Cocompactness and minimizers for inequalities of Hardy-Sobolev type involving N-Laplacian

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Abstract

The paper studies quasilinear elliptic problems in the Sobolev spaces $W^{1,p}(\Omega)$, $\Omega \subset \mathbb{R}^N$, with $p = N$, that is, the case of Pohozhaev-Trudinger-Moser inequality. Similarly to the case $p < N$ where the loss of compactness in $W^{1,p}(\mathbb{R}^N)$ occurs due to dilation operators $u \mapsto t^{(N-p)/p} u(tx)$, $t > 0$, and can be accounted for in decompositions of the type of Struwe's "global compactness" and its later refinements, this paper presents a previously unknown group of isometric operators that leads to loss of compactness in $W_0^{1,N}$ over a ball in $\mathbb{R}^N$.

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We give a one-parameter scale of Hardy-Sobolev functionals, a "$p = N$"-counterpart of the Hölder interpolation scale, for $p > N$, between the Hardy functional $\int \frac{|u|^p}{|x|^p}\,\mathrm{d}x$ and the Sobolev functional $\int |u|^{pN/(N-mp)}\,\mathrm{d}x$. Like in the case $p < N$, these functionals are invariant with respect to the dilation operators above, and the respective concentration-compactness argument yields existence of minimizers for $W^{1,N}$-norms under Hardy-Sobolev constraints.
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1 Introduction

Dirichlet forms in dimension 2, or, more generally, quasilinear elliptic problems set in the Sobolev space $W^{1,p}$ with $p = N$ in dimension N, are of major interest to researchers in partial differential equations, mathematical physics, calculus of variations and functional analysis, particularly because this case is not as well understood as and is quite different from the case of Sobolev spaces with $N > p$. The counterpart of Sobolev imbeddings in this case is the Pohozhaev-Trudinger-Moser inequality (see Pohozhaev [10], Trudinger [16], Moser [9]) and the study of problems involving the correspondent nonlinearity $e^{\lambda|u|^{\frac{N}{N-1}}}$ often cannot draw on properties of the critical Sobolev nonlinearity $|u|^{\frac{pN}{N-p}}$ for $N > p$. As an example of one of numerous pecularities of the Pohozhaev-Trudinger-Moser functional one can mention that it is a continuous functional on $W^{1,N}$, and there is a $R > 0$ such that it is bounded on balls of radius less than R and is unbounded on balls of radius greater than R.

In this paper we construct dilation operators which act isometrically in $W_0^{1,N}$ over a unit ball, and are a natural analog of the dilation operators $h_t u \stackrel{\mathrm{def}}{=} t^{\frac{N-p}{p}} u(t\cdot)$ in $W^{1,p}(\mathbb{R}^N)$, $N > p$. These dilations preserve the Sobolev norm and the weighted L^p-norm with the Hardy potential ($|x|^{-p}$ for $N > p$, $(|x| \log \frac{1}{|x|})^{-N}$ for $p = N$), as well as the L^{p^*} norm (for $N > p$) but they *do not* preserve the Pohozhaev-Trudinger-Moser term.

We give, in terms of the dilations above, a weak continuity statement for the Pohozhaev-Trudinger-Moser nonlinearity, Lemma 2.4. A similar result is proved also for the weighted critical nonlinearities of Hardy-Sobolev type. We also show existence of minimizers for the Hardy-Sobolev inequality. Similar existence for the Pohozhaev-Trudinger-Moser nonlinearity has been proved by Carleson and Chang [5] (see also de Figueiredo, do Ó and Ruf [6]).

We also prove a structural theorem for bounded sequences in $W_0^{1,N}(B)$, where $B = B_1(0)$ is a unit ball in $\mathbb{R}^N$, which is similar to Struwe's "global compactness" [12] and its subsequent refinements known in the case $N > p$. (Note that global compacntess results in [11][3] while providing asymptotic behavior of bounded sequences in $W_0^{1,N}$, use the different blowup transformations than this paper and end up with asymptotic profiles supported in the whole $\mathbb{R}^N$.) Our starting point for this result is the following version of global compactness, see [14, Theorem 5.1]:

Any bounded sequence u_k in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, $N > 2$, has a renumbered subsequence such that, defining $h_t u \stackrel{\mathrm{def}}{=} t^{\frac{N-p}{p}} u(t\cdot)$,

$$u_k - \sum_{n\in\mathbb{N}} h_{t_k^{(n)}}[w^{(n)}(\cdot - y_k^{(n)})] \to 0 \text{ in } L^{2N/(N-2)}, \tag{1.1}$$

with some $w^{(n)} \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, and some sequences $t_k^{(n)} > 0$, $y_k^{(n)} \in \mathbb{R}^N$, such that the terms in the expansion have asymptotically disjoint supports. Moreover,

$$w^{(n)} = \text{ w-lim } [h_{1/t_k^{(n)}} w^{(n)}](\cdot + y_k^{(n)})$$

and

$$\sum \|w^{(n)}\|^2_{\mathcal{D}^{1,2}} \le \limsup \|u_k\|^2_{\mathcal{D}^{1,2}}.$$

In other words, while the Sobolev imbedding of $\mathcal{D}^{1,2}(\mathbb{R}^N)$ into $L^{2N/(N-2)}(\mathbb{R}^N)$ is not compact, every bounded sequence in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ possesses a subsequence convergent in $L^{2N/(N-2)}(\mathbb{R}^N)$ after subtraction of "fugitive" terms, that is, the defect of compactness is structured by the means of sequences of translations and dilations acting upon arbitrary asymptotic profiles.

Note that in restriction to radially symmetric functions the decomposition (1.1) does not contain translations, that is, $y_k^{(n)} = 0$, and all dilations occur about the origin. If Ω is bounded, there are also no terms with $t_k^{(n)} \to 0$ (see [14, Proposition 5.1, Lemma 5.4]).

Let us fix the following equivalent norm in $W_0^{1,N}(\Omega)$:

$$\|u\|_{1,N}^N \stackrel{\mathrm{def}}{=} \int_\Omega |\nabla u|^N \,\mathrm{d}x.$$

Then

$$\sup_{u \in W_0^{1,N}(\Omega), \|u\|_{1,N} \le 1} \int_\Omega e^{\alpha_N |u|^{N/(N-1)}} \, dx < \infty, \tag{1.2}$$

where

$$\alpha_N = N \omega_{N-1}^{1/(N-1)}$$

is the largest constant for which (1.2) holds. By ω_{N-1} we denote the area of the unit $N-1$-dimensional sphere. The functional

$$\Phi(u) \stackrel{def}{=} \int_\Omega e^{\alpha_N |u|^{N/(N-1)}} \, dx \tag{1.3}$$

defines an Orlicz space, into which $W_0^{1,N}(\Omega)$ is imbedded continuously, but not compactly. Moreover, unlike in the case $N > p$, the functional Φ is continuous but unbounded on every ball $\|u\| \le r$ with $r > 1$, it is sequentially weakly continuous in the ball $\|u\| \le r$ for any $r < 1$, and is continuous on every sequence in the ball $\|u\| \le 1$ that converges weakly to a nonzero limit. An important question concerning weak continuity of Φ is then its behavior on sequences $\|u_k\| \le 1$ that converge weakly to zero. We refer to [8] for the discussion of weak convergence in this case and in particular for the statements above.

Let us denote the subspace of radial functions in $W_0^{1,N}(B)$ as $W_{0,r}^{1,N}(B)$.

We prove in this paper that for every sequence $u_k \rightharpoonup 0$ in $W_{0,r}^{1,N}(B)$ and every $\lambda > 0$,

$$\forall \mu_k^*, \langle \mu_k^*, u_k \rangle \to 0 \Rightarrow \int_B (e^{\lambda |u_k|^{N/(N-1)}} - 1) \, dx \to 0, \tag{1.4}$$

where μ_k^* is an arbitrary sequence of duality conjugates to the Moser functions μ_t (see definitions in the beginning of Section 2). Furthermore, the left hand side in (1.4) also yields that for every $p > N$,

$$Q_p(u_k) \stackrel{def}{=} \int_B \frac{|u_k|^p}{|x|^N \left(\log \frac{1}{|x|}\right)^{N+(p-N)\frac{N-1}{N}}} \, dx \to 0. \tag{1.5}$$

The latter expression is the left hand side of the natural $W^{1,N}$-counterpart of the Hardy-Sobolev inequality (the Hölder interpolation between the Hardy term $\int_{\mathbb{R}^N} |u|^p |x|^{-p} \, dx$ and the critical Sobolev functional $\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}} \, dx$). Note that, unlike in the case of higher dimension, Hardy-Sobolev inequality holds for $p \in [N, \infty]$, and at the right end of the scale, where in the higher dimension we have the critical Sobolev term, we have a weighted L^∞-norm, which in some respects can be regarded as *another critical nonlinearity*, alongside with the Pohozhaev-Trudinger-Moser term. Indeed, we observe in this paper that the transformations

$$g_s u(x) \stackrel{\text{def}}{=} s^{(1-N)/N} u(|x|^{s-1}x), s > 0, \tag{1.6}$$

form a group of isometries on $W_0^{1,N}(B)$, and, moreover, they preserve the functionals (1.5). Remarkably, Hardy inequality in $W_0^{1,N}(B)$ is readily available in Adimurthi and Sandeep [4], that contains the Hardy term $Q_p|_{p=N}$, the best constant, and further correction terms.

We conclude the paper with an existence proof, based on concentration compactness argument involving transformations (1.6), for the minimization problem

$$\kappa_p \stackrel{\text{def}}{=} \inf_{Q_p(u)=1} \int_B |\nabla u|^N \,\mathrm{d}x,\ p > N. \tag{1.7}$$

In Section 2 we prove the implication (1.4). In Section 3 verify (1.5). In Section 4 we show existence of minimizers in (1.7). In Section 5 we prove a "global compactness" statement for bounded sequences in $W_{0,r}^{1,N}(B)$. The main results of this paper are Lemma 3.3, Lemma 3.4, Theorem 4.2 and Theorem 5.1.

2 Weak continuity properties of the Pohozhaev-Trudinger-Moser functional

In what follows we will use the notations B for the unit ball $B_1(0)$, r for $|x|$, and set

$$\eta_r \stackrel{\text{def}}{=} \log \frac{1}{r}, \quad 0 < r < 1.$$

The following family of functions was introduced by Moser [9] in order to study the optimal parameters in the Pohozhaev-Trudinger-Moser inequality:

$$\mu_t(r) \stackrel{\text{def}}{=} (\omega_{N-1})^{-\frac{1}{N}} \eta_t^{\frac{N-1}{N}} \min\left\{\frac{\eta_r}{\eta_t}, 1\right\}, \quad r, t \in (0,1). \tag{2.1}$$

In what follows we define a continuous functional on $u \in W^{1,N}_{0,r}(B)$ associated with the function μ_t, $t \in (0,1)$:

$$\langle \mu_t^*, u\rangle \stackrel{\mathrm{def}}{=} \int_B \nabla \mu_t |\nabla \mu_t \cdot \nabla u|^{N-2}\,\mathrm{d}x.$$

Lemma 2.1. *Let $u \in W^{1,N}_{0,r}(B)$. Then for every $t \in (0,1)$,*

$$\langle \mu_t^*, u\rangle = \omega_{N-1}^{1/N} \eta_t^{(1-N)/N} u(t). \tag{2.2}$$

Proof. We have

$$\langle \mu_t^*, u\rangle = \omega_{N-1} \int_1^t u'(r)|\mu_t'(r)|^{N-1} r^{N-1}\mathrm{d}r$$

$$= \omega_{N-1}\omega_{N-1}^{\frac{1-N}{N}} \eta_t^{(1-N)/N} \int_1^t u'(r)\mathrm{d}r = \omega_{N-1}^{1/N} \eta_t^{(1-N)/N} u(t).$$

□

The well-known estimate below is an immediate consequence of Lemma 2.1.

Corollary 2.2. *Every function $u \in W^{1,N}_{0,r}(B)$ satisfies the inequality*

$$\sup_{r\in(0,1)} |u(r)| \eta_t^{(1-N)/N} \le \omega_{N-1}^{-1/N} \|u\|_{1,N} \tag{2.3}$$

and the constant $\omega_{N-1}^{-1/N}$ in the right hand side is optimal.

Proof. Apply Hölder inequaluty to (2.2). The best constant is attained on $u = \mu_t$. □

Lemma 2.3. *If $u_k \in W^{1,N}_{0,r}(B)$ and for every sequence $t_k \in (0,1)$, $\langle \mu_{t_k}^*, u_k\rangle \to 0$, then*

$$\sup_{0<r<1} |u_k(r)| \eta_r^{(1-N)/N} \to 0. \tag{2.4}$$

Proof. Let $t_k \in (0,1)$ be such that

$$|u_k(t_k)| \eta_{t_k}^{(1-N)/N} \ge \frac{1}{2} \sup_{0<r<1} |u_k(r)| \eta_r^{(1-N)/N}.$$

By Lemma 2.1, $\langle \mu_{t_k}^*, u_k\rangle \to 0$ implies

$$u_k(t_k) \eta_{t_k}^{(1-N)/N} \to 0,$$

and (2.4) follows. □

Functionals with the growth rate similar to that of (1.3), have the following continuity property related to convergence (2.4).

Proposition 2.4. *Assume now that* $u_k \rightharpoonup 0$ *in* $W^{1,N}_{0,r}(B)$*,and that for any sequence* $t_k \in (0,1)$*,* $\langle \mu^*_{t_k}, u_k\rangle \to 0$*. If* $\psi \in C(B \times \mathbb{R})$ *satisfies* $\psi(x,s) \le M|s|^{N/(N-1)}$ *with some* $M > 0$ *and* $\psi(x,0) = 0$ *for all* $x \in B$*, then*

$$\int_B (e^{\psi(x,u_k)} - 1)\,\mathrm{d}x \to 0.$$

Proof. By Fatou lemma

$$\liminf \int_B e^{\psi(x,u_k)}\,\mathrm{d}x \ge |B|. \tag{2.5}$$

By Lemma 2.3, there is a sequence $\epsilon_k \to 0$ such that $|u_k|^{N/(N-1)}(r) \le \epsilon_k \eta_r$. Then

$$\int_B e^{\psi(x,u_k)} dx \le \int_B e^{M|u_k|^{N/(N-1)}}\,\mathrm{d}x \le \int_\Omega r^{M\epsilon_k}\,\mathrm{d}x \to |B|. \tag{2.6}$$

Combine (2.5) and (2.6). □

3 Cocompactness of imbeddings of $W^{1,N}_{0,r}(B)$

We use the following definition introduced in [15].

Definition 3.1. *Let* X *be a Banach space and* D *a set of automorphisms of* X *One says that* $u_k \overset{D}{\rightharpoonup} 0$ *(or* u_k *converges to zero* D*-weakly) on* X *if for every* $g_k \in D$*,* $g_k u_k \rightharpoonup 0$*, and that an imbedding of* X *into a Banach space* Y *is (*D*-)cocompact if* $u_k \overset{D}{\rightharpoonup} 0$ *in* X *implies* $u_k \to 0$ *in the norm of* Y*.*

In what follows the group D will be the group of operators (1.6) on $W^{1,N}_{0,r}(B)$. In what follows we prove D-cocompactness of imbeddings of $W^{1,N}_{0,r}(B)$ into spaces $L^p(B, V_p dx)$.

We start with a formula that is verified by direct computation.

Proposition 3.2. *For every* $s > 0$ *and* $t \in (0,1)$*,*

$$g_s \mu_t = \mu_{t^{1/s}} \quad . \tag{3.1}$$

Lemma 3.3. *Assume that $u_k \in W^{1,N}_{0,r}(B)$ is D-weakly convergent to zero. Then for every sequence $s_k \in (0,1)$, $\langle \mu^*_{s_k}, u_k\rangle \to 0$ and consequently*

$$\sup_{r\in(0,1)} |u_k(r)|\eta_r^{(1-N)/N} \to 0. \tag{3.2}$$

Proof. Let $t_k \in (0,1)$ be an arbitrary sequence and let $s_k = \log \frac{1}{t_k}$. Since $u_k \stackrel{D}{\rightharpoonup} 0$, $\langle \mu^*_{1/e}, g_{s_k} u_k\rangle \to 0$. By Proposition 3.2, taking into account that operators g_{s_k} are isometries, we have

$$\langle \mu^*_{1/e}, g_{s_k} u_k\rangle = \langle \mu^*_{t_k}, u_k\rangle.$$

Then from Lemma 2.3 follows (3.2). □

Lemma 3.4. *Let Q_p, $p > N$, be the functional* (1.5). *If $u_k \in W^{1,N}_{0,r}(B)$, satisfies* (3.2) *(in particular, if $u_k \stackrel{D}{\rightharpoonup} 0$), then $Q_p(u_k) \to 0$.*

Proof. Taking into account the Hardy-type inequality (see Adimurthi and Sandeep, [4])

$$\int_B \frac{|u|^N}{r^N \eta_r^N} dx \le \left(\frac{N-1}{N}\right)^N \int_B |\nabla u|^N \, \mathrm{d}x, \tag{3.3}$$

by (3.2) we have

$$Q_p(u_k) \le \int_B \frac{|u_k|^N}{r^N \eta_r^N} \,\mathrm{d}x \left(\sup_{r\in(0,1)} \frac{|u_k(r)|}{\eta_r^{\frac{N-1}{N}}} \right)^{p-N} \to 0.$$

□

Remark 3.5. *Using the terminology of [15], Lemma 3.4 and Lemma 3.3 assert that $W^{1,N}_{0,r}(B)$ is D-cocompactly imbedded into the weighted space $L^p(B, V_p dx)$, where*

$$V_p(r) = \begin{cases} r^{-N} \left(\log \frac{1}{r}\right)^{-N-(p-N)\frac{N-1}{N}}, & N < p < \infty, \\ \left(\log \frac{1}{r}\right)^{\frac{1-N}{N}}, & p = \infty \end{cases} \tag{3.4}$$

Similarly, Lemma 3.3 combined with Proposition 2.4 can be interpreted as a cocompactness statement in connection to the correspondent Orlicz norm.

Remark 3.6. *It is easy to see that, conversely to Lemma 3.4 resp. Lemma 3.3, if* u_k *is a bounded sequence in* $W^{1,N}_{0,r}(B)$ *and* $Q_p(u_k) \to 0$ *for some* $p > N$*, resp.* $\sup_{r\in(0,1)} |u_k(r)|\eta_r^{(1-N)/N} \to 0$*, then* $u_k \overset{D}{\rightharpoonup} 0$*. Indeed, any such convergence yields weak convergence in* $W^{1,N}_{0,r}(B)$*, and since the functionals* Q_p *are invariant with respect to dilations* (1.6)*, this implies* $g_{s_k} u_k \rightharpoonup 0$ *in* $W^{1,N}_{0,r}(B)$ *for any sequence* $s_k > 0$.

4 Minimizers for Hardy-Sobolev inequalities in $W_0^{1,N}$

Lemma 4.1. *Let* $u_k \rightharpoonup u$ *in* $L^p(X, d\mu)$ *where* $(X, d\mu)$ *is a measure space. If* $q \geq 2$ *and* $u_k \geq 0$*, then*

$$\int u_k^q mathrmd\mu \geq \int u^q d\mu + \int |u_k - u|^q \mathrm{d}\mu + o(1). \tag{4.1}$$

Note that this result does not follow from Brezis-Lieb lemma since the latter has an additional requirement $u_k(x) \to u(x)$ for almost every $x \in X$.

Proof. Consider the elementary inequality

$$(1+t)^q \geq 1 + |t|^q + qt,\ t \geq -1, q \geq 2. \tag{4.2}$$

Note that this inequality becomes identity when $q = 2$. To verify the inequality, one considers cases $t \geq 0$ and $-1 \leq t < 0$. In both cases the inequality follows from the equality at zero and the derivative of constant sign, which is easy to verify. This inequality in turn implies

$$u_k^q \geq u^q + |u_k - u|^q + qu^{q-1}(u_k - u).$$

Integrating this inequality and noting that the integral of the last term tends to zero since $u_k \rightharpoonup 0$ we arrive at (4.1) □

Theorem 4.2. *The infimum*

$$\kappa_p \overset{def}{=} \inf\left\{\int_B |\nabla u|^N \,\mathrm{d}x :\ u \in W_0^{1,N}(B),\ \|u\|_{L^p(B,V_p\,\mathrm{d}x)} = 1\right\}, \quad p < N \leq \infty, \tag{4.3}$$

is attained on some positive function $w \in W^{1,N}_{0,r}(B)$*. Moreover, every radial minimizing sequence has a subsequence* u_k *such that, with some* $s_k > 0$*, the sequence* $s_k^{(1-N)/N} u_k(r^{s_k})$ *converges to a point of minimum in* $W_0^{1,N}(B)$*.*

Proof. In the case $p = \infty$ the theorem follows from (2.3) and the infimum is attained on $u = \mu_t$ for any $t \in (0, 1)$.

Consider now the case $N < p < \infty$. Let Q_p be the functional (1.5). Note that the weight (3.4) is decreasing. Then, by the standard rearrangement results, denoting as w^* the decreasing spherical rearrangement of a nonnegative function w, we have

$$Q_p(u) \le Q_p(|u|^*) \text{ and } \int_B |\nabla u|^N \,\mathrm{d}x \ge \int_B |\nabla |u|^*|^N \,\mathrm{d}x.$$

Consequently, the infimum in (4.3) does not increase if we restrict it to nonnegative monotone functions in the radial subspace $W_{0,r}^{1,N}(B)$. The positivity of κ_p follows from that of κ_N and the estimate (2.3):

$$Q_p(u) \le Q_N(u) \sup_r |u(r)|^{p-N} (\log \frac{1}{r})^{-(p-N)\frac{N-1}{N}} \le C\|u\|^p. \tag{4.4}$$

Let $u_k \in W_{0,r}^{1,N}(B)$ be a minimizing sequence, that is, $\|u_k\|_{1,N}^N \to \kappa_p$ and $Q_p(u_k) = 1$. If $u_k \overset{D}{\rightharpoonup} 0$, then by Lemma 3.3, $Q(u_k) \to 0$, a contradiction. Consequently, for a renamed subsequence, there exists a sequence $g_k \in D$ and a $w \in W_{0,r}^{1,N}(B) \setminus \{0\}$, such that $g_k u_k \rightharpoonup w$. Note that functions $g_k u_k$ remain decreasing radial functions. Then, by Brezis-Lieb lemma,

$$1 = Q_p(g_k u_k) = Q_p(w) + Q_p(g_k u_k - w) + o(1), \tag{4.5}$$

Let $v_k(r) = -(g_k u_k)'(r)$ and $v(r) = -w'(r)$. Then $v_k \rightharpoonup v$ in $L^N(B)$, $v_k \ge 0$ and $v \ge 0$. Applying Lemma 4.1 to the sequence v_k with $q = N$, we obtain:

$$\kappa_p = \|g_k u_k\|_{1,N}^N + O(1) = \|w\|_{1,N}^N + \|g_k u_k - w\|_{1,N}^N + o(1), \tag{4.6}$$

Let $t = Q_p(w)$ so that from (4.5) follows that $Q_p(g_k u_k - w) \to 1 - t$. Then from (4.6) follows $1 \ge t^{N/p} + (1-t)^{N/p}$, which, since $p > N$, is false unless $t = 0$ or $t = 1$. The first possibility, however, is excluded since $w \ne 0$. Then $\|w\|_{1,N}^N \ge \kappa_p = \|g_k u_k\|_{1,N}^N + o(1)$, which implies, by weak semicontinuity of the norm $\|g_k u_k\|_{1,N} \to \|w\|_{1,N}$ and, since $g_k u_k \rightharpoonup w$, we arrive at $g_k u_k \to w$ in $W_{0,r}^{1,N}(B)$. $\square$

Remark 4.3. *By* (2.3) *and* (3.3), $\kappa_N = \left(\frac{N-1}{N}\right)^N$, $\kappa_\infty = \omega_{N-1}$ *and therefore, by direct evaluation,*

$$\kappa_p \le \left(\frac{N-1}{N}\right)^{N^2/p} \omega_{N-1}^{\frac{p-N}{p}}.$$

The functional Q_p is an interpolation between the Hardy term in (3.3) *and* (2.3) *and since* (2.3) *lies at the right end of interpolation scale, the functions μ_t, on which κ_∞ is attained, may be regarded as the low-dimensional counterpart of the scaled Talenti solutions [13]. Similarly to the higher dimension, the best Hardy constant, κ_N, is not attained (Adimurthi and Sandeep, [4]).*

5 Global compactness theorem

In this section we derive a structural statement for bounded sequences in $W^{1,N}_{0,r}(B)$, $B \subset \mathbb{R}^N$, equipped with the group D of transformations (1.6).

Note that the operators (1.6) are unitary operators on $W^{1,N}_0(B)$ and satisfy

$$g_k \in D, g_k \not\rightharpoonup 0 \Rightarrow g_k \text{ has a strongly (elementwise) convergent subsequence.} \tag{5.1}$$

Indeed, it is easy to see that

$$g_{s_k} \in D, g_{s_k} \rightharpoonup 0 \Leftrightarrow |\log s_k| \to \infty. \tag{5.2}$$

If $s_k \to 0$, then for any $v \in C_0^\infty(B \setminus \{0\})$, $g_{s_k} v = 0$ for k sufficiently large since $|x|^{s_k} \to 1$ uniformly on $\operatorname{supp} v$. If $s_k \to \infty$, then

$$\left| \int u(x) g_{s_k} v(x)\, \mathrm{d}x \right| \le C s_k^{(1-N)/N} \to 0.$$

Consequently $(u, g_{s_k} v) \to 0$ in both cases, and by density this extends to all $v \in W^{1,N}_0(B)$. Then (5.1) follows from compactness of closed intervals on $\mathbb{R}$.

Theorem 5.1. *Let $u_k \rightharpoonup 0$ in $W^{1,2}_0(B)$ or let $u_k \rightharpoonup 0$ be a sequence of radial non-increasing functions in $W^{1,N}_0(B)$ for $N \ge 3$. There exist $s_k^{(n)} \in (0, \infty)$, $k \in \mathbb{N}$, $n \in \mathbb{N}$, such that for a renumbered subsequence,*

$$w^{(n)} = \text{ w-lim } \left(s_k^{(n)}\right)^{(1-N)/N} u_k(r^{-s_k^{(n)}}), \tag{5.3}$$

$$|\log(s_k^{(m)}/s_k^{(n)})| \to \infty \text{ for } n \ne m, \tag{5.4}$$

$$\sum_{n \in \mathbb{N}} \int_B |\nabla w^{(n)}|^N \,\mathrm{d}x \le \limsup \int_B |\nabla u_k|^N \,\mathrm{d}x, \tag{5.5}$$

$$u_k - \sum_{n \in \mathbb{N}} \left(s_k^{(n)}\right)^{(1-N)/N} w^{(n)}(r^{s_k^{(n)}}) \overset{D}{\rightharpoonup} 0, \tag{5.6}$$

and the series $\sum_{n\in\mathbb{N}} \left(s_k^{(n)}\right)^{(1-N)/N} w^{(n)}(r^{s_k^{(n)}})$ *converges in* $W_0^{1,N}(B)$ *uniformly in* k.

Furthermore, whenever functions u_k *are radial, the D-weak convergence in* (5.6) *is equivalent to the convergence in* $L^p(B, V_p)$ *for each* $p \in (N, \infty]$.

Proof. First note that the last assertion is a quote from the cocompactness results Lemma 3.3 and Lemma 3.4.

For $N = 2$ the theorem is an immediate application of Corollary 3.2 from [14], whose conditions are verified by (5.1). Interpretation of relation (3.9) from [14] by (5.2) gives (5.4).

For $N \geq 3$ the theorem is an immediate application of Theorem 2.6 from [15] with $F(u) = \|u\|_{1,N}^N$, whose conditions are verified by (5.1) and by Lemma 4.1 applied to positive sequences $-u_k'(r)$. □

This result is complementary to the blowup analysis in $W^{1,N}$ (e.g. [1],[2], [3], [7])) as it deals with general bounded sequences rather than with critical sequences of specific functionals and, more significantly, it suggests that the blowups for these problems are more naturally to define in terms of the transformations (1.6) rather than by an inflation on the linear scale.

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