepsilon(x)) \leqslant \frac{V_1}{2}\, w_\varepsilon\,(x), \qquad \forall\, |x| \geqslant R_0. \tag{29}$$

Set $\phi(x) = M \exp(-\zeta\,|x|)$ where $\zeta$ and $M$ are such that $2\zeta^2 < V_1$ and $M \exp(-\zeta R_0) \geqslant \omega_\varepsilon(x)$, for all $|x| = R_0$. It is easy to see that

$$\Delta\phi \leqslant \zeta^2\phi, \qquad \forall x \neq 0. \tag{30}$$

Also, from (29) and (30) we see that the function $\phi_\varepsilon = \phi - \omega_\varepsilon$ satisfies

$$-\Delta\phi_\varepsilon + \frac{V_1}{2}\,\phi_\varepsilon \geqslant 0 \qquad \text{in} \quad |x| \geqslant R_0,$$

$$\phi_\varepsilon \geqslant 0 \qquad \text{in} \quad |x| = R_0,$$

$$\lim_{|x| \to \infty} \phi_\varepsilon(x) = 0.$$

By the maximum principle, we have that $\phi_\varepsilon(x) \geqslant 0$ for all $|x| \geqslant R_0$. Hence, $\omega_\varepsilon(x) \leqslant M \exp(-\zeta\,|x|)$ for all $|x| \geqslant R_0$ and $\varepsilon \in (0, \varepsilon_0)$. This estimate implies easily that the last item of Theorem 1 holds.

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