Schrödinger equations with critical nonlinearity, singular potential and a ground state*

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Abstract

We study semilinear elliptic equations in a generally unbounded domain $\Omega \subset \mathbb{R}^N$ when the pertinent quadratic form is nonnegative and the potential is generally singular, typically a homogeneous function of degree -2. We prove solvability results based on the asymptotic behavior of the potential with respect to unbounded translations and dilations, while the nonlinearity is a perturbation of a selfsimilar, possibly oscillating, term f_∞ of critical growth satisfying $f_\infty(\lambda^j s) = \lambda^{\frac{N+2}{N-2}j} f_\infty(s)$, $j \in \mathbb{Z}$, $s \in \mathbb{R}$. This paper focuses on two qualitatively different cases of this problem, one when the quadratic form has a generalized ground state and another where the presence of potential does not change the energy space. In the latter case we allow nonlinearities with oscillatory critical growth. An important example of such quadratic form is the one on $\mathbb{R}^N$ with the radial Hardy potential $-\mu|x|^{-2}$ with $\mu = \mu_*$ in the first case, $\mu < \mu_*$ in the second case, where $\mu_* = \frac{(N-2)^2}{4}$ is the largest constant for which the energy form remains nonnegative.

Key words: Nonlinear Schrödinger equations, generalized ground state, Hardy potential, criticality theory, sign-changing solutions, linking geometry, minimax, critical points.

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1 Introduction

The main goal of this paper is to provide an approach to the stationary nonlinear Schrödinger equation $Lu = f(x,u)$, $L = -\Delta - V(x) \geq 0$, in generally unbounded domains, under somewhat minimal conditions on V and f. We consider two qualitatively different cases. In the first case it is assumed that the operator L possesses a ground state. A typical example of a problem admitting a ground state (which is not an eigenfunction in the classical sense, hence not a L^2-function) involves the Schrödinger operator $L = -\Delta - V(x)$ in $\mathbb{R}^N$, when the potential is the Hardy potential $V(x) = \frac{\lambda}{|x|^2}$ in the limiting situation $\lambda = (\frac{N-2}{2})^2$. Despite the fact that the generalized ground state might not belong to any standard functional space, it gives rise to a linking geometry involving 2-dimensional manifolds, which in turn yields a sign-changing solution (cf. [16]).

The other case that we study is focused on the potential V that does not affect the energy space, which remains $\mathcal{D}^{1,2}(\mathbb{R}^N)$. In this case we use the standard mountain pass argument.

Our starting point is the equation

$$-\Delta u = g(x,u) \quad \text{in} \quad \Omega\,, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a general domain and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $g(x,0) = 0$ and

$$g'_s(x,0) := V(x)$$

exists. Therefore, equation (1.1) can be written as

$$-\Delta u - V(x)u = f(x,u) \quad \text{in} \quad \Omega\,, \tag{1.2}$$

where $f'_s(x,0) := 0$. Let us assume that

$$Q(u) = \int_\Omega \left[|\nabla u|^2 - V(x)u^2\right]\, \mathrm{d}x \geq 0\,, \quad u \in C_0^\infty(\Omega)\,. \tag{1.3}$$

Schrödinger operators with "double criticality", that is, with a non-compact potential term involving a potential of positive homogeneity -2, in particular $\mu|x|^{-2}$, and with the critical nonlinearity $\lambda|u|^{2^*-2}u$, have been addressed already by Terracini [16]. Further results on the problem with some variations, in particular, its extension to different classes of unbounded domains, and the critical nonlinearity $\lambda(x)|u|^{2^*-2}u$, have been obtained in [4], [13] and [14].

The present paper can be regarded as a generalization of the above mentioned work. While we do not presume that the potentials are of Hardy type, the latter emerge naturally in the concentration compactness argument as asymptotic potentials under dilations in presence of a variational penalty condition that compares the potential with its asymptotic value. The second direction in which the prior work is generalized is that the nonlinearity may be regarded as a subcritical perturbation of a critical nonlinearity, where the latter is not restricted to a multiple of $|u|^{2^*-2}u$ (see the discussion below). The third direction of generalization here is that we allow the quadratic form of the potential to have a ground state (which is not necessarily an integrable first eigenfunction), the model situation being the Hardy inequality with the optimal constant. For the sake of simplicity we consider the problem in the whole $\mathbb{R}^N$, but most of the arguments can be extended to general open sets (using truncations like in the celebrated Brezis-Nirenberg problem [3]).

The problems considered here lack compactness due to both translations and dilations. The approach relies on eliminating various types of possible concentrations to obtain convergence of bounded sequences. The elimination mechanisms may include considerations of symmetry (cf. the pioneering paper [15] of Strauss) or, as in our case, arguments in the spirit of the concentration-compactness method of P.L. Lions [7, 8]. Here, we follow the version of concentration-compactness presented in [19] and which was applied in [18] to the case $V = 0$ of the present problem. By contrast with [18] this paper is focused on the role of the potential.

We do not restrict our study to energy spaces contained in $\mathcal{D}^{1,2}(\Omega)$, $N \geq 3$. Instead we allow potentials such that

$$\inf_{u \in C_0^\infty(\Omega),\ \int_B u = 1} Q(u) = 0\,, \tag{1.4}$$

where B is an open bounded set satisfying $\overline{B} \subset \Omega$. It has been shown by [10] that (1.4) implies that every minimizing sequence converges in H^1_{loc} to a unique (up to a scalar multiple) positive solution of

$$-\Delta u - Vu = 0, \quad \text{in} \quad \Omega\,, \tag{1.5}$$

which is called (generalized) ground state. A typical example of a problem that admits a ground state (which is not an eigenfunction in the sense that the terms in (1.3) are no longer integrable, and which is not a L^2-function) is given by the Hardy potential $V(x) = \frac{\lambda}{|x|^2}$ in the limiting situation $\lambda = (\frac{N-2}{2})^2$. The ground state in this case is $|x|^{\frac{2-N}{2}}$, $x \in \Omega = \mathbb{R}^N \setminus \{0\}$, $N \geq 3$.

Concerning coercivity properties of Schrödinger operators with singular potentials, we quote the following result in [12].

Theorem 1.1. *Let Q be a nonnegative functional on $C_0^\infty(\Omega)$ of the form* (1.3) *with a ground state φ. Then φ is the global positive solution (which is a unique supersolution) of* (1.5)*. Moreover, there exists a positive continuous function W such that for every bounded open set B, $\overline{B} \subset \Omega$, the following inequality holds:*

$$Q(u) + C\left|\int_B u\,\mathrm{d}x\right|^2 \geq \left(\int_\Omega W|u|^{2^*}\mathrm{d}x\right)^{2/2^*} \qquad \forall u \in C_0^\infty(\Omega), \tag{1.6}$$

for some suitable constant $C = C(B) > 0$.

In presence of a ground state, the natural energy space for such problems, sometimes denoted by $\mathcal{D}^{1,2}_V(\Omega)$, is the completion of $C_0^\infty(\Omega)$ in the norm

$$\|u\| = \left(Q(u) + \left(\int_B u\mathrm{d}x\right)^2\right)^{1/2}. \tag{1.7}$$

In this paper we find solutions of (1.2) as critical points of the functional

$$J(u) = \frac{1}{2}Q(u) - \int_\Omega F(x,u)\,\mathrm{d}x. \tag{1.8}$$

Note that if one attempts to define the functional (1.8) on the completion of $C_0^\infty(\Omega)$ with respect to the norm $Q(\cdot)^{1/2}$, then one immediately faces the consequences of the fact that the corresponding complete space is no longer continuously imbedded in L^1_{loc} and the ground state belongs to the zero element of the completion. On the other hand, it is known that $\mathcal{D}^{1,2}_V(\Omega) \subset H^1_{loc}(\Omega)$ (see e.g. Proposition 3.1 in [11]). Thus it is natural to consider the functional (1.8) on the space $\mathcal{D}^{1,2}_V(\Omega)$, and to make assumptions on $f(x,s)$ so that the functional (1.8) is continuously differentiable on $\mathcal{D}^{1,2}_V(\Omega)$, and has a linking geometry. In the case when the ground state is an eigenfunction in the classical sense, the appropriate linking geometry involves two-dimensional paths. Here we extend this construction to the case of general ground states. We note that the mountain pass argument (that is, the use of one-dimensional paths) would handle, in a uniform way, both nonnegative and sign-changing nonlinearities. On the other hand, (1.2) with a positive f cannot have a solution, as multiplication of it by a minimizing sequence for (1.4) followed by integration will result in a contradiction.

In the second case, where the quadratic form Q is bounded from above and from below by $\|\nabla u\|_2^2$, one may use the standard mountain pass argument, which allows autonomous nonlinearities of oscillatory critical growth (more general than $F(u) = \lambda|u|^{2^*}$), (see [19], Chapter 5). These are continuous functions verifying the relation

$$f(\lambda^j s) = \lambda^{(2^*-1)j} f(s)\,,\ j \in \mathbb{Z}\,,$$

whereby one starts with an arbitrary continuous function f (possibly oscillatory) defined on $[1,\lambda]$, $\lambda > 1$, and extends it continuously to $(0,+\infty)$ through the relation above, with a similar construction on $(-\infty,0)$ for f given on the interval $[-\lambda,-1]$. As it can easily be seen, such a function has the exact critical growth $|s|^{2^*-1}$, in the sense that there exists constants $C_1, C_2 > 0$ such that $C_1|s|^{2^*-1} \le |f(s)| \le C_2|s|^{2^*-1}$ for all $s \in \mathbb{R}$. Also, the terminology *selfsimilar* could be used to describe such a function as it 'reproduces' in variable scales on the intervals $[\lambda^{j-1}, \lambda^j]$ (and $[-\lambda^j, -\lambda^{j-1}]$) the original f given on $[-\lambda,-1]\cup[1,\lambda]$. An equivalent formulation of the selfsimilarity condition is that the functions

$$g_\pm(t) := \lambda^{-(2^*-1)t} f(\pm\lambda^t), \quad t \in \mathbb{R},$$

are 1-periodic.

Functionals with nonlinearities of critical growth are not weakly continuous and require the use of a concentration-compactness argument which, as the problem is not autonomous, involves comparison with asymptotic problems with regard to both translations and dilations. We use the penalty condition $V > V_\infty$, similar to that introduced by Lions in minimization problems and its adaptations to mountain pass problems. As V_∞ here corresponds to dilational limits as well, it follows from the penalty condition that $V \ge 0$, and moreover, unless all asymptotic limits of V are zero, there must exist a positive asymptotic limit V_∞ homogeneous of degree -2. Therefore, a corresponding concentration-compactness argument will involve comparison with solutions of nonlinear Schrödinger equations with potentials of Hardy type.

The paper is organized as follows. In Section 2 we state the main theorems in this paper. In Section 3 we survey properties of Schrödinger operators with a generalized ground state and of the associated energy space. In Section 4 we prove the existence result for the ground state case. In Section 5 we prove existence of solution of mountain pass type in the case of Hardy-type potentials and oscillatory critical nonlinearity.

2 Statements of the main theorems

For convenience of the reader, we comment below on the main theorems in this paper mentioned in the Introduction, namely Theorem 4.3 and Theorem 5.2.

Theorem 2.1. *Assume the Ambrosetti-Rabinowitz superlinearity condition*

(H_1) $s\,f(x,s) \geq \mu F(x,s) > 0$ *for all* $x \in \Omega$ *and* $|s| \neq 0$ *(and some* $\mu > 2$*)* ,

and

(H_2) $\lim_{|s|\to\infty} \frac{F(x,s)}{s^2} = +\infty$ *uniformly for* x *in compact subsets of* Ω*;*

(H_3') $|f(x,s)| \leq W_1(x)|s|^{p_1-1} + W_2(x)|s|^{p_2-1}$ *for all* $(x,s) \in \Omega \times \mathbb{R}$ *where* $2 < p_1 \leq p_2 < 2^*$ *and* W_i *satisfy*

$$\int_\Omega \left(\frac{W_i^{2^*}}{W^{p_i}}\right)^{\frac{1}{2^*-p_i}} \mathrm{d}x < \infty \quad \textit{for} \quad i = 1, 2\,,$$

(and we recall that W *is given in* (1.6)*);*

In case $-\Delta u - V(x)u = 0$ *has a (generalized) ground state (in particular when* $V(x) = \mu\frac{1}{|x|^2}$ *with* $\mu = (\frac{N-2}{2})^2$*), then problem*

$$-\Delta u - V(x)u = f(x,u) \quad \textit{in} \quad \Omega \tag{2.1}$$

has a (weak) nonzero solution in $D_V^{1,2}(\Omega)$.

Regarding our second main theorem, the following well-known result is an immediate corollary of Theorem 5.2:

Corollary 2.2. *Let* $N \geq 3$, $2 < p < 2^* := \frac{2N}{N-2}$, *and* $0 < \mu < \left(\frac{N-2}{2}\right)^2$. *Then problem*

$$-\Delta u - \mu\frac{1}{|x|^2}\,u = |u|^{2^*-2}u$$

has a non-zero solution $u \in D^{1,2}(\mathbb{R}^N)$.

We refer the reader to [17] for the endpoint case $\mu = \left(\frac{N-2}{2}\right)^2$.

Results of this paper also include a Sobolev-type inequality in the energy space of the optimal Hardy inequality, Proposition 3.1, and a condition of weak continuity of functionals in such energy spaces, Lemma 4.1.

3 A singular Schrödinger operator with a "large" ground state

Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded domain and let $D > \sup_{x\in\Omega}|x|$. The inequality below is due [5, Theorem A] by Filippas and Tertikas, with the correction [6] that excludes from the statment the endpoint value $D = \sup_{x\in\Omega}|x|$. The counterexample to the endpoint case was presented by Musina in [9], and the definitive version of the theorem, elaborating dependence of the constant (3.1) on D, was provided by Adimurthi, Filippas and Tertikas as [1, Theorem B].

For all $u \in C_0^\infty(\Omega)$,

$$\int_\Omega |\nabla u|^2 \mathrm{d}x - \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2}\mathrm{d}x \geq C_D \left(\int_\Omega |u|^{2^*}\left(\log\left(\frac{|x|}{D}\right)\right)^{\frac{2-2N}{N-2}} \mathrm{d}x\right)^{2/2^*}. \tag{3.1}$$

It is also proved in [5] that the exponent $(2N-2)/(N-2)$ cannot be decreased. Here we extend this estimate to the whole $\mathbb{R}^N$. @@We set the following bounded function that vanishes at zero and at infinity.

$$\eta_D(r) = 1/\log(D\max\{r, r^{-1}\}) \quad r > 0. \tag{3.2}$$

Proposition 3.1. *Let $D > 1$. There exists $C_D > 0$ such that for every $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$*

$$\left(\int_B u \mathrm{d}x\right)^2 + \int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x - \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{u^2}{|x|^2}\mathrm{d}x \geq C_D \left(\int_{\mathbb{R}^N} |u|^{2^*} \eta_D(|x|)^{\frac{2N-2}{N-2}} \mathrm{d}x\right)^{2/2^*}. \tag{3.3}$$

Proof: Let

$$Q_N(u) := \int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x - \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{u^2}{|x|^2}\mathrm{d}x.$$

Let us prove the inequality first for radially symmetric functions. Using (3.1) on a unit ball centered at the origin, repeating it for the function $r^{2-N}u(1/r)$ with u with support in the exterior of the ball, and adding the two inequalities, we immediately obtain (3.3) for all radial functions $u \in C_0^\infty(\mathbb{R}^N \setminus (S^{N-1} \cup \{0\}))$ where S^{N-1} is the $N-1$-dimensional unit sphere centered at the origin. Then, by an elementary density argument, (3.3) holds for all radial functions $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ such that $u(1) = 0$, and therefore,

$$u(1)^2 + Q_N(u) \geq C \left(\int_{\mathbb{R}^N} |u|^{2^*} \eta_D(r)^{\frac{2N-2}{N-2}} \mathrm{d}x\right)^{2/2^*} \tag{3.4}$$

@@is true for all radial functions $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$. Note, however, that for any bounded set $B \subset \mathbb{R}^N$, there exists a constant $C > 0$ such that

$$u(1)^2 \leq CQ(u) + C\left(\int_B u \mathrm{d}x\right)^2. \tag{3.5}$$

Indeed, the right hand side defines an equivalent $\mathcal{D}_V^{1,2}(\mathbb{R}^N)$-norm (with $V = \left(\frac{N-2}{2}\right)^2 \frac{1}{|x|^2}$), and $\mathcal{D}_V^{1,2}(\mathbb{R}^N)$ is continuously imbedded into $H^1_{\mathrm{loc}}(\mathbb{R}^N)$. In particular, this implies that restriction of u to an annulus $\lambda^{-1} < r < \lambda$ with any $\lambda > 0$ has a bounded H^1-norm, which implies that $u \mapsto u(1)$ is a continuous functional in the subspace of radial functions of $\mathcal{D}_V^{1,2}(\mathbb{R}^N)$, which yields (3.5). From here and (3.4) follows (3.3) for radial functions.

Let now $Pu(r) = \omega_N^{-1}\int_{S^{N-1}} u(r,\omega)\mathrm{d}\omega$, where ω_N is the area of S^{N-1}, and (r,ω) are polar coordinates in $\mathbb{R}^N$. Note that if $Pu = 0$, then

$$\int_{\mathbb{R}^N} |\nabla u|^2 \geq \left[\left(\frac{N-2}{2}\right)^2 + \lambda_1\right] \int_{\mathbb{R}^N} \frac{u^2}{|x|^2}\mathrm{d}x,$$

where $\lambda_1 > 0$ is the first positive eigenvalue of Laplace-Beltrami operator on S^{N-1}. This implies that, whenever $Pu = 0$,

$$Q_N(u) \geq C \left(\int_{\mathbb{R}^N} |u|^{2^*} \mathrm{d}x \right)^{2/2^*} . \tag{3.6}$$

Inequality (3.3) for general functions u follows now from its restriction to the radial functions Pu combined with (3.6) for $(I - P)u$. ■

4 Generalized ground state and linking geometry

Consider a generalized ground state φ of (1.5) and the functional $J(u)$ defined in (1.8) for $u \in \mathcal{D}_V^{1,2}(\Omega)$. The following assumptions will be made on the nonlinearity $F(x,s) = \int_0^s f(x,t)\,\mathrm{d}t$, where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous:

(H_1) $s\,f(x,s) \geq \mu F(x,s) > 0$ for all $x \in \Omega$ and $|s| \neq 0$ (and some $\mu > 2$) ;

(H_2) $\lim_{|s|\to\infty} \frac{F(x,s)}{s^2} = +\infty$ uniformly for x in compact subsets of Ω;

(H_3) $|F(x,s)| \leq W_1(x)|s|^{p_1} + W_2(x)|s|^{p_2}$ for all $(x,s) \in \Omega \times \mathbb{R}$, where $2 < p_1 \leq p_2 < 2^*$ and W_i satisfy

$$\int_\Omega \left(\frac{W_i^{2^*}}{W^{p_i}} \right)^{\frac{1}{2^*-p_i}} \mathrm{d}x < \infty \quad \text{for} \quad i = 1,2,$$

and we recall that W is given in (1.6);

Let $\varphi_j \in C_0^\infty(\Omega)$ is an approximating sequence for the generalized ground state φ satisfying

$$Q(\varphi_j) \to 0 \quad \text{and} \quad 0 \leq \varphi_j \to \varphi \quad \text{uniformly on conpact subsets of} \quad \Omega. \tag{4.1}$$

For existence of such a sequence see [11].

Lemma 4.1. *Under the assumption (H_3) the functional $K(u) = \int_\Omega F(x,u)\mathrm{d}x$ is weakly continuous on $\mathcal{D}_V^{1,2}(\Omega)$.*

Proof: We first note that, for any measurable set $A \subset \Omega$, we have by (1.6), (1.7) and Hölder inequality that

$$\begin{aligned}\int_A W_i|u|^{p_i}\,\mathrm{d}x &= \int_A \frac{W_i}{W^{\frac{p_i}{2^*}}} W^{\frac{p_i}{2^*}} |u|^{p_i}\,\mathrm{d}x \\ &\leq \left(\int_A \left(\frac{W_i^{2^*}}{W^{p_i}} \right)^{\frac{1}{2^*-p_i}} \mathrm{d}x \right)^{\frac{2^*-p_i}{2^*}} \left(\int_\Omega W|u|^{2^*}\,\mathrm{d}x \right)^{\frac{p_i}{2^*}} \\ &\leq \left(\int_A \left(\frac{W_i^{2^*}}{W^{p_i}} \right)^{\frac{1}{2^*-p_i}} \mathrm{d}x \right)^{\frac{2^*-p_i}{2^*}} \|u\|^{p_i} .\end{aligned} \tag{4.2}$$

Let $u_n \rightharpoonup u$ weakly in $\mathcal{D}_V^{1,2}(\Omega)$. Then $\|u_n\| \le C$ and, given $\alpha > 0$, we pick $R > 0$ sufficiently large so that, in view of (H_3) with $A_R = \Omega \setminus (\Omega \cap B_R)$, we have

$$\left(\int_{A_R} \left(\frac{W_i^{2^*}}{W^{p_i}} \right)^{\frac{1}{2^*-p_i}} \mathrm{d}x \right)^{\frac{2^*-p_i}{2^*}} < \frac{\varepsilon}{6\, C^{p_i}} \,, \quad i = 1, 2 \,. \tag{4.3}$$

It follows from (H_3), (4.2) and (4.3) that

$$\begin{aligned} |K(u_n) - K(u)| &= \left| \int_\Omega F(x, u_n) - F(x, u) \,\mathrm{d}x \right| \\ &\le \int_{\Omega \cap B_R} |F(x, u_n) - F(x, u)| \,\mathrm{d}x + \frac{2\varepsilon}{3} \,. \end{aligned} \tag{4.4}$$

Finally, since $\Omega \cap B_R$ is bounded and again in view of (H_3), there exists $N \in \mathbb{N}$ such that the last integral above is less than $\varepsilon/3$, and it follows from (4.4) that

$$|K(u_n) - K(u)| < \varepsilon \quad \forall\, n \ge \mathbb{N} \,.$$

The proof is complete. ■

Remark 4.1. Similarly to the above lemma, one can show that $K'(u) \in (D_V^{1,2}(\Omega))'$ is well defined for all $u \in \mathcal{D}_V^{1,2}(\Omega)$ and $u \mapsto K'(u)$ is a compact map from $\mathcal{D}_V^{1,2}(\Omega)$ to $(\mathcal{D}_V^{1,2}(\Omega))'$ provided that the following condition holds:

(H_3') $|f(x,s)| \le W_1(x)|s|^{p_1-1} + W_2(x)|s|^{p_2-1}$ for all $(x,s) \in \Omega \times \mathbb{R}$ where, as before, $2 < p_1 \le p_2 < 2^*$ and W_i satisfy

$$\int_\Omega \left(\frac{W_i^{2^*}}{W^{p_i}} \right)^{\frac{1}{2^*-p_i}} \mathrm{d}x < \infty \quad \text{for} \quad i = 1, 2 \,.$$

Lemma 4.2. *Assume $(H_1)-(H_3)$. Let B be an open set, $\overline{B} \subset \Omega$, $\omega \in C_0^\infty(B)$ is a nonzero function satisfying*

$$\int_B \omega(x) \,\mathrm{d}x = 0 \,, \tag{4.5}$$

and let φ_j be as in (4.1) *and $D_{j,R}^+ := \{ u = t\varphi_j + s\omega \,|\, s \ge 0 \,,\, s^2 + t^2 \le R^2 \}$. Then*

(i) There exist $\rho > 0$ and $\alpha > 0$ such that $J(u) \ge \alpha > 0$ if $\|u\| = \rho$ and $\int_B u \,\mathrm{d}x = 0$;

(ii) $\max_{u \in \partial D_{j,R}^+} J(u) \le \frac{\alpha}{2}$ if $j \in \mathbb{N}$ and $R > 0$ are sufficiently large, where $\partial D_{j,R}^+$ denotes the boundary of $D_{j,R}^+$ in span $\{\varphi_j \,,\, \omega\}$.

Proof: (*i*) Let $W := \{\, u \in D_V^{1,2}(\Omega) \mid \int_B u \ \mathrm{d}x = 0 \,\}$ and consider the sphere $S_\rho := \{\, u \in D_V^{1,2}(\Omega) \mid \|u\| = \rho$ and $\int_B u \ \mathrm{d}x = 0 \,\}$ in W. Using (H_3) and the estimate (4.2), we have for some constant $C_{1,2} > 0$ and $p = \min\{\, p_1,\, p_2 \,\}$ that

$$J(u) \geq \frac{1}{2}\rho^2 - C_{1,2}\rho^p \,,$$

for all $u \in S_\rho$, provided $\rho < 1$. By taking $\rho > 0$ suitably small, it follows that

$$J(u) \geq \alpha > 0 \quad \text{for all} \quad u \in S_\rho \,. \tag{4.6}$$

(*ii*) Let $u = t\varphi_j + s\omega \in \partial D^+_{j,R}$. Consider first $s = 0$, $|t| \leq R$. From (H_1) and (4.1), we have

$$J(t\varphi_j) = \frac{1}{2}t^2 Q(\varphi_j) - \int_\Omega F(x, t\varphi_j) \ \mathrm{d}x \leq \frac{1}{2}R^2 Q(\varphi_j) \longrightarrow 0 \quad \text{as} \quad j \to \infty \,.$$

Fix $R > \rho$. From the above estimate we can take $j = j(R) \in \mathbb{N}$ sufficiently large so that

$$J(t\varphi_j) \leq \frac{\alpha}{2} \,. \tag{4.7}$$

Next, we consider $u = t\varphi_j + s\omega \in \partial D^+_{j,R}$ with $s \geq 0$ and $s^2 + t^2 = R^2$, and define the set $A_\delta := \{\, x \mid \operatorname{sgn}(t)\omega(x)\varphi_j(x) \geq \delta \,\} \subset \Omega$. Note that $A_\delta \neq \emptyset$ by picking $\delta > 0$ small (say $0 < \delta < \min\{\, \sup(\omega^+, \varphi_j)\,,\, \sup(\omega^-, \varphi_j) \,\}$). Using (H_1) and (H_2) it follows that

$$\begin{aligned} J(t\varphi_j + s\omega) &\leq \frac{1}{2}Q(t\varphi_j + s\omega) - \int_{A_\delta} F(x, t\varphi_j + s\omega) \ \mathrm{d}x \\ &\leq C\,R^2 - \int_{A_\delta} F(x, t\varphi_j + s\omega) \ \mathrm{d}x \\ &\leq C\,R^2 - M\,R^2 \\ &\leq 0\,, \end{aligned} \tag{4.8}$$

by taking $M = M(R) \geq C$. The proof of (*ii*) (hence, of Lemma 4.2) is complete in view of (4.7) and (4.8) . ■

Theorem 4.3. *Assume conditions* $(H_1), (H_2), (H_3')$. *Then problem* (1.2) *has a (weak) non-zero solution in* $D_V^{1,2}(\Omega)$*, i.e.,* (1.8) *has a critical point* $u \in D_V^{1,2}(\Omega) \setminus \{0\}$.

Proof: As observed in Remark 4.1, the functional $K(u) = \int_\Omega F(x,u) \ \mathrm{d}x$ is continuously differentiable on the space $\mathcal{D}_V^{1,2}(\Omega)$ with $u \mapsto K'(u)$ being a compact mapping. Consider the following class Φ of deformations of the two-dimensional half-disk $D^+_{j,R}$, where j and R are chosen as in Lemma 4.2, and φ_j and ω are as in (4.1) and (4.5) respectively:

$$\Phi := \{\ h \in C(D^+_{j,R}, D_V^{1,2}) \mid h(u) = u \text{ if } u \in \partial D^+_{j,R} \ \} \,.$$

As is well-known, the minimax value

$$c := \inf_{h\in\Phi} \max_{u\in D^+_{j,R}} J(h(u))$$

satisfies $c \geq \alpha > 0$ and there exists a Palais-Smale sequence u_k at the level c, i.e., one has

$$J(u_k) \longrightarrow c > 0\ ,\ J'(u_k) \longrightarrow 0\,.$$

Since we have $J'(u) = u - K'(u)$, with K' a compact mapping by (H_3'), a standard argument shows that there exists $u \in \mathcal{D}^{1,2}_V(\Omega)$ such that $u_k \longrightarrow u$. Thus $u \in \mathcal{D}^{1,2}_V(\Omega)$ is a critical point of J, i.e., u is a (weak) solution of (1.8). As usual, $u \neq 0$ since $J(u) = c > 0$. The proof is complete. ■

5 Schrödinger equation with oscillatory critical nonlinearity and singular potential

Let $\lambda > 1$ and $N \geq 3$. Consider $f(s) \in C(\mathbb{R})$ satisfying the following assumptions:

(F_1) $f(s) = f_\infty(s) + o(|s|^{2^*-1})$ as $|s| \to 0$ or $|s| \to \infty$, where f_∞ satisfies $\sup_{s>0} f_\infty(s) > 0$ and the selfsimilar relation

(F_2) $f_\infty(\lambda^j s) := \lambda^{\frac{N+2}{N-2}j} f_\infty(s)$, $j \in \mathbb{Z}$;

(F_3) $F(s) = \int_0^s f$ and $F_\infty(s) = \int_0^s f_\infty$ are such that

$$F(s) > F_\infty(s)\,, s \neq 0;$$

(F_4) There exists $\delta > 0$ such that $\frac{f(s)}{s|s|^\delta}$ is a nondecreasing function for all $s > 0$ and is nonincreasing for $s < 0$.

Remark 5.1. In what follows we assume that $V > 0$. This is not a substantial restriction, since from a particular case of the penalty condition (V_1) below, namely, $V(x) \geq \lim_{s\to 0} 4^s V(2^s x)$, it follows that $V \geq 0$, provided that V has a point of continuity (which we readily assume and set, without loss of generality, at the origin).

Let $\mathcal{M}$ denote the Banach space of continuous multipliers from $\mathcal{D}^{1,2}(\mathbb{R}^N)$ to $L^2(\mathbb{R}^N)$. We assume that $V^{\frac{1}{2}} \in \mathcal{M}$ and that for any sequence @@$(y_k, s_k) \in \mathbb{R}^N \times \mathbb{R}$ with $|s_k| \to \infty$, there exists $\widehat{y} \in \mathbb{R}^N$ such that, a renamed subseqence of $2^{s_k} V^{1/2}(2^{s_k}(x + \widehat{y}) - y_k)$ converges in the metric of $\mathcal{M}$ (and, consequently, almost everywhere). In what follows we will say that a sequence of potentials $V_k \geq 0$ converges in the sense of $\mathcal{M}$ if the sequence $V_k^{1/2}$ converges in $\mathcal{M}$.

We will assume the following penalty conditions on V:

(V_1) $V(x) \geq V_\infty(x) := \lim 4^{s_k} V(2^{s_k}(x + \widehat{y}) - y_k)$, where V_∞ is any of the subsequential limits defined above;

(V_2) $\lim_{|y|\to\infty} V(\cdot - y) = 0$ in the sense of $\mathcal{M}$;

(V_3) there exists an $\epsilon \in (0,1)$ such that, for every $u \in C_0^\infty(\Omega)$,

$$(1-\epsilon)\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} V(x)u^2 \,\mathrm{d}x \geq 0.$$

Above as well as and in what follows we use the notations V_∞ and $\hat{y}$ with the understanding that these are relative to the sequence (y_k, s_k). @@Note that $V_\infty < \infty$ a.e., since otherwise it is easy to show that the quadratic from Q has negative values, contrary to our assumption.

Example 5.1. *Conditions (V_1) - (V_3) are satisfied, in particular, by the Hardy potential*

$$V_\mu(x) = \mu \frac{1}{|x|^2}\,,\ 0 < \mu < \left(\frac{N-2}{2}\right)^2.$$

Note that (V_1) and (V_2) are also satisfied by finite sums of Hardy potentials of the form $V_\mu(\cdot - y)$, $y \in \mathbb{R}^N$, possibly perturbed by positive lower order potentials (i.e., $h(\cdot - y)$ satisfying $h(x) = o(H_\nu(x))$ as either $|x| \to 0$ or $|x| \to \infty$). If any such potential is multiplied by a sufficiently small scalar, condition (V_1) is also satisfied. In particular, we point out that all the conditions (F_1)-(F_4) and (V_1)-(V_3) are satisfied in the model problem

$$-\Delta u - V_\mu(x)\, u = |u|^{2^*-2}u, \quad u \in D^{1,2}(\mathbb{R}^N)\,,$$

where $N \geq 3$, $2 < p < 2^* := \frac{2N}{N-2}$, and $0 < \mu < \left(\frac{N-2}{2}\right)^2$.

Remark 5.2. A connection of potentials satisfying (V_1) with the Hardy potential is not accidental, since every potential V_∞ given by (V_1) is a positive homogeneous function of degree -2 relative to the origin at $\widehat{y}$. Indeed, the limit in (V_1) does not change if we replace s with $s+r$, $r \in \mathbb{R}$. From this it follows immediately that $V_\infty(2^r(x+\widehat{y})) = 4^r V_\infty(x+\widehat{y})$.

On the other hand, condition (V_2) excludes some well-studied Hardy-type potentials such as $V(x) = \frac{\mu}{x_1^2}$, $\mu \in (0, \frac{1}{4})$. This is not a shortcoming of the method used in the paper, but serves the authors' intention to study here the core problem rather than the one in full generality.

Now let us consider a limiting quadratic form

$$Q_\infty(u) = \int_{\mathbb{R}^N} \left(|\nabla u|^2 - V_\infty(x)u^2\right) \,\mathrm{d}x\,. \tag{5.1}$$

¿From (V_3) and the positivity of V it follows that the norms associated with the respective quadratic forms (1.3) and (5.1) in the spaces $\mathcal{D}_V^{1,2}(\mathbb{R}^N)$ and $\mathcal{D}_{V_\infty}^{1,2}(\mathbb{R}^N)$ are equivalent to the usual gradient norm in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Define

$$\psi(u) = \int_{\mathbb{R}^N} F(u)\,\mathrm{d}x\ ,\ \psi_\infty(u) = \int_{\mathbb{R}^N} F_\infty(u)\,\mathrm{d}x$$

and

$$J(u) = \frac{1}{2}Q(u) - \psi(u)\ ,\ J_\infty(u) = \frac{1}{2}Q_\infty(u) - \psi_\infty(u). \tag{5.2}$$

Also, from (F_1), (F_2) and (V_3) it follows that J and J_∞ are C^1-functionals on $\mathcal{D}^{1,2}(\mathbb{R}^N)$. It is easy to conclude from (F_1) and $\sup F_\infty > 0$ in (F_3), that there exists a point $e \in \mathcal{D}^{1,2}$ such that $J_\infty(e) \leq 0$. We define

$$\Phi = \{v_t \in C([0,1], \mathcal{D}^{1,2}) \mid v_0 = 0, v_1 = e\}, \tag{5.3}$$

and

$$c := \inf_{v_t \in \Phi} \max_{t \in [0,1]} J(v_t) \ , \ c_\infty := \inf_{v_t \in \Phi} \max_{t \in [0,1]} J_\infty(v_t). \tag{5.4}$$

Remark 5.3. Note that even without assuming the penalty conditions (F_3), (V_1) and (V_2) one always has the non-strict inequality $c \leq c_\infty$ (we recall that generally these are multiple inequalities), since one can consider the mountain path statement (5.4) for c restricted to nearly-optimal paths for c_∞, under suitable dislocations, that is, paths of the form $\gamma^{\frac{N-2}{2}j} v_t(\gamma^j \cdot + y)$ with large $|j|$ or $|y|$. The proof of this statement is analogous to that given in [18] in the case $V = 0$, and is omitted.

Theorem 5.2. *Assume* (F_1) - (F_4) *and* (V_1) - (V_3). *Then there exists* $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ *such that* $J(u) = c$ *and* $J'(u) = 0$.

Proof: 1. The standard mountain pass argument implies that there exists a sequence $u_k \in D^{1,2}(\mathbb{R}^N)$ such that $J(u_k) \to c$ and $J'(u_k) \to 0$. It follows from (F_4) (which implies the Ambrosetti-Rabinowitz condition of the form (H_1)) that u_k is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

2. Consider now the renamed subsequence of u_k given by Theorem 6.1 below together with the corresponding $w^{(n)}$, sequences $j_k^{(n)} \in \mathbb{Z}$ and $y_k^{(n)} \in \mathbb{R}^N$. Due to (6.3), Lemma 6.2 and Lemma 6.3, we have the following lower estimate for c:

$$c = \lim J(u_k) \geq J(w^{(1)}) + \sum_{n \geq 2} J_\infty^{(n)}(w^{(n)}) \,. \tag{5.5}$$

Note that $w^{(1)}$ is necessarily a critical point of J, and the functions $w^{(n)}$, $n \geq 2$, are critical points of corresponding functionals $J_\infty^{(n)}$, which are the asymptotic functionals J_∞ from (5.2) relative to the sequences $(j_k^{(n)}, y_k^{(n)})$. When $j_k^{(n)} = 0$, the functional $J_\infty^{(n)}$ has the nonlinearity $F_\infty^{(n)} = F$ and the potential $V_\infty^{(n)} = 0 < V$ due to (V_2). In the remaining case, when $|j_k^{(n)}| \to \infty$, the nonlinearity is $F_\infty^{(n)} = F_\infty < F$ by (F_3) while the potential $V_\infty^{(n)} = 0 \leq V$ by (V_1). Therefore, for any $n \geq 2$, and any $w \neq 0$, $J(w) < J_\infty^{(n)}(w)$.

3. From the Ambrosetti-Rabinowitz condition (H_1) which is here a consequence of (F_4), it follows that $J_\infty^{(n)}(w^{(n)}) \geq 0$ for every n. Assume now that there is $m \geq 2$ such that $w^{(m)} \neq 0$. Then, using (F_4) in order to show that the function $t \mapsto J_\infty^{(m)}(tw^{(m)})$ has a unique critical point (that of maximum), which is necessarily $t = 1$ since $w^{(m)}$ is a critical point of $J_\infty^{(m)}$, we infer from (5.5) that

$$J_\infty^{(m)}(w^{(m)}) \leq c \leq \max_t J(tw^{(m)}) < \max_t J_\infty^{(m)}(tw^{(m)}) = J_\infty^{(m)}(w^{(m)}) \,,$$

which is a contradiction. Consequently, by (6.4) we have that $u_k \to w^{(1)}$ in L^{2^*}. Finally, from the relation $J'(u_k) \to 0$ in $\mathcal{D}^{1,2}$, it follows that u_k converges in $\mathcal{D}^{1,2}$, which concludes the proof. ■

6 Appendix: Weak convergence decomposition in $\mathcal{D}^{1,2}(\mathbb{R}^N)$

The following theorem is Theorem 5.1 in [19], with the dilation factor 2 replaced by general γ.

Theorem 6.1. *Let $u_k \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, $N \geq 3$, be a bounded sequence. Let $\gamma > 1$. There exist $w^{(n)} \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, $y_k^{(n)} \in \mathbb{R}^N$, $j_k^{(n)} \in \mathbb{Z}$ with $k, n \in \mathbb{N}$, and disjoint sets $\mathbb{N}_0, \mathbb{N}_{+\infty}, \mathbb{N}_{-\infty} \subset \mathbb{N}$, such that, for a renumbered subsequence of u_k,*

$$w^{(n)} = \text{weak } \lim \gamma^{-\frac{N-2}{2} j_k^{(n)}} u_k(\gamma^{-j_k^{(n)}}(\cdot + y_k^{(n)})),\ n \in \mathbb{N}, \tag{6.1}$$

$$|j_k^{(n)} - j_k^{(m)}| + |y_k^{(n)} - y_k^{(m)}| \to \infty \text{ for } n \neq m, \tag{6.2}$$

$$\sum_{n\in\mathbb{N}} \|w^{(n)}\|^2_{\mathcal{D}^{1,2}} \leq \limsup \|u_k\|^2_{\mathcal{D}^{1,2}}, \tag{6.3}$$

$$u_k - \sum_{n\in\mathbb{N}} \gamma^{\frac{N-2}{2} j_k^{(n)}} w^{(n)}(\gamma^{j_k^{(n)}} \cdot - y_k^{(n)}) \to 0 \ \text{ in } L^{2^*}(\mathbb{R}^N), \tag{6.4}$$

and the series above converges uniformly in k.

Furthermore, $1 \in \mathbb{N}_0$, $y_k^{(1)} = 0$; $j_k^{(n)} = 0$ whenever $n \in \mathbb{N}_0$; $j_k^{(n)} \to -\infty$ (resp. $j_k^{(n)} \to +\infty$) whenever $n \in \mathbb{N}_{-\infty}$ (resp. $n \in \mathbb{N}_{+\infty}$); and $y_k^{(n)} = 0$ whenever $|y_k^{(n)}|$ is bounded.

The following statement is an elementary modification of Lemma 5.6 in [19].

Lemma 6.2. *Assume (F_1) and (F_2) with $\lambda = \gamma^{(N-2)/2}$. Let $u_k, w^{(n)} \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, $y_k^{(n)} \in \mathbb{R}^N$, $j_k^{(n)} \in \mathbb{Z}$, and $\mathbb{N}_0, \mathbb{N}_{+\infty}, \mathbb{N}_{-\infty} \subset \mathbb{N}$, be as provided by Theorem 6.1. Then*

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} F(u_k) = \sum_{n\in\mathbb{N}_0} \int_{\mathbb{R}^N} F(w^{(n)}) + \sum_{n\in\mathbb{N}_{+\infty}\cup\mathbb{N}_{-\infty}} \int_{\mathbb{R}^N} F_\infty(w^{(n)}). \tag{6.5}$$

Lemma 6.3. *Let $V \geq 0$ and $V^{\frac{1}{2}} \in \mathcal{M}$. Assume ($V_2$) and assume that the limit $V_\infty(x)$ in (V_1) exists for every x. Let $u_k, w^{(n)} \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, $y_k^{(n)} \in \mathbb{R}^N$ and $j_k^{(n)} \in \mathbb{Z}$, $y_k^{(n)} \in \mathbb{R}^N$ be as provided by Theorem 6.1. Let $Q^{(n)} = Q_\infty$ be as in (5.1) relative to the sequence $(j_k^{(n)}, y_k^{(n)})$. Then*

$$\limsup_{k\to\infty} Q(u_k) \geq \sum_{n\in\mathbb{N}} \int_{\mathbb{R}^N} Q^{(n)}(w^{(n)})^2\,. \tag{6.6}$$

The proof of the lemma follows easily from the bilnear expansion of the form Q evaluated on the left hand side of (6.4) By continuity it suffices to consider finitely many terms. It is easy to see that the mixed terms in the expansion vanish in the limit. The limits of $Q(\gamma^{\frac{N-2}{2} j_k^{(n)}} w^{(n)}(\gamma^{j_k^{(n)}} \cdot - y_k^{(n)}))$ can be easily evaluated by a linear change of variable in the integral and the use of the definition of V_∞ in (V_1).

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