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On perturbations of a class of a periodic m-Laplacian equation with critical growth

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1. Introduction

The main purpose of this paper is to study the existence of weak solutions of the quasilinear elliptic problem

$$\begin{aligned} &-\Delta_m u + a(x)u^{m-1} = h(x)u^q + k(x)u^{m^*-1}, \quad x \in \mathbb{R}^N, \\ &u \in H^{1,m}(\mathbb{R}^N), \quad u(x) > 0, \quad x \in \mathbb{R}^N, \end{aligned} \tag{1}$$

where $1 < m < N$, $m - 1 < q < m^* - 1$, $m^* = Nm/(N - m)$ and $a, h, k : \mathbb{R}^N \to \mathbb{R}$ are continuous functions.

The study of this class of problems has received considerable attention in recent years. First, we would like to mention the progress involving the Laplacian operator, which corresponds to the case $m = 2$. Benci and Cerami [8] assuming that $h \equiv 0$, $k \equiv 1$, $a(x)$ is a nonnegative function and $|a|_{L^{N/2}}$ is sufficiently small, have proved that problem (1) has at least one solution. Pan [24] have considered the case $a \equiv 0$

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and $h \in L^p(\mathbb{R}^N)$ with $p \in [p_1, p_2]$ where $p_1 < 2N/(N+2-q(N-2)) < p_2$ if $N > 3$ and $p_2 < 3$ if $N = 3$. Problems with autonomous asymptotic behavior at infinite, more precisely, $a(x)$ is a nonnegative function, $a(x) \to a_0 > 0$ as $|x| \to +\infty$ and $h \equiv k \equiv 1$, have been treated by Jianfu and Xi Ping [18]. Rabinowitz [25], among other results, obtained a nontrivial solutions to the problem $-\Delta u + a(x)u = f(x,u)$ in $\mathbb{R}^N$, under the assumption that $a(x)$ was a nonnegative coercive function, that is, $a(x) \to +\infty$ as $|x| \to +\infty$ and that the potential $F(x,u) = \int_0^s f(x,t)\,dt$ was superquadratic and $f(x,u)$ had subcritical growth. Miyagaki [23] studied this problem involving critical Sobolev exponent namely for $f(x,u) = \lambda |u|^{q-1}u + |u|^{2^*-2}u$ with $\lambda > 0$. Alves et al. [4] have considered the case when $a(x)$ is a radial function, $h \equiv \lambda$ and $k \equiv 1$. Problems involving the m-Laplacian operator have been studied in [19] assuming autonomous asymptotic behavior at infinite. In [5] it was obtained a existence result with $0 < q < m^* - 1$, $a \equiv 0$, $h \in L^\theta((\mathbb{R}^N)$ $(1/\theta + (q+1)/m^* = 1)$ and $k \equiv 1$. In [2], assuming also that $|h|_{L^\theta}$ is sufficiently small and $1 < q < m$ it was obtained a multiplicity result. Ben-Naoum et al. [9] using minimization techniques showed the existence of a nontrivial solution under the assumption that the function a is negative on a set of positive measure and $h \equiv 0$, $k \equiv 1$.

After the well-known results of Brezis–Nirenberg [10], problems involving elliptic equations with critical growth in bounded domains have been studied by several authors, see for example [7,15,11,13,17] and the references therein.

The special features of this class of problems, considered in this paper, is that it is defined in $\mathbb{R}^N$, involve critical growth and a nonlinear operator. To overcome these difficulties that has arisen from these features we combine concentration compactness principle due to Lions [20,21], appropriate estimates for the levels associated with the mountain-pass theorem and the comparison arguments involving the Nehari manifold. Some of these ideas we adapt from [1,3,12,15,19]. Our main result improves the existence conditions in [3,18], since we are considering here a more general class of operator and nonlinearities. We also improve the results contained in [19], as we are concerning a class of nonlinearity which have periodic asymptotic behavior at infinite. Furthermore, it is not clear that an straightforward application of their arguments works to our class of problems.

We are now describing our main assumptions in a more precise way:

(H_1) There exists a continuous $\mathbb{Z}$-periodic function $A : \mathbb{R}^N \to \mathbb{R}$ such that
1. $A(x) \geq a_0 > 0$ for all $x \in \mathbb{R}^N$,
2. $a(x) \leq A(x)$, for all $x \in \mathbb{R}^N$,
3. $a(x) - A(x) \to 0$ as $|x| \to +\infty$.

(H_2) There exists a continuous $\mathbb{Z}$-periodic function $H : \mathbb{R}^N \to \mathbb{R}$ such that
1. $H(x) \geq 0$ for all $x \in \mathbb{R}^N$ and $\inf_{\overline{B_2(0)}} H = \eta_0 > 0$,
2. $h(x) \geq H(x)$, for all $x \in \mathbb{R}^N$,
3. $h(x) - H(x) \to 0$ as $|x| \to +\infty$.

(H_3) There exists a continuous $\mathbb{Z}$-periodic function $K : \mathbb{R}^N \to \mathbb{R}$ such that
1. $K(x) > 0$ for all $x \in \mathbb{R}^N$,
2. $k(x) \geq K(x)$, for all $x \in \mathbb{R}^N$,
3. $k(x) = \|k\|_\infty + O(|x|^{(N-m)/(m-1)})$, for all $x \in B_1(0)$,
4. $k(x) - K(x) \to 0$ as $|x| \to +\infty$.

(H_4) At least one of the nonnegative continuous functions

$$A(x) - a(x), \quad h(x) - H(x) \quad \text{and} \quad k(x) - K(x)$$

is positive on a subset of positive measure.

Our main result in this paper is stated as follows.

Theorem 1. *Assume* $1 < m < N$, $m < q < m^* - 1$ *and* (H_1)–(H_4). *Then problem* (1) *has a solution provided that one of the following conditions holds*:
(i) $N \geq m^2$,
(ii) $m < N < m^2$ *and* $m^* - m/(m+1) - 1 < q$,
(iii) $m < N < m^2$, $m^* - m/(m+1) - 1 \geq q$ *and* $\inf_{\overline{B_2(0)}} H$ *is sufficiently large.*

This paper is organized as follows: Section 2 contains some preliminary facts including a existence result to a periodic problem; in Section 3, we proved our main result.

2. Existence of solutions for a limit problem

In this section we will discuss the existence of a nontrivial solution for the following problem:

$$\begin{aligned} &-\Delta_m u + A(x)u^{m-1} = H(x)u^q + K(x)u^{m^*-1} \quad \text{for } x \in \mathbb{R}^N, \\ &u \in H^{1,m}(\mathbb{R}^N), \quad u(x) \geq 0 \quad \text{for } x \in \mathbb{R}^N, \end{aligned} \tag{2}$$

where $A, H, K : \mathbb{R}^N \to \mathbb{R}$ are continuous $\mathbb{Z}$-periodic functions with $A(x) \geq a_0 > 0$, $H(x) \geq 0$, $K(x) > 0$ for all $x \in \mathbb{R}^N$ and $\inf_{\overline{B_2(0)}} H = \eta_0 > 0$.

The solutions of problem (2) will be found as the critical points of the Fréchet differentiable functional given by

$$J_\infty(u) = \frac{1}{m}\int_{\mathbb{R}^N} [|\nabla u|^m + A(x)|u|^m] - \frac{1}{q+1}\int_{\mathbb{R}^N} H(x)u_+^{q+1} - \frac{1}{m^*}\int_{\mathbb{R}^N} K(x)u_+^{m^*},$$

where $u_\pm = \max\{\pm u, 0\}$, defined on the Sobolev space $H^{1,m}(\mathbb{R}^N)$, endowed with the equivalent norm

$$\|u\| = \left\{\int_{\mathbb{R}^N} [|\nabla u|^m + A(x)|u|^m]\right\}^{1/m}.$$

We set in the associated functional J_∞ the integral $\int_{\mathbb{R}^N} K(x)u_+^{m^*}$ in order to obtain nonnegative solutions.

It is standard to prove that J_∞ verifies the mountain-pass geometrical conditions as we state in the following result (cf. [6]).

Lemma 2 (Mountain-pass geometry). *The functional* J_∞ *satisfies the following conditions*:
- *there exist* $\alpha, \beta > 0$, *such that* $J_\infty(u) \geq \beta$ *if* $\|u\| = \alpha$,
- *for any* $u \in H^{1,m}(\mathbb{R}^N)$ *with* u_+ *nontrivial, we have* $J_\infty(tu) \to -\infty$ *as* $t \to +\infty$.

Now, in view of Lemma 2, we may apply a version of mountain-pass theorem without a compactness condition such as the one of Palais–Smale (cf. [14,22]), to obtain a Palais–Smale sequence of functional J_∞, more precisely, $(u_n) \subset H^{1,m}(\mathbb{R}^N)$ such that

$$J_\infty(u_n) \to C^* \geq \alpha > 0 \quad \text{and} \quad J'_\infty(u_n) \to 0,$$

where

$$C^* = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} J_\infty(\gamma(t)) \tag{3}$$

and

$$\Gamma = \{\gamma \in C([0,1], H^{1,m}(\mathbb{R}^N)) \colon J_\infty(\gamma(0)) \leq 0 \text{ and } J_\infty(\gamma(1)) \leq 0\}.$$

In this work we are denoting by S the best Sobolev constant to the Sobolev embedding, $D^{1,m}(\mathbb{R}^N) \hookrightarrow L^{m^*}(\mathbb{R}^N)$, that is,

$$S = \inf\{\|\nabla u\|^m_{L^m}/\|u\|^m_{L^{m^*}} \colon u \in D^{1,m}(\mathbb{R}^N) - \{0\}\}. \tag{4}$$

We recall that $D^{1,m}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\|^m_{D^{1,m}} = \|\nabla u\|_{L^m}$.

According to Lemma 2 in [26] the infimum S is attained by the functions w_ε given by

$$w_\varepsilon(x) = \frac{C(N,m)\varepsilon^{(N-m)/m^2}}{[\varepsilon + |x|^{m/(m-1)}]^{(N-m)/m}}, \quad C(N,m) = \left[N\left(\frac{N-m}{m-1}\right)^{m-1}\right]^{(N-m)/m^2} \tag{5}$$

for any $x \in \mathbb{R}^N$ and any $\varepsilon > 0$.

As it was done in [10,15], we are using in the next lemma, the extremal functions (5), in order to get a more precise estimate about the minimax level C^* obtained by the mountain-pass theorem in (3).

Lemma 3. *There exists* $v \in H^{1,m}(\mathbb{R}^N) - \{0\}$ *such that*

$$\max_{t \geq 0} J_\infty(tv) < \|K\|_\infty^{(m-N)/m} \frac{S^{N/m}}{N}, \tag{6}$$

provided that one of conditions (i), (ii) or (iii) *in Theorem* 1 *holds.*

Proof. Consider the functions

$$\beta_\varepsilon(x) = \phi(x) w_\varepsilon(x) \quad \text{and} \quad v_\varepsilon(x) = \frac{\beta_\varepsilon(x)}{\left(\int_{|x|\leq 2} K(x)\beta_\varepsilon^{m^*}\,\mathrm{d}x\right)^{1/m^*}},$$

where $\phi \in C_0^\infty(\mathbb{R}^N, [0,1])$, $\phi(x) = 1$ if $|x| \leq 1$ and $\phi(x) = 0$ if $|x| \geq 2$. By a similar argument to that used in [1,13,15], we show that the functions β_ε and v_ε satisfy the following estimates:

(A) $\int_{|x|\geq 1} |\nabla\beta_\varepsilon(x)|^m\,\mathrm{d}x = O(\varepsilon^{(N-m)/m})$,

(B) $k_1 < \int_{|x|\leq 2} K(x)\beta_\varepsilon^{m^*}(x)\,\mathrm{d}x < k_2$ for ε sufficiently small,

(C) $\int_{|x|\le 1} |x|^{(N-m)/(m-1)} w_\varepsilon^{m^*}(x)\,dx = O(\varepsilon^{(N-m)/m})$,

(D) $\int_{\mathbb{R}^N} |\nabla v_\varepsilon(x)|^m\,dx \le \|K\|_\infty^{(m-N)/N} S + O(\varepsilon^{(N-m)/m})$.

In view of Lemma 2, for each $\varepsilon > 0$ small, there exists $t_\varepsilon > 0$ such that

$$J_\infty(t_\varepsilon v_\varepsilon) = \max\{J_\infty(tv_\varepsilon)\colon t \ge 0\}.$$

Note that, at $t = t_\varepsilon$, we have $(d/dt)J_\infty(tv_\varepsilon) = 0$, that is,

$$t_\varepsilon^{m-1} \int_{\mathbb{R}^N} [|\nabla v_\varepsilon|^m + A(x)|v_\varepsilon|^m]\,dx = t_\varepsilon^{q-1} \int_{\mathbb{R}^N} H(x) v_\varepsilon^{q+1}\,dx - t_\varepsilon^{m^*-1}.$$

From this and estimates (A)–(D), it follows that $t_\varepsilon > \alpha_0 > 0$, for all $0 < \varepsilon < \varepsilon_0$, where α_0 is a positive constant independent of ε. Furthermore, by straightforward calculations we find

$$\begin{aligned} J_\infty(t_\varepsilon v_\varepsilon) \le{} & \|K\|_\infty^{(m-N)/m} \frac{S^{N/m}}{N} + O(\varepsilon^{(N-m)/m}) + \int_{|x|\le 2} [c_1 a(x) v_\varepsilon^m - c_2 \eta_0 v_\varepsilon^{q+1}]\,dx \\ & + \|K\|_\infty^{(m-N)/m} \frac{S^{N/m}}{N} \\ & + \varepsilon^{(N-m)/m} \left[M + \varepsilon^{(m-N)/m} \int_{|x|\le 2} [c_1 a(x) v_\varepsilon^m - c_2 \eta_0 v_\varepsilon^{q+1}]\,dx \right], \end{aligned}$$

where $\eta_0 \doteq \inf_{\overline{B_2(0)}} H(x) > 0$ and M, c_1 and c_2 are positive constants independent of ε.

Claim 1. *There is $\varepsilon > 0$ sufficiently small such that*

$$M + \varepsilon^{(m-N)/m} \int_{|x|\le 2} [c_1 a(x) v_\varepsilon^m - c_2 H v_\varepsilon^{q+1}]\,dx < 0.$$

From this claim we easily see that

$$\max_{t\ge 0} J_\infty(tv_\varepsilon) = J_\infty(t_\varepsilon v_\varepsilon) < \|K\|_\infty^{(m-N)/m} \frac{S^{N/m}}{N},$$

and taking $u = t_\varepsilon v_\varepsilon$ we obtain (6).

Verification of Claim 1. Once more, using estimates (A)–(D), and the expression of v_ε we have

$$\varepsilon^{(m-N)/m} \int_{|x|\le 2} [c_1 a(x) v_\varepsilon^m - c_2 \eta_0 v_\varepsilon^{q+1}]\,dx \le \Phi(\varepsilon) + \Psi(\varepsilon),$$

where

$$\begin{aligned} \Phi(\varepsilon) &= \varepsilon^{(m-N)/m} \int_{|x|\le 1} [c_3 a(x) v_\varepsilon^m - c_4 \eta_0 v_\varepsilon^{q+1}]\,dx \\ &\le c_5 \varepsilon^{(m^2-N)/m} K_\varepsilon - c_6 \eta_0 \varepsilon^{(q+1)[(N-m)/m^2 - (N-m)/m] + [(m-1)/m]N + (m-N)/m}, \end{aligned}$$

$$K_\varepsilon = \int_0^{\varepsilon^{(1-m)/m}} \frac{s^{N-1}}{(1 + s^{m/(m-1)})^{N-m}}\,ds,$$

$$\Psi(\varepsilon) = \varepsilon^{(m-N)/m} \int_{1 \le |x| \le 2} [c_3 a(x) v_\varepsilon^m - c_4 \eta_0 v_\varepsilon^{q+1}]\, dx$$

$$\le c_5 \varepsilon^{(m-N)/m} \int_{1 \le |x| \le 2} w_\varepsilon^m$$

$$\le c_6 \int_{1 \le |x| \le 2} |x|^{m(m-N)/(m-1)}\, dx \le c_7$$

for some positive constants c_3, c_4, c_5, c_6 and c_7 independent of ε. Finally, using these estimates and studying separately conditions (i)–(iii) in Theorem 1, we prove that there exists $\varepsilon > 0$ such that

$$\Phi(\varepsilon) < c_7 - M.$$

Thus, the Claim 1 follows. □

The next result gives an important aspect of the behavior of Palais–Smale sequences associated to the functional J_∞ at level $b < \|K\|_\infty^{(m-N)/m} S^{N/m}/N$.

Lemma 4. *Let $(u_n) \subset H^{1,m}(\mathbb{R}^N)$ be a $(PS)_b$ sequence associated to the functional J_∞ such that $u_n \rightharpoonup 0$ in $H^{1,m}(\mathbb{R}^N)$ and $b < \|K\|_\infty^{(m-N)/m} S^{N/m}/N$. Thus, one of the following conditions holds*:

(a) *$u_n \to 0$ strong in $H^{1,m}(\mathbb{R}^N)$*,

(b) *there is a sequence $(y_n) \subset \mathbb{R}^N$, and $\rho, \eta > 0$ such that*

$$\limsup_{n \to +\infty} \int_{B_\rho(y_n)} |u_n|^m\, dx \ge \eta.$$

Proof. Suppose that condition (b) does not hold. Using Lemma 1.1 in [20], it follows:

$$\int_{\mathbb{R}^N} (u_n)_+^{q+1}\, dx \to 0 \quad \text{as } n \to +\infty$$

for $m < q+1 < m^*$. From $J'_\infty(u_n)u_n = o_n(1)$, we conclude

$$\|u_n\|^m = \int_{\mathbb{R}^N} K(x)(u_n)_+^{m^*}\, dx + o_n(1).$$

Assume that $\|u_n\|^m \to \ell$ $(\ell > 0)$. Since $J_\infty(u_n) \to b$ it follows that $b = \ell/N$. Now, according to (4),

$$\|u_n\|^m \ge S \left(\int_{\mathbb{R}^N} u_n^{m^*}\, dx \right)^{m/m^*}$$

$$\ge S \left(\int_{\mathbb{R}^N} (u_n)_+^{m^*}\, dx \right)^{(N-m)/N}$$

$$\ge S \|K\|_\infty^{(m-N)/m} \left(\int_{\mathbb{R}^N} K(x)(u_n)_+^{m^*}\, dx \right)^{m/m^*}.$$

Thus, passing to the limit we achieve

$$b \geq \|K\|_\infty^{(m-N)/N} \frac{S^{N/m}}{N},$$

which is a contradiction with our assumption, then (a) holds. □

Now we are ready to state the following existence result.

Theorem 5. *Assume that* $A, H, K : \mathbb{R}^N \to \mathbb{R}$ *are continuous* $\mathbb{Z}$*-periodic functions with* $A(x) \geq a_0 > 0$, $H(x) \geq 0$, $K(x) > 0$ *for all* $x \in \mathbb{R}^N$ *and* $\inf_{\overline{B_2(0)}} H = \eta_0 > 0$. *Then, problem* (2) *has a nontrivial solution provided that one of the conditions* (i), (ii) *or* (iii) *in Theorem* 1 *holds.*

Proof. We know that there is a bounded sequence $(u_n) \subset H^{1,m}(\mathbb{R}^N)$ such that

$$J_\infty(u_n) \to C^*, \quad 0 < C^* < \|K\|_\infty^{(m-N)/m} S^{N/m}/N \quad \text{and} \quad J'_\infty(u_n) \to 0.$$

Then, up to the subsequence, $u_n \rightharpoonup u_0$ weakly in $H^{1,m}(\mathbb{R}^N)$. Now, using the same kind of ideas contained in [1,15], we can prove that, for all $\phi \in H^{1,m}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |\nabla u_n|^{m-2} \nabla u_n \nabla \phi \, dx \to \int_{\mathbb{R}^N} |\nabla u_0|^{m-2} \nabla u_0 \nabla \phi \, dx,$$

$$\int_{\mathbb{R}^N} A(x) |u_n|^{m-2} u_n \phi \, dx \to \int_{\mathbb{R}^N} A(x) |u_0|^{m-2} u_0 \phi \, dx,$$

$$\int_{\mathbb{R}^N} H(x)(u_n)_+^q \phi \, dx \to \int_{\mathbb{R}^N} H(x)(u_0)_+^q \phi \, dx,$$

$$\int_{\mathbb{R}^N} K(x)(u_n)_+^{m^*-1} \phi \, dx \to \int_{\mathbb{R}^N} K(x)(u_0)_+^{m^*-1} \phi \, dx.$$

From this, together with $J'_\infty(u_n) \to 0$, passing to the limit, it is easy to prove that

$$\int_{\mathbb{R}^N} [|\nabla u_0|^{m-2} \nabla u_0 \nabla \phi + A |u_0|^{m-2} u_0 \phi] = \int_{\mathbb{R}^N} H(u_0)_+^q \phi + \int_{\mathbb{R}^N} K(u_0)_+^{m^*-1} \phi \qquad (7)$$

for all $\phi \in H^{1,m}(\mathbb{R}^N)$. Let $u_0 = (u_0)_+ + (u_0)_-$ and taking as test function $\phi(x) = (u_0)_-$ in (7) we have

$$\int_{\mathbb{R}^N} [|\nabla (u_0)_-|^m + A(x)(u_0)_-^m] \, dx = 0.$$

Hence $(u_0)_- \equiv 0$, thus $u_0 \geq 0$. If u_0 is nontrivial we have finished, otherwise, that is $u_0 \equiv 0$, from $C^* > 0$ we see that u_n does not go to zero strongly in $H^{1,m}(\mathbb{R}^N)$. Therefore, from Lemma 4, there are $\rho, \eta > 0$ and $(y_n) \subset \mathbb{R}^N$ (which without loss of generality, we may assume that $(y_n) \subset \mathbb{Z}^N$) such that

$$\limsup_{n \to +\infty} \int_{B_{\rho+1}(y_n)} |u_n|^m \, dx \geq \eta > 0. \qquad (8)$$

Note that, if we set $v_n(x) = u_n(x - y_n)$, since A, H and K are $\mathbb{Z}$-periodic functions, by a simple change of variable we can show that

$$\|v_n\| = \|u_n\|, \quad J'_\infty(v_n) = J'_\infty(u_n) + o_n(1) \quad \text{and} \quad J_\infty(v_n) = J_\infty(u_n).$$

Therefore, $v_n \rightharpoonup v$ in $H^{1,m}(\mathbb{R}^N)$ and $J'_\infty(v)\phi = 0$, for all $\phi \in H^{1,m}(\mathbb{R}^N)$. Thus, v is a weak solution of problem (2). Furthermore, from the weak lower semicontinuity of the norm and (8) (taking a bigger ball if necessary) we conclude that v is non-trivial. □

3. Proof of Theorem 1

In this section we are going to prove the existence of a solutions to problem (1) as critical point of the associated Fréchet differentiable functional given by

$$J(u) = \frac{1}{m}\int_{\mathbb{R}^N} [|\nabla u|^m + a(x)|u|^m] - \frac{1}{q+1}\int_{\mathbb{R}^N} h(x)u_+^{q+1} - \frac{1}{m^*}\int_{\mathbb{R}^N} k(x)u_+^{m^*},$$

defined on the Sobolev space $H^{1,m}(\mathbb{R}^N)$, with the equivalent norm

$$\|u\| = \left\{\int_{\mathbb{R}^N} [|\nabla u|^m + a(x)|u|^m]\,\mathrm{d}x\right\}^{1/m}.$$

Some arguments used here were adapted from [3,12,19]. By M_∞ we denote the following Nehari manifold:

$$M_\infty = \{u \in E - \{0\}\colon J'_\infty(u)u = 0\}$$

and we consider the constrained problem

$$C_\infty = \inf\{J_\infty(u)\colon u \in M_\infty\}. \tag{9}$$

Note that M_∞ is non empty from Theorem 5.

We begin stating the following fundamental result.

Proposition 6. (a) $C_\infty > 0$.

(b) *There exists $u \in M_\infty$ such that $J_\infty(u) = C_\infty$.*

Proof. Assume for the sake of contradiction that $C_\infty = 0$, then there exists $(z_n) \subset M_\infty$ such that $J_\infty(z_n) \to 0$. Since $z_n \in M_\infty$, it follows that

$$\|z_n\|^m = \int_{\mathbb{R}^N} H(x)(z_n)_+^{q+1} + \int_{\mathbb{R}^N} K(x)(z_n)_+^{m^*} \tag{10}$$

and

$$J_\infty(z_n) = \left(\frac{1}{m} - \frac{1}{q+1}\right)\int_{\mathbb{R}^N} H(x)(z_n)_+^{q+1} + \frac{1}{N}\int_{\mathbb{R}^N} K(x)(z_n)_+^{m^*} \to 0$$

as $n \to +\infty$. Since $m < q+1$ and H, K are nonnegative functions we have that

$$\int_{\mathbb{R}^N} H(x)(z_n)_+^{q+1} \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} K(x)(z_n)_+^{m^*} \to 0$$

as $n \to +\infty$, which together with (10) implies that

$$\|z_n\| \to 0. \tag{11}$$

On the order hand, dividing (10) by $\|z_n\|^m$ and using the Sobolev inequality, we achieve

$$1 \le C_1\|z_n\|^{q+1-m} + C_2\|z_n\|^{m^*-m},$$

which proves that (11) does not occur. Therefore (a) holds.

To prove (b), first we observe that, adapting an argument used in [12, p. 288], we have

$$\max_{t\ge 0} J_\infty(tu) = J_\infty(u)$$

for each $u \in M_\infty$. In view of Lemma 2, we can choose $t_0 \in (0,+\infty)$ such that $J_\infty(t_0u) < 0$ and we set $\bar{u} = t_0u$. Note that $\gamma(t) = t\bar{u} \in \Gamma$ and

$$J_\infty(u) = \max_{0\le t\le 1} J_\infty(\gamma(t)) \ge C^*.$$

Therefore,

$$C_\infty \ge C^*. \tag{12}$$

Now, as in the proof of Theorem 5, we consider $(u_n) \subset H^{1,m}(\mathbb{R}^N)$ such that

$$u_n \rightharpoonup u_0, \quad J_\infty(u_n) \to C^* \quad \text{and} \quad J'_\infty(u_n) \to 0.$$

For each u_n consider $t_n \in (0,+\infty)$ such that $t_nu_n \in M_\infty$, that is,

$$J'_\infty(t_nu_n)(t_nu_n) = 0. \tag{13}$$

Thus,

$$\|u_n\|^m = t_m^{q+1-m}\int_{\mathbb{R}^N} H(x)(u_n)_+^{q+1}\,\mathrm{d}x + t_n^{m^*-m}\int_{\mathbb{R}^N} K(x)(u_n)_+^{m^*}\,\mathrm{d}x. \tag{14}$$

Since $J'_\infty(u_n)u_n \to 0$ as $n \to +\infty$, we have

$$\|u_n\|^m = \int_{\mathbb{R}^N} H(x)(u_n)_+^{q+1}\,\mathrm{d}x + \int_{\mathbb{R}^N} K(x)(u_n)_+^{m^*}\,\mathrm{d}x + o_n(1). \tag{15}$$

Claim 2. $t_n \to 1$ *as* $n \to +\infty$.

Verification of Claim 2. First we prove that (t_n) is bounded. Assume by contradiction that $t_n \to +\infty$. Dividing (14) by $t_n^{m^*-m}$, we obtain

$$\frac{\|u_n\|^m}{t_n^{m^*-m}} = \frac{1}{t_n^{m^*-(q+1)}}\int_{\mathbb{R}^N} H(x)(u_n)_+^{q+1}\,\mathrm{d}x + \int_{\mathbb{R}^N} K(x)(u_n)_+^{m^*}\,\mathrm{d}x.$$

Since (u_n) is bounded in $H^{1,m}(\mathbb{R}^N)$ and $t_n \to +\infty$ as $n \to +\infty$, we conclude that

$$\int_{\mathbb{R}^N} K(x)(u_n)_+^{m^*}\,\mathrm{d}x \to 0 \quad \text{as } n \to +\infty. \tag{16}$$

Since K is a $\mathbb{Z}$-periodic continuous functions and $K(x) > 0$, for all $x \in \mathbb{R}^N$, it follows that $K(x) \ge k_0 > 0$, for all $x \in \mathbb{R}^N$, which implies that

$$\int_{\mathbb{R}^N} (u_n)_+^{m^*}\,\mathrm{d}x \to 0 \quad \text{as } n \to +\infty.$$

Using the interpolation inequality, we prove that

$$\int_{\mathbb{R}^N} (u_n)_+^{q+1}\,\mathrm{d}x \to 0 \quad \text{as } n \to +\infty.$$

Since H is $\mathbb{Z}$-periodic, we obtain

$$\int_{\mathbb{R}^N} H(x)(u_n)_+^{q+1}\,\mathrm{d}x \to 0 \quad \text{as } n \to +\infty,$$

which together with (15) and (16) imply that

$$\|u_n\| \to 0 \quad \text{as } n \to +\infty,$$

which is a contradiction with $J_\infty(u_n) \to C^* \geq \alpha > 0$. Therefore, up to subsequence, we have $t_n \to t_1$. Now, subtracting (14) from (15), we achieve

$$o_n(1) = (t_m^{q+1-m} - 1)\int_{\mathbb{R}^N} H(x)(u_n)_+^{q+1}\,\mathrm{d}x + (t_n^{m^*-m} - 1)\int_{\mathbb{R}^N} K(x)(u_n)_+^{m^*}\,\mathrm{d}x.$$

Passing to the limit we obtain that

$$0 = (t_1^{q+1-m} - 1)l_1 + (t_1^{m^*-m} - 1)l_2,$$

where

$$l_1 = \lim \int_{\mathbb{R}^N} H(x)(u_n)_+^{q+1}\,\mathrm{d}x \geq 0 \quad \text{and} \quad l_2 = \int_{\mathbb{R}^N} K(x)(u_n)_+^{m^*}\,\mathrm{d}x > 0.$$

Therefore, $t_1 = 1$ and Claim 1 holds.

Note that

$$\begin{aligned} C_\infty &\leq J_\infty(t_n u_n) \\ &= t_n^m \left[J_\infty(u_n) + \frac{1}{q+1}(1 - t_n^{q+1-m})\int_{\mathbb{R}^N} H(x)(u_n)_+^{q+1}\,\mathrm{d}x \right. \\ &\quad \left. + \frac{1}{m^*}(1 - t_n^{m^*-m})\int_{\mathbb{R}^N} K(x)(u_n)_+^{m^*}\,\mathrm{d}x \right] \\ &= t_n^m J_\infty(u_n) + o_n(1) \\ &= (t_n^m - 1)J_\infty(u_n) + J_\infty(u_n) + o_n(1). \end{aligned}$$

Once more passing to the limit we get

$$C_\infty \leq C^*,$$

which together with (12) imply that

$$C_\infty = C^*.$$

Finally, we are going to prove that minimization problem (9) has a solution, more precisely, we shall prove that $J_\infty(u_0) = C_\infty$. Note that

$$C_\infty = J_\infty(u_n) + o_n(1)$$

$$= J_\infty(u_n) - \frac{1}{m} J'_\infty(u_n)u_n + o_n(1)$$

$$= \left(\frac{1}{m} - \frac{1}{q+1}\right) \int_{\mathbb{R}^N} H(x)(u_n)_+^{q+1}\,\mathrm{d}x + \frac{1}{N}\int_{\mathbb{R}^N} K(x)(u_n)_+^{m^*}\,\mathrm{d}x + o_n(1).$$

Thus, using the Fatou's Lemma, we conclude that

$$C_\infty \geq \left(\frac{1}{m} - \frac{1}{q+1}\right) \int_{\mathbb{R}^N} H(x)(u_0)_+^{q+1}\,\mathrm{d}x + \frac{1}{N}\int_{\mathbb{R}^N} K(x)(u_0)_+^{m^*}\,\mathrm{d}x$$

$$= J_\infty(u_0) - \frac{1}{m} J'_\infty(u_0)u_0 = J_\infty(u_0).$$

Therefore,

$$C_\infty = J_\infty(u_0),$$

since $u_0 \in M_\infty$. Thus (b) holds. □

It is easy to check that the functional J has a geometry of the mountain-pass theorem. Therefore, applying the mountain-pass theorem without Palais–Smale condition together with the arguments from Section 2, we obtain a bounded sequence $(v_n) \subset H^{1,m}(\mathbb{R}^N)$ such that

$$J(v_n) \to C^{**}, \quad 0 < C^{**} < \|k\|_\infty^{(m-N)/m} \frac{S^{N/m}}{N} \quad \text{and} \quad J'(v_n) \to 0 \quad \text{as } n \to +\infty.$$

Furthermore, $v_n \rightharpoonup v_0$ weakly in $H^{1,m}(\mathbb{R}^N)$. Arguing as in the proof of Theorem 5, we conclude that v_0 is a critical point of functional J and $v_0 \geq 0$.

Claim 3. *v_0 is nontrivial.*

Verification of Claim 3. Assume for the sake of contradiction that $v_0 \equiv 0$. Thus, from the fact that $h(x) - H(x) \to 0$ as $|x| \to \infty$ and the Sobolev compact embedding theorem, we find

$$\int_{\mathbb{R}^N} [h(x) - H(x)](v_n)_+^{q+1}\,\mathrm{d}x \to 0 \quad \text{as } n \to +\infty \tag{17}$$

and

$$\int_{\mathbb{R}^N} [A(x) - a(x)]|v_n|^m\,\mathrm{d}x \quad \text{as } n \to +\infty. \tag{18}$$

Next, we assume the following result, which will be proved later.

Claim 4.

$$\int_{\mathbb{R}^N} (k(x) - K(x))(v_n)_+^{m^*} \to 0 \quad as\ n \to +\infty. \tag{19}$$

Note that

$$|J_\infty(v_n) - J(v_n)| \le \frac{1}{m} \int_{\mathbb{R}^N} [A(x) - a(x)]|v_n|^m \,\mathrm{d}x$$
$$+\frac{1}{q+1} \int_{\mathbb{R}^N} [h(x) - H(x)](v_n)_+^{q+1} \,\mathrm{d}x$$
$$+\frac{1}{m^*} \int_{\mathbb{R}^N} [k(x) - K(x)](v_n)_+^{m^*} \,\mathrm{d}x.$$

Using Hölder and Sobolev inequalities, we have that

$$|(J'_\infty(v_n) - J'(v_n))\phi| \le C_1 \int_{\mathbb{R}^N} [A(x) - a(x)]|v_n|^m \,\mathrm{d}x$$
$$+C_2 \int_{\mathbb{R}^N} [h(x) - H(x)](v_n)_+^{q+1} \,\mathrm{d}x$$
$$+C_3 \int_{\mathbb{R}^N} [k(x) - K(x)](v_n)_+^{m^*} \,\mathrm{d}x$$

for all $\phi \in H^{1,m}(\mathbb{R}^N)$ with $\|\phi\| \le 1$, where C_1, C_2 and C_3 are constants independent of n. This estimate together with (17)–(19) imply that

$$|J_\infty(v_n) - J(v_n)| \to 0 \quad \text{and} \quad \|J'_\infty(v_n) - J'(v_n)\| \to 0 \quad \text{as } n \to +\infty.$$

Therefore,

$$J_\infty(v_n) \to C^{**}, \quad 0 < C^{**} < \|k\|_\infty^{(m-N)/m} \frac{S^{N/m}}{N} \quad \text{and} \quad J'_\infty(v_n) \to 0 \text{ as } n \to +\infty.$$

Repeating the same idea in the proof of Proposition 6, we conclude that

$$C_\infty \le C^{**}.$$

On the other hand,

$$C^{**} \le \max_{t \ge 0} J(tu_0) = J(t^* u_0) < J_\infty(t^* u_0) \le \max_{t \ge 0} J_\infty(tu_0) = J_\infty(u_0) = C^* = C_\infty,$$

which it is a contradiction. Therefore, Claim 3 holds. □

Verification of Claim 4. Taking a subsequence, we may suppose that

$$k(v_n)_+^{m^*} \rightharpoonup v \quad \text{and} \quad |\nabla(v_n)_+|^m \rightharpoonup \mu \quad (\text{weak}^*\text{-sense of measure}). \tag{20}$$

Using the so-called Concentration-Compactness Principle II, due to Lions (cf. [21]), there exists an at most countable index set Υ, sequences $(x_i) \subset \mathbb{R}^N$, (v_i), $(\mu_i) \subset (0, \infty)$, $i \in \Upsilon$ such that

$$v = \sum_{i \in \Upsilon} v_i \delta_{x_i} \quad \text{and} \quad \mu \ge \sum_{i \in \Upsilon} \mu_i \delta_{x_i}$$

where δ_{x_i} is Dirac mass at $x_i \in \mathbb{R}^N$. Furthermore,

$$S\left(\frac{v_i}{\|k\|_\infty}\right)^{m/m^*} \le \mu_i. \tag{21}$$

Let $\Psi_\varepsilon(x) \doteq \Psi((x-x_i)/\varepsilon)$, $x\in\mathbb{R}^N$, $\varepsilon > 0$, where $\Psi \in C_0^\infty(\mathbb{R}^N,[0,1])$ is such that $\Psi \equiv 1$ on $B_1(0)$, $\Psi \equiv 0$ on $\mathbb{R}^N - B_2(0)$ and $|\nabla\Psi| \le 2$. Using that

$$J'(v_n)(\Psi_\varepsilon(v_n)_+) \to 0 \quad \text{as } n \to +\infty$$

and (20), we obtain

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} |\nabla(v_n)_+|^{m-2}(v_n)_+\nabla\Psi_\varepsilon\nabla(v_n)_+\,\mathrm{d}x$$

$$= \int_{\mathbb{R}^N} hv_0^{q+1}\Psi_\varepsilon\,\mathrm{d}x + \int_{\mathbb{R}^N}\Psi_\varepsilon\,\mathrm{d}v - \int_{\mathbb{R}^N}\Psi_\varepsilon\,\mathrm{d}\mu.$$

Taking the limit in this last expression as $\varepsilon \to 0$ and using that

$$\lim_{\varepsilon\to 0}\left[\lim_{n\to+\infty}\int_{\mathbb{R}^N} |\nabla(v_n)_+|^{m-2}(v_n)_+\nabla\Psi_\varepsilon\nabla(v_n)_+\,\mathrm{d}x\right] = 0,$$

we conclude that

$$\mu_i = v_i.$$

Thus, from (21), we have

$$v_i \ge \|k\|_\infty^{(m-N)/m} S^{N/m}. \tag{22}$$

Therefore, Υ is at most a finite set, that is, there are at most a finite number of singularities, because v is a bounded measure. Now we shall prove that (22) does not occur and consequently we have that $\Upsilon = \emptyset$. Assume for the sake of contradiction that we have (22) for $i \in \Upsilon$. Since

$$\begin{aligned}
C^{**} &= J(v_n) + o_n(1)\\
&= J(v_n) - \frac{1}{m}J'(v_n)v_n + o_n(1)\\
&= \frac{1}{N}\int_{\mathbb{R}^N} k(x)(v_n)_+^{m^*}\,\mathrm{d}x + \left(\frac{1}{m} - \frac{1}{q+1}\right)\int_{\mathbb{R}^N} h(x)(v_n)_+^{q+1}\,\mathrm{d}x + o_n(1)\\
&\ge \frac{1}{N}\int_{\mathbb{R}^N} k(x)(v_n)_+^{m^*}\,\mathrm{d}x + o_n(1)\\
&\ge \frac{1}{N}\int_{\mathbb{R}^N} \Psi_\varepsilon(x)k(x)(v_n)_+^{m^*}\,\mathrm{d}x + o_n(1),
\end{aligned}$$

taking the limit as $n \to \infty$, we obtain

$$C^{**} \ge \frac{1}{N}\sum_{i\in\Upsilon}\Psi_\varepsilon(x_i)v_i = \frac{1}{N}\sum_{i\in\Upsilon} v_i \ge \|k\|_\infty^{(m-N)/m}\,\frac{S^{N/m}}{N},$$

which is a contradiction. Therefore,

$$\int_{\mathbb{R}^N} \phi(x)k(x)(v_n)_+^{m^*}\,\mathrm{d}x \to 0 \quad \text{as } n \to +\infty \text{ for all } \phi \in C_0^\infty(\mathbb{R}^N),$$

that is,

$$k(v_n)_+^{m^*} \to 0 \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^N) \text{ as } n \to +\infty.$$

This fact, together with assumption (H_3) and the estimate

$$\left|\int_{\mathbb{R}^N} (k(x) - K(x))(v_n)_+^{m^*}\right|$$

$$\leq \left|\int_{|x|\leq R} (k(x) - K(x))v_{n+}^{m^*}\right| + \left|\int_{|x|\geq R} (k(x) - K(x))v_{n+}^{m^*}\right|,$$

implies that Claim 4 holds. □

Using elliptic regularity theory, as it was done in [28], we may show that $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ (see also [13,16], for an adaptation of the results by Trudinger to quasilinear equations). Finally, by the maximum principle or Harnack's inequality, it is standard to prove that $u > 0$ in $\mathbb{R}^N$ (see Theorem 1.2 in [27]).

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