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Nontrivial solutions for a class of semilinear biharmonic problems involving critical exponents☆

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1. Introduction

In this paper we will discuss the existence of a nontrivial solution of the following class of semilinear biharmonic problem involving critical exponents

$$\Delta^2 u + a(x)u = h(x)|u|^{q-1}u + k(x)|u|^{p-1}u \quad \text{in } \mathbb{R}^N,$$
$$u \in H^2(\mathbb{R}^N), \quad N \geq 5, \tag{P}$$

where $1 < q < p \leq 2^{**}-1 = (N+4)/(N-4)$ and $a, h, k : \mathbb{R}^N \to \mathbb{R}$ are bounded, non-negative and continuous functions.

We recall that semilinear and quasilinear problems for second order equations have been extensively studied in the last years. For Laplace operator and subcritical case, they have been studied for instance by Rabinowitz [30], Coti-Zelati and Rabinowitz [14], Kryszewski and Szulkin [21], Alama and Li [1] when a, h and k are 1-periodic, while in the paper [25] Montecchiari has dropped the periodicity condition on a, h and

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k and proved the existence of an uncountable set of bounded solutions provided that the functions a, h and k are small perturbations of the 1-periodic functions in the sense that there exist periodic continuous function a_∞, h_∞ and k_∞ and a set $A \subset \mathbb{R}^N$ "large at infinity", that is, $\forall R > 0$, $\exists x \in A$ such that $B_R(x) = \{y \in \mathbb{R}^N \colon \|y - x\| < R\} \subset A$, and $\forall \varepsilon > 0$, $\exists R > 0$ such that

$$\max\left\{\sup_{A\setminus B_R(0)} |a - a_\infty|, \sup_{A\setminus B_R(0)} |h - h_\infty|, \sup_{A\setminus B_R(0)} |k - k_\infty|\right\} < \varepsilon.$$

On the other hand, Brezis and Nirenberg [8] have been the first to study semilinear elliptic problem in bounded domains involving critical exponents. After this paper, many authors have considered the problem above for Laplacian or p-Laplacian operators, exploring either the approach used by [8] or a combination of this technique with the concentration compactness principle of Lions ([22,23]), among others, we would like to mention Garcia and Peral [18], Ambrosetti, Garcia and Peral [4], Ambrosetti and Struwe [3], Capozzi et al. [10] and references therein when domain is bounded, and Noussair et al. [27], Benci and Cerami [5], Cao et al. [9], Pan [28], Ben-Naoum et al. [6] and references therein in whole space $\mathbb{R}^N$. Still in the critical case, Jianfu and Xiping [19] have treated a quasilinear elliptic problem where the function a satisfies the following condition:

there exist $a_\infty > 0$ such that

$$a(x) \to a_\infty, \quad \text{as } |x| \to \infty, \ a(x) \lneqq a_\infty, \tag{a$_0$}$$

where the last inequality is strict on a subset of positive measure in $\mathbb{R}^N$. Recently, motived by the paper [19], Alves et al. [2] studied this kind of problems for the Laplacian operator with $h = \lambda$ and $k = 1$, as a perturbation of the 1-periodic problem.

For the biharmonic operators, in the case where $a = 0$ and $p = 2^{**}-1$, adapting a compactness result due to Egnell [17], problem (P) was studied by Noussair et al. [26]. For other results involving biharmonic operators with critical growth, we refer to van der Vorst [31], Bernis et al. [7], Edmunds et al. [16], Pucci and Serrin [29] and references therein.

Here, we basically extend the results in [2] for the biharmonic operators, but different from the case $k = 1$, the coefficient k brings another difficulty with respect to the lack of compactness, but this was overcame by using a version of the concentration compactness principle for the biharmonic operator.

First of all we are going to prove that the 1-periodic problem (P) possesses a nontrivial solution assuming that the functions a, h and k satisfy the following hypothesis:

$$a(x + p) = a(x), \quad h(x + p) = h(x), \quad k(x + p) = k(x), \ x \in \mathbb{R}^N, \ p \in \mathbb{Z}^N, \tag{h1}$$

there exists $a_0 > 0$ such that

$$a(x) \geq a_0 > 0, \quad x \in \mathbb{R}^N, \tag{h2}$$

$$h(x) \geq 0, \quad \inf_{x \in B(0,\delta)} h = \eta_0 > 0, \quad \delta > 0 \tag{h3}$$

and

$$k(x) > 0, \quad k(x) = |k|_{L^\infty} + O(|x|^4) \quad \text{near at the origin.} \tag{h4}$$

Then, we shall prove the existence of a nontrivial solution for the nonperiodic problem

$$\Delta^2 u + A(x)u = H(x)|u|^{q-1}u + K(x)|u|^{2^{**}-1}u \quad \text{in } \mathbb{R}^N,$$

$$u \in H^2(\mathbb{R}^N), \; N \geq 5, \tag{P$_\#$}$$

when the functions $A, H, K : \mathbb{R}^N \to \mathbb{R}$ are small perturbations of the 1-periodic functions a, h and k, respectively. More precisely, A, H, K are nonnegative and continuous functions verifying the following assumptions:
there exists $A_0 > 0$ such that

$$A(x) = a(x) - W(x) \geq A_0 \quad \text{with} \quad W(x) \geq 0, \; x \in \mathbb{R}^N, \tag{h5}$$

$$H(x) - h(x) \to 0 \quad \text{as } |x| \to \infty,$$

$$H(x) \geq h(x), \quad x \in \mathbb{R}^N, \tag{h6}$$

$$K(x) - k(x) \to 0 \quad \text{as } |x| \to \infty,$$

$$K(x) \geq k(x) \quad K(x) = |K|_{L^\infty} + \mathrm{O}(|x|^4) \quad \text{near at the origin,} \tag{h7}$$

where at least one of the inequalities above is strict on a subset of positive measure in $\mathbb{R}^N$.

We are now ready to state our main result.

Theorem 1.1. *Assume that* $W \in L^{N/4}(\mathbb{R}^N)$, $p = 2^{**} - 1$, (h1)–(h7) *hold. In addition, suppose either* $N \geq 8$ *and* $q > 1$ *or* $5 \leq N \leq 7$ *and* $p - 2 < q < p$. *Then, problem* (P$_\#$) *has a nontrivial solution* $u \in H^2(\mathbb{R}^N)$. *Problem* (P$_\#$) *still possesses a nontrivial solution* $u \in H^2(\mathbb{R}^N)$ *provided* $5 \leq N \leq 7$, $1 < q \leq p - 2$ *and* η_0 *is sufficiently large.*

2. Preliminary results

Let E denote the Sobolev space $H^2(\mathbb{R}^N)$ endowed with norm

$$\|u\|^2 = \int_{\mathbb{R}^N} (|\Delta u|^2 + au^2),$$

which is equivalent to usual norm on E. Also $|u|_q$ means the norm in $L^q(\mathbb{R}^N)$.

We shall state the following auxiliary result.

Theorem 2.1. *Assume* $p = 2^{**} - 1$, (h1)–(h4). *Suppose either*

$$N \geq 8 \text{ and } q > 1 \text{ or } 5 \leq N \leq 7 \text{ and } p - 2 < q < p. \tag{2.1}$$

Then, problem (P) *has a nontrivial solution* $u \in H^2(\mathbb{R}^N)$. *Problem* (P) *still possesses a nontrivial solution* $u \in H^2(\mathbb{R}^N)$ *provided*

$$5 \leq N \leq 7, \quad 1 < q \leq p - 2 \quad \text{and} \quad \eta_0 \text{ is sufficiently large.} \tag{2.2}$$

Remark 1. Even this result it extends or complements those of some papers above mentioned in the sense that here the nonlinearity involves the Sobolev's critical exponent.

3. Proof of Theorem 2.1

Consider the C^1 functional I on E given by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\Delta u|^2 + au^2 - \frac{1}{q+1}\int_{\mathbb{R}^N} h|u|^{q+1} - \frac{1}{2^{**}}\int_{\mathbb{R}^N} k|u|^{2^{**}}$$

with Fréchet derivative

$$I'(u)v = \int_{\mathbb{R}^N} (\Delta u \Delta v + auv) - \int_{\mathbb{R}^N} h|u|^{q-1}uv - \int_{\mathbb{R}^N} k|u|^{2^{**}-2}uv.$$

First of all, we recall that it is standard to prove that I verifies the following.

Lemma 3.1 (Mountain Pass Geometry). *I satisfies the following conditions*
(i) *There exist* $\rho,\ \beta > 0$, *such that* $I(u) \geq \beta,\ \|u\| = \rho$.
(ii) *There exist* $e \in E,\ \|e\| > \rho$ *such that* $I(e) \leq 0$.

From the Lemma above, by using the Ambrosetti–Rabinowitz Mountain Pass Theorem without $(\mathrm{PS})_c$ condition (see [24]), it follows that there exists a $(\mathrm{PS})_c$ sequence $\{u_n\}$, that is, $\{u_n\} \subset E$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$, where c is characterized by

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t))$$

and

$$\Gamma = \{\gamma \in C([0,1],E)\colon I(\gamma(0)) = 0 \text{ and } I(\gamma(1)) = e\}.$$

Since (2.1)–(2.2) hold, applying the same argument used in [8] (see also [11]), we conclude that the level c defined above verifies the inequalities

$$0 < c < \frac{2}{N}|k|_{\infty}^{(4-N)/4} S^{N/4},$$

where S is the best Sobolev constant defined by

$$S = \inf\left\{\int_{\mathbb{R}^N} |\Delta u|^2\colon u \in D^{2,2}(\mathbb{R}^N),\ |u|_{L^{2^{**}}} = 1\right\}.$$

We remark that the infimum above is achieved by functions

$$u_\varepsilon(x) = C_N\left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{(N-4)/2}, \quad \varepsilon > 0,$$

where C_N is chosen so that u_ε satisfies the equation

$$\Delta^2 u = u^{2^{**}-1} \quad \text{in } \mathbb{R}^N,\ N \geq 5,$$

see (e.g. [26] or [16]).

Also, we will need the following lemma that gives a behavior of $(\mathrm{PS})_c$ sequence.

Lemma 3.2. *Let $\{u_n\}$ be a $(\mathrm{PS})_c$ sequence such that $u_n \rightharpoonup 0$ weakly in E, with $c < 2/N|k|_\infty^{(4-N)/4} S^{N/4}$. Then, $k^{1/2^{**}} u_n \to 0$ in $L^{2^{**}}_{\mathrm{loc}}(\mathbb{R}^N)$, in addition the sequence $\{u_n\}$ verifies either*
(a) *$u_n \to 0$ strongly in E or*
(b) *There exist ρ, $\eta > 0$ and $\{y_n\} \in \mathbb{R}^N$ such that*

$$\limsup_{n\to\infty} \int_{B_\rho(y_n)} |u_n|^2 \geq \eta,$$

where $B_r(y)$ denotes the ball in $\mathbb{R}^N$ of center at y and radius r.

Proof. Since $u_n \rightharpoonup 0$ weakly in E, we may suppose that

$$k|u_n|^{2^{**}} \rightharpoonup v \quad \text{and} \quad |\Delta u_n|^2 \rightharpoonup \mu \quad (\text{weak}^*\text{-sense of measure.})$$

Using a version of the concentration compactness principle due to Lions [23] (see e.g. [31] for a bounded domains version, also [12]), there exist an at most countable index set J, distinct points $\{x_j\}_{j\in J} \subset \mathbb{R}^N$, and nonnegative weights $\{\mu_j, v_j\}_{j\in J}$ such that

$$v = \sum_{j\in J} v_j \delta_{x_j} \quad \text{and} \quad \mu \geq \sum_{j\in J} \mu_j \delta_{x_j},$$

where δ_{x_i} is the Dirac mass at $x_i \in \mathbb{R}^N$. Furthermore

$$S\left(\frac{v_i}{|k|_\infty}\right)^{(N-4)/N} \leq \mu_i.$$

It can be proved (see [18, p. 881]) that

$$v_i \geq |k|_\infty^{(4-N)/4} S^{N/4} \text{ or } v_i = 0.$$

Then

$$J = \Phi \text{ (empty set) or finite.}$$

In our case $J = \Phi$. In fact, suppose on the contrary that $J \neq \Phi$. Then

$$\begin{aligned} c + o_n(1) &= I(u_n) - \tfrac{1}{2} I'(u_n)u_n \\ &= \left(\frac{1}{2}\frac{1}{q+1}\right) \int_{\mathbb{R}^N} h|u_n|^{q+1} + \frac{2}{N} \int_{\mathbb{R}^N} k|u_n|^{2^{**}} \\ &\geq \frac{2}{N} \int_{\mathbb{R}^N} k|u_n|^{2^{**}} \\ &\geq \frac{2}{N} \int_{\mathbb{R}^N} \Psi k|u_n|^{2^{**}}, \end{aligned}$$

where Ψ is a cut-off function, $\Psi \in C_c^\infty(\mathbb{R}^N)$, $0 \leq \Psi \leq 1$, $\Psi \equiv 1$ on $B_\varepsilon(x_i)$, $\Psi \equiv 0$ on $\mathbb{R}^N \backslash B_{2\varepsilon}(x_i)$, $|\nabla \Psi| \leq 2/\varepsilon$ and $|\Delta\Psi| \leq 2/\varepsilon^2$.

Passing to the limit in the inequality above, we obtain

$$c \geq \frac{2}{N} \sum_{i \in J} \Psi(x_i) v_i = \frac{2}{N} \sum_{i \in J} v_i \geq \frac{2}{N} |k|_{\infty}^{(4-N)/4} S^{N/4},$$

which is a contradiction. Hence, since $J = \Phi$, we have

$$\int_{\mathbb{R}^N} k|u_n|^{2^{**}} \Psi \to 0, \quad \forall \Psi \in C_c^{\infty}(\mathbb{R}^N),$$

that is,

$$k^{1/2^{**}} u_n \to 0 \quad \text{in } L_{\text{loc}}^{2^{**}}(\mathbb{R}^N).$$

This proves the first part. Suppose that (b) does not hold. Using Lion's result ([22, Lemma 1.1] or [14, Lemma 2.18]) follows:

$$\int_{\mathbb{R}^N} h|u_n|^{q+1} \to 0.$$

From $I'(u_n)u_n = o_n(1)$, we conclude

$$\|u_n\|^2 = \int_{\mathbb{R}^N} k|u_n|^{2^{**}} + o_n(1).$$

Assume that $\|u_n\|^2 \to l (l > 0)$. Since $I(u_n) \to c$ and $\int_{\mathbb{R}^N} k|u_n|^{2^{**}} \to l$, it follows that $c = 2/Nl$.

Arguing as in [8] and observing that

$$||u_n||^2 \geq \int_{\mathbb{R}^N} |\Delta u_n|^2 \geq S \left(\int_{\mathbb{R}^N} |u_n|^{2^{**}} \right)^{2/2^{**}}$$

$$\geq \frac{S}{|k|_{\infty}^{2/2^{**}}} \left(\int_{\mathbb{R}^N} k|u_n|^{2^{**}} \right)^{2/2^{**}},$$

we get that $c \geq 2/N|k|_{\infty}^{(4-N)/4} S^{N/4}$, which is a contradiction, then (a) holds. □

Let be $\{u_n\}$ a $(\text{PS})_c$ sequence which was obtained above such that

$$0 < c < \frac{2}{N} |k|_{\infty}^{(4-N)/4} S^{N/4}.$$

Using the inequality,

$$I(u_n) - \frac{1}{q+1} I'(u_n)u_n \leq M + \|u_n\|, \quad n \text{ sufficiently large},$$

we conclude that $\{u_n\}$ is bounded in E. Then we can assume that $u_n \rightharpoonup u$ weakly in E, (up to a subsequence).

Now, applying a Brezis and Lieb's result (see [20]) we have

$$\int_{\mathbb{R}^N} k|u_n|^{2^{**}-2} u_n v = \int_{\mathbb{R}^N} k|u|^{2^{**}-2} uv + o_n(1), \quad \forall v \in E$$

and

$$\int_{\mathbb{R}^N} h|u_n|^{q-1}u_n v = \int_{\mathbb{R}^N} h|u|^{q-1}uv + o_n(1), \quad \forall v \in E.$$

From these equalities and keeping in mind that $I'(u_n)v = o_n(1)$, $\forall v \in E$, we obtain

$$I'(u)v = 0, \quad \forall v \in E.$$

If $u \neq 0$, we have finished. Next, consider the case $u = 0$.

By Lemma 3.2 and noting that $c > 0$ we have that there exist $\rho,\ \eta > 0$ and $\{y_n\} \subset \mathbb{R}^N$ (we may assume, without loss of generality, that $\{y_n\} \subset \mathbb{Z}^N$) such that

$$\limsup_{n\to\infty} \int_{B_{\rho+1}(y_n)} |u_n|^2 \geq \eta > 0. \tag{3.1}$$

Now, letting $v_n(x) = u_n(x - y_n)$, since a, h and k are 1-periodic functions and by an easy computation we obtain

$$\|v_n\| = \|u_n\|, \quad I(v_n) = I(u_n) \quad \text{and} \quad I'(v_n) \to 0.$$

Then, there exists $v \in E$ such that $v_n \rightharpoonup v$ weakly in E and it follows that $I'(v) = 0$.

We claim that

Claim 1. $v \neq 0$.

Verification of Claim 1. By (3.1), taking a subsequence if necessary, we get

$$\sqrt{\eta} \leq |v_n|_{L^2(B_\rho(0))} \leq |v_n|_{L^2(B_{\rho+1}(0))} \leq |v|_{L^2(B_{\rho+1}(0))} + |v_n - v|_{L^2(B_{\rho+1}(0))},$$

thus, from the Sobolev's compact imbedding Theorem we conclude that $v \neq 0$. This proves Claim 1 and Theorem 2.1. □

4. Proof of Theorem 1.1

Some arguments of this proof were adapted from the articles [2,15,19]. Consider the functional $I_W : E_W \to \mathbb{R}$ associated to problem ($\mathrm{P}_\#$), defined by

$$I_W(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\Delta u|^2 + (a - W)u^2 - \frac{1}{q+1}\int_{\mathbb{R}^N} H|u|^{q+1} - \frac{1}{2^{**}}\int_{\mathbb{R}^N} K|u|^{2^{**}},$$

where E_W denote the Sobolev space $H^2(\mathbb{R}^N)$ endowed with the norm equivalent to usual norm in $H^2(\mathbb{R}^N)$ given by

$$\|u\|_W^2 = \|u\|^2 - \int_{\mathbb{R}^N} Wu^2,$$

and we denote $I_o = I$, $\|\cdot\|_o = \|\cdot\|$ and $E_o = E$ for $W = 0$.

Also, define

$$J_v = \inf_{u \in M} \{I(u)\},$$

where

$$M = \{u \in E \backslash \{0\} : I'(u)u = 0\},$$

which is nonempty from Theorem 2.1. The following result is crucial on our proof.

Lemma 4.1. (i) $J_v > 0$.
(ii) *There exists* $u \in M$ *such that* $I(u) = J_v$.

Proof of Lemma 4.1. (i) Suppose by contradiction that $J_v = 0$, then there exists $u_n \in M$ such that $I(u_n) \to J_v = 0$.

But, since $u_n \in M$ we have

$$I(u_n) = \left(\frac{1}{2} - \frac{1}{q+1}\right) \int_{\mathbb{R}^N} h|u_n|^{q+1} + \frac{2}{N} \int_{\mathbb{R}^N} k|u_n|^{2^{**}} + o_n(1) \to 0.$$

Hence $\int_{\mathbb{R}^N} k|u_n|^{2^{**}} \to 0$ and $\int_{\mathbb{R}^N} h|u_n|^{q+1} \to 0$. Combining with $I'(u_n)u_n = 0$, we conclude that

$$\|u_n\| \to 0. \tag{4.1}$$

On the other hand, since $0 \neq u_n \in M$, we have

$$1 \leq C_1 \|u_n\|^{q-1} + C_2 \|u_n\|^{2^{**}-2}.$$

Combining this inequality with (4.1) we reach a contradiction. This proves (i).

(ii) We claim that

Claim 2. $c = J_v$.

Verification of claim 2. Taking $u \in M$ and adapting an argument used in [15, p. 288], we have

$$\max_{t \geq 0} I(tu) = I(u).$$

Choose $t_o \in \mathbb{R}$ and $u_o = t_o u$ such that $I(u_o) < 0$. Then $\gamma(t) = tu_o \in \Gamma$ and it follows that $I(u) \geq c$, that is

$$J_v \geq c. \tag{4.2}$$

Now, we are going to prove the reversed inequality. Let $\{u_n\} \subset E$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$. From the boundedness of $\{u_n\}$ we get $I'(u_n)u_n \to 0$ and in addition, arguing as in [19] or [15], there exists a sequence $\{t_n\} \in \mathbb{R}^+$ such that

$$I'(t_n u_n) t_n u_n = 0. \tag{4.3}$$

Hence $t_n u_n \in M$. Using (4.3) we have

$$\|u_n\|^2 = t_n^{q-1} \int_{\mathbb{R}^N} h|u_n|^{q+1} + t_n^{2^{**}-2} \int_{\mathbb{R}^N} k|u_n|^{2^{**}}. \tag{4.4}$$

Hence t_n does not converge to 0, otherwise, since $\{u_n\}$ is bounded, using (4.4) we have $\|u_n\| \to 0$, which is impossible since $c > 0$. Also, t_n does not go to infinity. In fact, dividing (4.4) by $t_n^{2^{**}-2}$, we get

$$\frac{\|u_n\|^2}{t_n^{2^{**}-2}} = t_n^{q-1-2^{**}} \int_{\mathbb{R}^N} h|u_n|^{q+1} + \int_{\mathbb{R}^N} k|u_n|^{2^{**}}. \tag{4.5}$$

On the other hand, since $I'(u_n)u_n \to 0$, we obtain

$$\|u_n\|^2 = \int_{\mathbb{R}^N} h|u_n|^{q+1} + \int_{\mathbb{R}^N} k|u_n|^{2^{**}} + o_n(1). \tag{4.6}$$

Now, assuming that $t_n \to \infty$, by (4.5) we get that $k^{1/2^{**}} u_n \to 0$ in $L^{2^{**}}$. Then using the interpolation inequality it follows that

$$\int_{\mathbb{R}^N} h|u_n|^{q+1} \to 0,$$

which, together with (4.6), leads to $\|u_n\| \to 0$, contradicting $c > 0$.

Hence, the sequence $\{t_n\}$ is bounded and there exists $t_o \in (0, \infty)$ such that $t_n \to t_o$ (passing to a subsequence if necessary).

Subtracting (4.4) from (4.6), we have

$$o_n(1) = (t_n^{q-1} - 1) \int_{\mathbb{R}^N} h|u_n|^{q+1} + (t_n^{2^{**}-2} - 1) \int_{\mathbb{R}^N} k|u_n|^{2^{**}}. \tag{4.7}$$

Passing to the limit into (4.7), we obtain

$$0 = (t_o^{q-1} - 1)l_1 + (t_o^{2^{**}-2} - 1)l_2,$$

where $\lim_{n\to\infty} \int_{\mathbb{R}^N} h|u_n|^{q+1} = l_1 \geq 0$ and $\lim_{n\to\infty} \int_{\mathbb{R}^N} k|u_n|^{2^{**}} = l_2 > 0$.

Therefore, $t_o = 1$, that is,

$$t_n \to 1. \tag{4.8}$$

Note that, by (4.8) and recalling that $t_n u_n \in M$, we have

$$\begin{aligned} J_v &\leq I(t_n u_n) \\ &= t_n^2 \left[I(u_n) + \frac{1}{q+1}(1 - t_n^{q-1}) \int_{\mathbb{R}^N} h|u_n|^{q+1} + \frac{1}{2^{**}}(1 - t_n^{2^{**}-2}) \int_{\mathbb{R}^N} k|u_n|^{2^{**}} \right] \\ &= t_n^2 I(u_n) + o_n(1) \\ &= (t_n^2 - 1)I(u_n) + I(u_n) + o_n(1). \end{aligned}$$

Passing to the limit we obtain $J_v \leq c$. This concludes the verification of claim 2.

Since I verifies the mountain-pass geometry, there is a bounded $(\mathrm{PS})_c$ sequence $\{u_n\}$, so that there exists $u \in E$ such that $u_n \rightharpoonup u$ weakly in E (up to a subsequence). Since V is periodic and $c = J_v < 2/N|k|_\infty^{(4-N)/4} S^{N/4}$, arguing as in the proof of Theorem 2.1, we can assume $u \neq 0$.

Claim 3.

$$I(u) = J_v \quad (\equiv c < \tfrac{2}{N}|k|_\infty^{(4-N)/4} S^{N/4}).$$

Verification of claim 3. First, we recall that u is a nontrivial solution of (P), that is, $u \in M$. Then

$$I(u) \geq J_v. \tag{4.9}$$

On the other hand, since $I'(u_n)u_n \to 0$, we get

$$\begin{aligned} J_v &= c = I(u_n) + o_n(1) \\ &= I(u_n) - \frac{1}{2} I'(u_n)u_n + o_n(1) \\ &= \left(\frac{1}{2} - \frac{1}{q+1}\right) \int_{\mathbb{R}^N} h|u_n|^{q+1} + \frac{2}{N} \int_{\mathbb{R}^N} k|u_n|^{2^{**}} + o_n(1). \end{aligned}$$

Now, taking the limit and using Fatou's Lemma, we obtain

$$\begin{aligned} J_v &\geq \left(\frac{1}{2} - \frac{1}{q+1}\right) \int_{\mathbb{R}^N} h|u|^{q+1} + \frac{2}{N} \int_{\mathbb{R}^N} k|u|^{2^{**}} \\ &= I(u) - \frac{1}{2} I'(u)u \\ &= I(u), \end{aligned}$$

which together with (4.9) concludes the proof of Lemma 4.1. □

Remark 2. From the proof of Lemma 4.1, we can choose $u \in M$ satisfying

$$J_v = I(u) \quad \text{and} \quad I'(u)\phi = 0, \quad \forall \phi \in E.$$

Note that from (h5)–(h7), it follows that I_W satisfies the mountain-pass geometry (see Lemma 3.1), so that, there exists a $(\mathrm{PS})_{c_W}$ sequence $\{u_n\}$ in E_W, that is

$$I_W(u_n) \to c_W \quad \text{and} \quad I'_W(u_n) \to 0, \tag{4.10}$$

where

$$c_W = \inf_{\gamma \in \Gamma_W} \max_{t \in [0,1]} I_W(\gamma(t))$$

and

$$\Gamma_W = \{\gamma \in C([0,1], E_W)\colon I_W(\gamma(0)) = 0 \text{ and } I_W(\gamma(1)) \leq 0\},$$

(see [13]). From Remark 2, let be $u \in M$ such that $I(u) = J_v$ and $I'(u)\phi = 0$, $\forall \phi \in E$.

We may assume that the Lebesgue's measure of the set

$$\operatorname{supp} u \cap \{x \in \mathbb{R}^N\colon W(x) \neq 0\}, \tag{4.11}$$

is positive, otherwise from Remark 2, we have

$$I'(u)\phi = I'_W(u)\phi = 0, \quad \forall \phi \in C_o^\infty(\mathbb{R}^N),$$

therefore, u is a nontrivial solution of $(\mathrm{P}_\#)$. Then, choosing $t^* \in \mathbb{R}$ such that

$$0 < c_W \leq \sup_{t \geq 0} I_W(tu) = I_W(t^*u),$$

by using (h5)–(h7), (4.11) and recalling that $u \in M$, we have

$$0 < c_W \leq I_W(t^*u) < I(t^*u) \leq \sup_{t \geq 0} I(tu) = J_v.$$

Hence

$$c_W < J_v = c < \frac{2}{N}|k|_\infty^{(4-N)/4} S^{N/4}.$$

It is standard to prove that the sequence $\{u_n\}$ in (4.10) is bounded, then up to a subsequence, we have $u_n \rightharpoonup \bar{u}$ weakly in E_W and arguing as in the proof of Theorem 2.1, we conclude that $\bar{u}$ is a weak solution of $(\mathrm{P}_\#)$.

Finally we are going to prove that $\bar{u} \neq 0$.

Suppose on the contrary that $\bar{u} = 0$, that is, $u_n \rightharpoonup 0$ weakly in E_W.

Since $W \in L^{N/4}(\mathbb{R}^N)$, using a result by Brezis–Lieb ([20]) we have

$$\int_{\mathbb{R}^N} W u_n^2 \to 0. \tag{4.12}$$

Note that

$$\begin{aligned}|I(u_n) - I_W(u_n)| \le \Bigg| \frac{1}{2}\int_{\mathbb{R}^N} W u_n^2 + \frac{1}{q+1}\int_{\mathbb{R}^N} |(H-h)u_n^{q+1}| \\ + \frac{1}{2^{**}}\int_{\mathbb{R}^N} |(K-k)u_n^{2^{**}}\Bigg| = J_1 + J_2 + J_3.\end{aligned}$$

From (4.12), (h6) and the Sobolev compact embedding theorem, the first two integrals go to zero, and by Lemma 3.2, more precisely, using the fact that $k^{1/2^{**}} u_n \to 0$, in $L^{2^{**}}_{\mathrm{loc}}(\mathbb{R}^N)$ and (h7) the last one also goes to zero. Then

$$|I(u_n) - I_W(u_n)| = o_n(1).$$

Therefore,

$$I(u_n) \to c_W < \frac{2}{N}|k|_\infty^{(4-N)/4} S^{N/4}.$$

On the other hand, noting that $W \ge 0$ and taking $\phi \in E \subset E_W$ with $\|\phi\| \le 1$, we obtain

$$\begin{aligned}|(I'(u_n) - I'_W(u_n))\phi| &= \left(\le \int_{\mathbb{R}^N} W u_n^2 \right)^{1/2} C + \int_{\mathbb{R}^N} |(H-h)u_n^q \phi| \\ &\quad + \int_{\mathbb{R}^N} |(K-k)u_n^{2^{**}-1}\phi| \\ &= I_1 + I_2 + I_3\end{aligned}$$

for some constant $C > 0$ and on the first inequality we used the Hölder's inequality.

As before, by Lemma 3.2, (h6)–(h7) and (4.12), the integral $I_i (i = 1, 2, 3)$ goes to zero. It follows that

$$I'(u_n) \to 0$$

so that,

$$I(u_n) \to c_W < J_v \quad \text{and} \quad I'(u_n) \to 0.$$

Now, arguing as in the verification of Claim 2, there exists a sequence $\{t_n\} \subset \mathbb{R}$ satisfying

$$t_n \to 1, \quad I'(t_n u_n) t_n u_n = 0$$

and

$$c_W \geq J_v,$$

which is a contradiction.

This completes the proof of Theorem 1.1. □

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