Properties of positive harmonic functions on the half space with a nonlinear boundary condition *

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Abstract

In this paper we study existence and properties of solutions of the problem $\Delta w = 0$ on the half-space $\mathbb{R}^N_+$ with nonlinear boundary condition $\partial w/\partial\eta + w = |w|^{p-2}w$ where $2 < p < 2(N-1)/(N-2)$ and $N \geq 3$. We obtain a ground state solution $w = w(x_1, ..., x_{N-1}, t)$ which is radial and has exponential decay in the first $N-1$ variables. Moreover, w has sharp polynomial decay in the variable t.

Key words: Minimax, critical points, moving planes, harmonic function, ground state, asymptotic behaviors.
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1 Introduction

This article is concerned with the nonlinear boundary value problem

$$\begin{cases} \Delta w = 0 & \text{in} \quad \mathbb{R}^N_+, \\ \dfrac{\partial w}{\partial \eta} = |w|^{p-2}w - w & \text{on} \quad \partial\mathbb{R}^N_+, \end{cases} \tag{P}$$

where $\mathbb{R}^N_+$ $(N \geq 3)$ is the Euclidean half space, η is the unit outer normal to the boundary $\partial\mathbb{R}^N_+$ and $2 < p < 2_* = 2(N-1)/(N-2)$. Recall that 2_* is the critical Sobolev exponent for the trace embedding

$$H^1(\mathbb{R}^N_+) \hookrightarrow L^q(\partial\mathbb{R}^N_+), \quad 2 \leq q \leq 2_*.$$

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Our main goal here is to study the existence, symmetry and asymptotic behavior of positive solutions of (P).

The interest in this problem comes from the fact that it appears naturally after blow up when studying solutions of the nonlinear boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega_\epsilon, \\ \dfrac{\partial u}{\partial \eta} = |u|^{p-2}u - u & \text{on} \quad \partial\Omega_\epsilon, \end{cases} \tag{1.1}$$

where ϵ is a positive parameter, $\Omega_\epsilon := \{\epsilon^{-1}z \ : \ z \in \Omega\}$ and Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. More precisely, if we stand at a point on the boundary $\partial\Omega$ and take $\epsilon \to 0$, then the domain Ω_ϵ becomes a half space which, after a convenient rotation and translation, may be assumed to be $\mathbb{R}^N_+$. Note that $u \equiv 1$ is a positive solution of (1.1). By using an approach of [10], the authors in [1] have used (P) as a limit problem in order to obtain a nontrivial positive solution of (1.1). We note also that problem (1.1) is related to the steady state of a parabolic problem introduced by Steklov [13].

For convenience, we write $z = (x,t) \in \mathbb{R}^N_+$ with $x \in \mathbb{R}^{N-1}$ and $t > 0$. Hereafter, we identify $\partial\mathbb{R}^N_+ = \mathbb{R}^{N-1}$ and we use the notation $\overline{\mathbb{R}^N_+} := \mathbb{R}^N_+ \cup \mathbb{R}^{N-1}$.

In order to prove the existence of solution of (P), we shall use a minimax argument to the energy associated functional with (P),

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N_+} |\nabla u|^2 \, \mathrm{d}z + \frac{1}{2}\int_{\mathbb{R}^{N-1}} u^2 \, \mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^{N-1}} (u^+)^p \, \mathrm{d}x, \tag{1.2}$$

defined on the natural space

$$H = \left\{ u \in \mathcal{D}^{1,2}(\mathbb{R}^N_+) : u|_{\mathbb{R}^{N-1}} \in L^2(\mathbb{R}^{N-1}) \right\},$$

(where $u|_{\mathbb{R}^{N-1}}$ is understood in the sense of trace) endowed with the inner product

$$< u, v > = \int_{\mathbb{R}^N_+} \nabla u \nabla v \, \mathrm{d}z + \int_{\mathbb{R}^{N-1}} uv \, \mathrm{d}x$$

and the corresponding norm

$$\|u\|^2_\partial = \int_{\mathbb{R}^N_+} |\nabla u|^2 \, \mathrm{d}z + \int_{\mathbb{R}^{N-1}} u^2 \, \mathrm{d}x. \tag{1.3}$$

One can see that H is a Hilbert space and since $C^\infty_0(\mathbb{R}^N)$ is dense in $\mathcal{D}^{1,2}(\mathbb{R}^N_+)$ it follows that the restrictions to $\mathbb{R}^N_+$ of functions in $C^\infty_0(\mathbb{R}^N)$ are dense in H.

Now let us define what we mean by a solution of (P) in H. We say that $w \in H$ is a $H-$weak solution of (P) if, for all $\varphi \in H$,

$$\int_{\mathbb{R}^N_+} \nabla w \nabla \varphi \, \mathrm{d}z + \int_{\mathbb{R}^{N-1}} w\varphi \, \mathrm{d}x = \int_{\mathbb{R}^{N-1}} |w|^{p-2} w \varphi \, \mathrm{d}x. \tag{1.4}$$

In what follows, we mention some known results on nonexistence for problem (P). Note that if w is a $H-weak$ solution of (P) with $2 < p \leq 2_*$, for each $\alpha > 0$ the function $\alpha^{1/(p-2)} w(\alpha z)$ is a solution of:

$$\begin{cases} \Delta w = 0 & \text{in} \quad \mathbb{R}^N_+, \\ \dfrac{\partial w}{\partial \eta} = |w|^{p-2}w - \alpha w & \text{on} \quad \mathbb{R}^{N-1}. \end{cases} \tag{1.5}$$

Using a convenient sequence of cut-off functions, we can prove that the following *Pohozaev's identity* holds:

$$\frac{N-2}{2}\int_{\mathbb{R}^N_+}|\nabla w|^2dz=(N-1)\int_{\mathbb{R}^{N-1}}\left(\frac{|w|^p}{p}-\alpha\frac{w^2}{2}\right)dx,$$

which implies that, if $\alpha>0$ and $p=2_*$ then problem (1.5) does not have H-weak solution. However, if $\alpha=0$ and $p=2_*$, then

$$w(z)=\left(\frac{-(N-2)\bar{t}}{|z-\bar{z}|^2}\right)^{(N-2)/2}$$

is a positive solution of (1.5) for each fixed $\bar{z}\in\mathbb{R}^N_-=\{(x,t)\in\mathbb{R}^N\ :\ t<0\}$ (see [5]). Problem (P) is related to the Yamabe problem with boundary, namely, to find a Riemaniann metric conformal to Euclidean metric whose scalar curvature is zero and the mean curvature of the boundary is constant, see more details in Adimurthi-Yadava [2]. In the subcritical case $2<p<2_*$, Hu in [4] showed that if $\alpha=0$, then problem (1.5) does not admit any classical bounded positive solution. On the other hand, our result asserts that lower order terms reverse this situation. In fact, we prove that $c_p(\mathbb{R}^N_+)$, the least energy level of the functional I, is achieved. Moreover, we can see that

$$c_p(\mathbb{R}^N_+)=\frac{p-2}{2p}S_p(\mathbb{R}^N_+)^{p/(p-2)},$$

where

$$S_p(\mathbb{R}^N_+)=\inf\left\{\|\nabla u\|^2_{L^2(\mathbb{R}^N_+)}+\|u\|^2_{L^2(\mathbb{R}^{N-1})}:u\in H,\|u\|_{L^p(\mathbb{R}^{N-1})}=1\right\}.$$

Here we are interested in finding a *ground state solution* of (P) that is, a positive solution $w\in H$ whose energy is minimal among the energy of all nontrivial solutions of (P) in H. Let us point out that Lions prove in [9, page 275] the existence of a ground state solution of (P) in the Sobolev space $H^1(\mathbb{R}^N_+)$. Since $H^1(\mathbb{R}^N_+)\hookrightarrow H$, the solution found by Lions might not be a ground state solution in H. We shall analyze the behavior of w and prove that in fact $w\in H^1(\mathbb{R}^N_+)$. To our knowledge few properties about the asymptotic behavior of solution of (P) are known. Here, in order to obtain the asymptotic behavior of solutions of (P), we combine a new Harnack's inequality with a comparison argument. (see also [6] for related results).

As a consequence of [14, Theorem 0.1], all positive solution in H of (P) are radially symmetric with respect the first $N-1$ variables provided that $p\in[2N/(N-1),2_*)$. Here, we complement this result, since by our argument this result holds for all $p\in(2,2_*)$. We remark that this improvement was obtained thanks to the polynomial decay of w proved in Section 3. As we will see, the symmetry will allow us to improve the decay of w in the first $N-1$ variables. In fact, we will prove that all positive solutions in H have exponential decay in the first $N-1$ variables. In order to prove this fact, some non standard and sharp decay estimates are carefully obtained.

Our main result is summarized by the following theorem.

Theorem 1.6. *Problem (P) has a ground state solution $w\in C^\infty(\mathbb{R}^N_+)\cap C^{2,\alpha}(\overline{\mathbb{R}^N_+})\cap H$ such that*

(i) w is radially symmetric with respect to the variable $x\in\mathbb{R}^{N-1}$, that is, $w(x,t)=w(r,t)$ if $r=|x|$. Moreover, $w_r(r,t)<0$ in $(0,+\infty)\times[0,+\infty)$.

(ii) w has exponential decay in the variable x and polynomial decay in the variable t; more precisely, there exist positive numbers c_1 and c_2 such that

$$w(z)\le c_1\exp(-c_2|x|)\frac{1}{(1+t^2)^{(N-2)/2}},\qquad\forall\ z=(x,t)\in\overline{\mathbb{R}^N_+}.$$

Moreover, for each $x\in\mathbb{R}^{N-1}$ fixed, there exist positive numbers c_3,c_4 and t_0 such that

$$\frac{c_3}{t^N}\le w(x,t)\le\frac{c_4}{(1+t^2)^{(N-2)/2}}\quad\textit{for all}\quad t\ge t_0.$$

2 Existence results

We note that the family of functions $w(x,t) = at + b$, with $a, b \in \mathbb{R}$ satisfying $-a + b = b^{p-1}$, are classical solutions of (P), but not $H-$weak solutions. In this section we establish the existence of a ground state solution of (P). We first require several technical results.

Lemma 2.1. *For each $p \in [2, 2_*]$ we have the continuous embedding*

$$H \hookrightarrow L^p(\mathbb{R}^{N-1}).$$

Proof. From the classical trace embedding

$$\mathcal{D}^{1,2}(\mathbb{R}^N_+) \hookrightarrow L^{2_*}(\mathbb{R}^{N-1}), \tag{2.2}$$

(see [8]) together with the interpolation inequality and $\theta \in (0,1)$, there exists $C > 0$ such that

$$\begin{aligned}
\int_{\mathbb{R}^{N-1}} |u|^p \, \mathrm{d}x \leq & \left(\int_{\mathbb{R}^{N-1}} |u|^2 \, \mathrm{d}x\right)^{\theta p/2} \left(\int_{\mathbb{R}^{N-1}} |u|^{2_*} \, \mathrm{d}x\right)^{(1-\theta)p/2_*} \\
\leq & \; C \left(\int_{\mathbb{R}^{N-1}} |u|^2 \, \mathrm{d}x\right)^{\theta p/2} \left(\int_{\mathbb{R}^N_+} |\nabla u|^2 \, \mathrm{d}z\right)^{(1-\theta)p/2} \\
\leq & \; C \|u\|_\partial^{\theta p/2} \|u\|_\partial^{(1-\theta)p/2},
\end{aligned}$$

which completes the proof. ■

It follows from Lemma 2.1 that the functional I is well defined in H and belongs to the class $C^2(H, \mathbb{R})$. Moreover, one can see that nonnegative $H-$weak solutions of (P) are critical points of I and conversely.

By using Lemma 2.1 it is standard to show that the functional I has the mountain pass structure on the space H. Thus, the minimax level

$$c_p(\mathbb{R}^N_+) = \inf_{g \in \mathfrak{F}} \max_{t \in [0,1]} I(g(t))$$

is positive, where

$$\mathfrak{F} := \{g \in C([0,1], H) : \; g(0) = 0 \text{ and } \; I(g(1)) < 0\}.$$

Therefore, there exists a Palais-Smale sequence (PS sequence for short) $(u_m) \subset H$ at the level $c_p(\mathbb{R}^N_+)$, that is,

$$I(u_m) \to c_p(\mathbb{R}^N_+) \quad \text{and} \quad I'(u_m) \to 0.$$

Lemma 2.3. *If $(u_m) \subset H$ is a (PS) sequence at the level $c_p(\mathbb{R}^N_+)$, then (u_m) is bounded and there exists $b > 0$ such that*

$$\|u_m^+\|_{L^p(\mathbb{R}^{N-1})} \geq b > 0, \tag{2.4}$$

for m suficiently large.

Proof. If (u_m) is a (PS) sequence at level $c_p(\mathbb{R}^N_+)$, one can see that

$$\left(\frac{p}{2} - 1\right) \|u_m\|_\partial^2 = pI(u_m) - I'(u_m)u_m \leq pc_p(\mathbb{R}^N_+) + C_1 \|u_m\|_\partial + C_2,$$

which implies that $\|u_m\|_\partial$ is bounded. Since (u_m) is bounded for m sufficiently large we have

$$\frac{c_p(\mathbb{R}^N_+)}{2} \leq I(u_m) - \frac{1}{2} I'(u_m)u_m = \left(\frac{1}{2} - \frac{1}{p}\right) \|u_m^+\|^p_{L^p(\mathbb{R}^{N-1})}.$$

This yields that (2.4) holds. ■

In order to prove the existence of a nontrivial critical point of I at the minimax level $c_p(\mathbb{R}^N_+)$ we establish some technical lemmata.

Lemma 2.5. *For each $q \in [2, 2_*]$ and $y \in \mathbb{R}^{N-1}$ there exists a constant $C = C(N, q) > 0$ such that*

$$\|u\|_{L^q(\Gamma(y))} \leq C \left(\|\nabla u\|^2_{L^2(B_1^+(y))} + \|u\|^2_{L^2(\Gamma(y))} \right)^{1/2}, \quad u \in H, \tag{2.6}$$

where

$$B_1^+(y) = \{x \in \mathbb{R}^N_+ : |z - (y, 0)| < 1\} \text{ and } \Gamma(y) = \{x \in \mathbb{R}^{N-1} : |x - y| < 1\}.$$

Proof. As a consequence of Friedrichs inequality we have

$$\|u\|_{L^2(B_1^+(y))} \leq C \left(\|\nabla u\|^2_{L^2(B_1^+(y))} + \|u\|^2_{L^2(\Gamma(y))} \right)^{1/2}, \quad u \in H, \tag{2.7}$$

which together with the trace embedding $H^1(B_1^+(y)) \hookrightarrow L^q(\partial B_1^+(y))$ implies that (2.6) holds. ■

Lemma 2.8. *If $(u_m) \subset H$ is a (PS) sequence, then there exists $C = C(N, p) > 0$ such that*

$$\sup_{y \in \mathbb{R}^{N-1}} \int_{\Gamma(y)} (u_m^+)^2 \, \mathrm{d}x \geq C > 0. \tag{2.9}$$

Proof. For $q \in (p, 2_*)$ fixed and by interpolation we have

$$\|u\|^p_{L^p(\Gamma(y))} \leq \|u\|^{(1-\alpha)p}_{L^2(\Gamma(y))} \|u\|^{\alpha p}_{L^q(\Gamma(y))}, \quad u \in H, \tag{2.10}$$

where $\alpha = pq/[(p-2)(q-2)]$. Now, we consider two cases.

Case 1: $q_* = 4(q-1)/q \leq p$. In this case we have $\alpha p/2 \geq 1$. Then, setting

$$\|u\|_{B_1^+, \Gamma, y} := \left(\|\nabla u\|^2_{L^2(B_1^+(y))} + \|u\|^2_{L^2(\Gamma(y))} \right)^{1/2}$$

and using (2.10) together with Lemma 2.5 we obtain

$$\begin{aligned} \|u\|^p_{L^p(\Gamma(y))} &\leq C_1 \|u\|^{(1-\alpha)p}_{L^2(\Gamma(y))} \|u\|^{\alpha p}_{B_1^+, \Gamma, y} \\ &\leq C_1 \left(\sup_{y \in \mathbb{R}^{N-1}} \int_{\Gamma(y)} u^2 \, \mathrm{d}x \right)^{(1-\alpha)p/2} \|u\|^{\alpha p - 2}_{B_1^+, \Gamma, y} \|u\|^2_{B_1^+, \Gamma, y} \\ &\leq C_2 \left(\sup_{y \in \mathbb{R}^{N-1}} \int_{\Gamma(y)} u^2 \, \mathrm{d}x \right)^{(1-\alpha)p/2} \|u\|^{\alpha p - 2}_{\partial} \|u\|^2_{B_1^+, \Gamma, y}, \end{aligned} \tag{2.11}$$

where C_1, C_2 are positive constants which depend only on N and p. Now we choose a family $\{B_1^+(y)\}$ covering $\mathbb{R}^{N-1}$ and such that each point of $\mathbb{R}^{N-1}$ is contained in at most N such balls. Summing up inequalities (2.11) over this family, we find

$$\int_{\mathbb{R}^{N-1}} |u|^p \, \mathrm{d}x \quad \leq NC \left(\sup_{y \in \mathbb{R}^{N-1}} \int_{\Gamma(y)} u^2 \, \mathrm{d}x \right)^{(1-\alpha)p/2} \|u\|^{\alpha p - 2}_{\partial} \|u\|^2_{\partial}. \tag{2.12}$$

Now, setting $u = u_m^+$ in (2.12) and using the fact that (u_m) is bounded we obtain

$$\|u_m^+\|_{L^p(\mathbb{R}^{N-1})} \leq C \left\{ \sup_{y \in \mathbb{R}^{N-1}} \int_{\Gamma(y)} (u_m^+)^2 \, \mathrm{d}x \right\}^{(1-\alpha)/2},$$

which together with Lemma 2.3 implies that (2.9) holds.

Case 2: $q_* = 4(q-1)/q > p$. In this case we have that

$$\|u\|_{L^p(\mathbb{R}^{N-1})} \leq \|u\|_{L^2(\mathbb{R}^{N-1})}^{1-\beta} \|u\|_{L^{4(q-1)/q}(\mathbb{R}^{N-1})}^{\beta},$$

for some $\beta \in (0,1)$. Therefore, (2.10) follows by using (2.12) with $p = q_*$ and $\alpha = [qq_*/[(q-2)](q_*-2)]$. ■

Now we are ready to prove the existence of a nontrivial H-weak solution of (P).

Proposition 2.13. *There exists a ground state solution at the level* $c_p(\mathbb{R}_+^N)$.

Proof. By Lemma 2.8, there exits a sequence of points $(y_m) \subset \mathbb{R}^{N-1}$ such that

$$\int_{\Gamma(y_m)} (u_m^+)^2 \, \mathrm{d}x \geq \frac{b}{2}.$$

Thus, considering the new sequence $w_m(\cdot) = u_m(\cdot + y_m)$, it follows from (2.14) that

$$\int_{\Gamma(0)} (w_m^+)^2 \, \mathrm{d}x \geq \frac{b}{2}. \tag{2.14}$$

Using the invariance by translation, it is easy to show that $I(w_m) \to c_p(\mathbb{R}_+^N)$ and $I'(w_m) \to 0$. Using again Lemma 2.3 we obtain that (w_m) is bounded. Since H is reflexive, we can take a subsequence (still denoted in the same way) such that $w_m \rightharpoonup w$ in H. Thus, $w_m \to w$ in $L^2_{loc}(\mathbb{R}^{N-1})$ and hence it follows from (2.14) that w is nontrivial.

Claim 2.15. $c_p(\mathbb{R}_+^N) = I(w)$ *and* $I'(w) = 0$.

Indeed, since $I'(w_m) \to 0$ in H' (dual space) and $w_m \rightharpoonup w$ in H, taking the limit we obtain $I'(w)\varphi = 0$ for all $\varphi \in H$. Thus, taking $\varphi = w^-$ as testing function, it follows that w is a nonnegative $H-$weak solution of (P).

Since $I'(w)w = 0$ and the norm is weakly lower semicontinuous, we obtain

$$I(w) = \left(\frac{1}{2} - \frac{1}{p}\right) \|w\|_\partial^2 \leq \lim \left(\frac{1}{2} - \frac{1}{p}\right) \|w_m\|_\partial^2 = c_p(\mathbb{R}_+^N). \tag{2.16}$$

Next, using the fact that the mountain pass level is equal to the infimum of I on the Nehary manifold

$$\mathcal{N} = \{u \in H \setminus \{0\} : I'(u)u = 0\},$$

that is, $c_p(\mathbb{R}_+^N) = \inf_{w \in \mathcal{N}} I(w)$, and since $w \in \mathcal{N}$, we get $c_p(\mathbb{R}_+^N) \leq I(w)$. Therefore, $c_p(\mathbb{R}_+^N) = I(w)$ and we conclude that w is a ground state solution of (P). ■

3 Regularity and polynomial decay

In this section we shall prove some regularity and decay properties for ground state solutions of (P).

Proposition 3.1. *Let v be a $H-$weak solution of the nonlinear boundary value problem*

$$\begin{cases} \Delta v = 0 & in \quad \mathbb{R}^N_+, \\ \dfrac{\partial v}{\partial \eta} = a(x)|v|^{q-1}v - v & on \quad \mathbb{R}^{N-1}, \end{cases} \tag{3.2}$$

with $a \in L^\infty(\mathbb{R}^{N-1})$ and $1 \le q < 2_* - 1$, that is,

$$\int_{\mathbb{R}^N_+} \nabla v \nabla \varphi \,\mathrm{d}z + \int_{\mathbb{R}^{N-1}} v\varphi \,\mathrm{d}x = \int_{\mathbb{R}^{N-1}} a(x)|v|^{q-1} v\varphi \,\mathrm{d}x, \quad \forall\, \varphi \in H. \tag{3.3}$$

Then $v \in L^\infty(\mathbb{R}^N_+)$ and its trace $v|_{\mathbb{R}^{N-1}}$ belongs to $L^\infty(\mathbb{R}^{N-1})$. In particular, any $H-$weak solution of (P) enjoys the same properties.

Proof. Let v be a $H-$weak solution of (3.2). We can assume without lost of generality that v is nonnegative, by changing the test function. For each $k > 0$, we define $\varphi_k = v_k^{2(\beta-1)}v$ and $w_k = v v_k^{\beta-1}$ with $\beta > 1$ to be determined later, where $v_k = \min\{v, k\}$. Note that $0 \le v_k \le v$, $< \nabla v_k, \nabla v > \ge 0$ and $|\nabla v_k| \le |\nabla v|$. Taking φ_k as a test function in (3.3), we get

$$\begin{aligned} \int_{\mathbb{R}^N_+} v_k^{2(\beta-1)} |\nabla v|^2 \,\mathrm{d}z \le & - \int_{\mathbb{R}^{N-1}} v_k^{2(\beta-1)} v^2 \,\mathrm{d}x - 2(\beta-1) \int_{\mathbb{R}^N_+} v_k^{2(\beta-1)-1} v \nabla v_k \nabla v \,\mathrm{d}z \\ & + |a|_\infty \int_{\mathbb{R}^{N-1}} v^{q+1} v_k^{2(\beta-1)} \,\mathrm{d}x. \end{aligned}$$

Now, observing that the first and the second terms on the right–hand side of the inequality above are non positive, we obtain

$$\int_{\mathbb{R}^N_+} v_k^{2(\beta-1)} |\nabla v|^2 \,\mathrm{d}z \le C \int_{\mathbb{R}^{N-1}} v^{q+1} v_k^{2(\beta-1)} \,\mathrm{d}x = C \int_{\mathbb{R}^{N-1}} v^{q-1} w_k^2 \,\mathrm{d}x.$$

This together with the trace imbedding (2.2) implies that

$$\begin{aligned} \left(\int_{\mathbb{R}^{N-1}} w_k^{2_*} \,\mathrm{d}z \right)^{2/2_*} & \le C_1 \int_{\mathbb{R}^N_+} |\nabla w_k|^2 \,\mathrm{d}x \\ & \le C_2 \int_{\mathbb{R}^N_+} \left[v_k^{2(\beta-1)} |\nabla v|^2 + (\beta-1)^2 v^2 v_k^{2(\beta-2)} |\nabla v_k|^2 \right] \mathrm{d}z \\ & \le C_4 \beta^2 \int_{\mathbb{R}^N_+} v_k^{2(\beta-1)} |\nabla v|^2 \,\mathrm{d}z \\ & \le C_5 \beta^2 \int_{\mathbb{R}^{N-1}} v^{q-1} w_k^2 \,\mathrm{d}x, \end{aligned}$$

where we used that $1 + (\beta-1)^2 \le \beta^2$ for $\beta \ge 1$. By Hölder inequality we get

$$\left(\int_{\mathbb{R}^{N-1}} w_k^{2_*} \,\mathrm{d}x \right)^{2/2_*} \le \beta^2 C_5 \left(\int_{\mathbb{R}^{N-1}} v^{2_*} \,\mathrm{d}x \right)^{(q-1)/2_*} \left(\int_{\mathbb{R}^{N-1}} w_k^{22_*/(2_*-q+1)} \,\mathrm{d}x \right)^{(2_*-q+1)/2_*}.$$

Using that $|w_k| \le |v|^\beta$ and the continuous embedding $H \hookrightarrow L^{2_*}(\mathbb{R}^{N-1})$ we have

$$\left(\int_{\mathbb{R}^{N-1}} |vv_k^{\beta-1}|^{2_*}\,\mathrm{d}x\right)^{2/2_*} \le \beta^2 C_6 \|v\|_\partial^{q-1} \left(\int_{\mathbb{R}^{N-1}} v^{\beta 2 2_*/(2_*-q+1)}\,\mathrm{d}x\right)^{(2_*-q+1)/2_*}.$$

Choosing $\beta = 2^{-1}(2_* - q + 1) > 1$, we have $2\beta(2_* - q + 1)^{-1} = 1$. Thus,

$$\left(\int_{\mathbb{R}^{N-1}} |vv_k^{\beta-1}|^{2_*}\,\mathrm{d}x\right)^{2/2_*} \le \beta^2 C_6 \|v\|_\partial^{q-1} \|v\|_{\beta\alpha^*}^{2\beta},$$

where $\alpha^* = 2(2_* - q + 1)^{-1} 2_*$. By Fatou's Lemma, we obtain

$$\|v\|_{\beta 2_*} \le (C_6 \beta^2 \|v\|_\partial^{q-1})^{1/2\beta} \|v\|_{\beta\alpha^*}. \tag{3.4}$$

Taking $\beta_0 = \beta$ and inductively $\beta_{m+1}\alpha_* = 2_*\beta_m$ for $m = 1, 2, \ldots$ and applying the previous processes for β_1, we have that by (3.4)

$$\begin{aligned}\|v\|_{\beta_1 2_*} &\le (\beta_1^2 C_6 \|v\|_\partial^{q-1})^{1/2\beta_1} \|u\|_{\beta_1\alpha^*}\\ &\le (\beta_1^2 C_6 \|v\|_\partial^{q-1})^{1/2\beta_1} (\beta^2 C_6 \|v\|_\partial^{q-1})^{1/2\beta} \|v\|_{\beta\alpha^*}\\ &\le (C_6 \|v\|_\partial^{q-1})^{1/2\beta_1 + 1/2\beta} (\beta)^{1/\beta} (\beta_1)^{1/\beta_1} \|v\|_{2_*}.\end{aligned}$$

Observing that $\beta_m = \chi^m \beta$ where $\chi = 2_*/\alpha^*$, we obtain by iteration

$$\|v\|_{\beta_m 2_*} \le (C_6 \|v\|_\partial^{q-1})^{1/2\beta \sum_{i=0}^m \chi^{-i}} \beta^{1/\beta \sum_{i=0}^m \chi^{-i}} \chi^{1/\beta \sum_{i=0}^m i\chi^{-i}} \|v\|_{2_*}.$$

Since $\chi > 1$ and

$$\lim_{m\to\infty} \frac{1}{2\beta} \sum_{i=0}^m \chi^{-i} = \frac{1}{2_* - q - 1},$$

we can take the limit as $m \to \infty$ to get

$$\|v\|_\infty \le C_7 (\|v\|_\partial^{q-1})^{1/(2_*-q-1)} \|v\|_\partial.$$

Thus, we concluded that $v \in L^\infty(\mathbb{R}^{N-1})$.

Now, for each $k \in \mathbb{N}$ define

$$\Omega(k) = \{z = (x,t) \in \overline{\mathbb{R}_+^N} \ : \ v(z) > k\}.$$

Note that $\Omega(k)$ has finite Lebesgue measure because $v \in L^{2^*}(\mathbb{R}_+^N)$ and its trace $v|_{\mathbb{R}^{N-1}}$ belongs $L^2(\mathbb{R}^{N-1})$. Thus, the function

$$\varphi(z) = \begin{cases} (v-k)(z), & \text{if } \ z \in \Omega(k),\\ 0, & \text{if } \ z \in \overline{\mathbb{R}_+^N} \backslash \Omega(k),\end{cases}$$

belongs to the space H and $\nabla\varphi = \nabla v$ in $\Omega(k)$.

Since $v \in L^\infty(\mathbb{R}^{N-1})$, there exist a constant $M > 0$ such that $\|v\|_{L^\infty(\mathbb{R}^{N-1})} \le M$. Therefore, taking $k > M$ we obtain that $\varphi(x,0) = 0$ for all $x \in \mathbb{R}^{N-1}$. Hence, choosing φ as a testing function in (3.3) we get

$$\int_{\Omega(k)} |\nabla v|^2\,\mathrm{d}z = 0 \tag{3.5}$$

which implies that v is constant in $\Omega(k)$ or $|\Omega(k)| = 0$. In any case, we have $v \in L^\infty(\mathbb{R}_+^N)$ and the proof is complete. ■

Remark 3.6. *1) As a consequence of Lemma 3.1 and Harnack inequality (see [11] or [16, Theorem 1.1]), we obtain that nonnegative $H-$weak solutions of (P) are indeed positive in $\mathbb{R}^N_+$.*

2) From Lemma 3.1 and regularity results proved in [7, 15], we obtain that $H-$weak solutions of (P) belong to $C^{1,\alpha}_{loc}(\overline{\mathbb{R}^N_+})$. By a maximum principle due to Vazquez [17] we obtain in fact that $w > 0$ in $\overline{\mathbb{R}^N_+}$.

Next, using some ideas of [16]) and [11], we prove a Harnack type inequality, which will be useful in order to prove some decay properties of the ground state solutions of (P).

For fixed $y \in \mathbb{R}^{N-1}$ and $r < \rho$, we denote $B^+_\rho = B^+_\rho(y)$, $\Gamma_\rho = \Gamma_\rho(y)$ and, let $B^+_r \subset B^+_\rho$, $\Gamma^+_r \subset \Gamma^+_\rho$ be concentric balls, where

$$B^+_\rho(y) = \{z \in \mathbb{R}^N_+ : |z - (y, 0)| < \rho\} \text{ and } \Gamma_\rho(y) = \{x \in \mathbb{R}^{N-1} : |x - y| < \rho\}.$$

Lemma 3.7. *Let w be a $H-$weak solution of (P) with $0 < w \leq M$ in $B^+_{3\rho}$. Then there exist $C = C(N, M) > 0$ and $\theta_0 > 1$ such that*

$$\max_{B^+_\rho} w + \max_{\Gamma_\rho} w \leq C\rho^{-(N-1)/\theta_0} \left(\rho^{-1}\|w\|^{\theta_0}_{L^{\theta_0}(B^+_{2\rho})} + \|w\|^{\theta_0}_{L^{\theta_0}(\Gamma_{2\rho})}\right)^{1/\theta_0}.$$

In particular, we have

$$\lim_{|z|\to+\infty} w(z) = 0, \quad \forall z \in \overline{\mathbb{R}^N_+}.$$

Proof. In what follows C denote an arbitrary constant. Assume that $w \geq \epsilon > 0$ on $\overline{\mathbb{R}^N_+ \cap B^+_{3\rho}}$. Let us define the function φ by

$$\varphi = \eta^2 w^\beta,$$

where $\beta > 1$, $0 \leq \eta(z) \leq 1$, $\eta \in C^1(B_{3\rho})$ and $supp(\eta) \subset B^+_\rho$. Note that

$$\nabla\varphi = \beta\eta^2 w^{\beta-1}\nabla w + 2\eta w^\beta \nabla\eta.$$

Taking φ as a test function in (1.4) we obtain

$$\int_{B^+_\rho} \left[\beta\eta^2 w^{\beta-1}|\nabla w|^2 + 2\eta w^\beta\,(\nabla\eta\cdot\nabla w)\right]\,\mathrm{d}z = \int_{\Gamma_\rho} \eta^2 w^\beta (w^{p-1} - w)\,\mathrm{d}x. \tag{3.8}$$

This yields

$$\int_{B^+_\rho} \beta\eta^2 w^{\beta-1}|\nabla w|^2\,\mathrm{d}z \leq 2\int_{B^+_\rho} \eta w^\beta|\nabla\eta||\nabla w|\,\mathrm{d}z + M^{p-2}\int_{\Gamma_\rho} \eta^2 w^{\beta+1}\,\mathrm{d}x. \tag{3.9}$$

From (3.9), using Young's inequality

$$cd \leq \frac{1}{2}\epsilon^2 c^2 + \frac{1}{2}\epsilon^{-2} d^2,$$

with $c = \eta w^{(\beta-1)/2}|\nabla w|$, $d = w^{(\beta+1)/2}|\nabla\eta|$, after some straightforward calculations we get

$$\int_{B^+_\rho} \eta^2 w^{\beta-1}|\nabla w|^2\,\mathrm{d}z \leq C\left(1 - \frac{\epsilon^2}{\beta}\right)^{-1}\beta^{-1}\left(\epsilon^{-2}\int_{B^+_\rho} w^{\beta+1}|\nabla\eta|^2\,\mathrm{d}z + \int_{\Gamma_\rho}\eta^2 w^{\beta+1}\,\mathrm{d}x\right). \tag{3.10}$$

Now, choosing β large enough and defining the function

$$v = w^s, \quad \text{where} \quad 2s = \beta + 1,$$

we have

$$\left(\frac{1}{s}\right)^2 \int_{B_\rho^+} (\eta|\nabla v|)^2 \, \mathrm{d}z \leq C\beta^{-1} \left(\int_{B_\rho^+} (|\nabla \eta| v)^2 \, \mathrm{d}z + \int_{\Gamma_\rho} (\eta v)^2 \, \mathrm{d}x \right). \tag{3.11}$$

After adding the term $\int_{\Gamma_\rho} (\eta v)^2$ to both side of (3.11) we obtain

$$\left(\|\eta|\nabla v|\|^2_{L^2(B_\rho^+)} + \|\eta v\|^2_{L^2(\Gamma_\rho)} \right)^{1/2} \leq Cs(1+\beta^{-1}) \left(\|v|\nabla \eta|\|^2_{L^2(B_\rho^+)} + \|\eta v\|^2_{L^2(\Gamma_\rho)} \right)^{1/2}. \tag{3.12}$$

Taking $\eta(z) = 1$ in B_{r_2} and $\eta(z) = 0$ outside B_{r_1} where $1 \leq r_2 < \rho \leq r_1 \leq 2,\ |\nabla \eta| \leq 2/(r_1 - r_2),\ 2\gamma = 2_*$ and $(1 + \beta^{-1}) < C$, we obtain from (3.12) that

$$\left(\|\nabla v\|^2_{L^2(B_{r_2}^+)} + \|v\|^2_{L^2(\Gamma_{r_2})} \right)^{1/2} \leq \frac{2sC}{(r_1 - r_2)} \left(\|v\|^2_{L^2(B_{r_1}^+)} + \|v\|^2_{L^2(\Gamma_{r_1})} \right)^{1/2}. \tag{3.13}$$

Using (2.6) and (2.7) we obtain

$$\|v\|_{L^{2\gamma}(B_{r_2}^+)} + \|v\|_{L^{2\gamma}(\Gamma_{r_2})} \leq C \left(\|\nabla v\|^2_{L^2(B_{r_2}^+)} + \|v\|^2_{L^2(\Gamma_{r_2})} \right)^{1/2},$$

which together with (3.13) implies that

$$\left(\|v\|^{2\gamma}_{L^{2\gamma}(B_{r_2}^+)} + \|v\|^{2\gamma}_{L^{2\gamma}(\Gamma_{r_2})} \right)^{1/(2\gamma)} \leq \frac{2sC}{(r_1 - r_2)} \left(\|v\|^2_{L^2(B_{r_1}^+)} + \|v\|^2_{L^2(\Gamma_{r_1})} \right)^{1/2}.$$

Since $v = w^s$ we get

$$\left(\int_{B_{r_2}^+} |w|^{2s\gamma} \, \mathrm{d}z + \int_{\Gamma_{r_2}} |w|^{2s\gamma} \, \mathrm{d}x \right)^{1/(2\gamma)} \leq \frac{2sC}{(r_1 - r_2)} \left(\int_{B_{r_1}^+} |w|^{2s} \, \mathrm{d}z + \int_{\Gamma_{r_1}} |w|^{2s} \, \mathrm{d}x \right)^{1/2}. \tag{3.14}$$

Moreover, taking the $s - th$ root in (3.14) and setting $\theta = 2s$ we obtain

$$\phi(\theta\gamma, r_2) \leq (C\theta(r_1 - r_2)^{-1})^{2/\theta} \phi(\theta, r_1), \tag{3.15}$$

where

$$\phi(q, r) = \left(\int_{B_r^+(y)} |w|^q \, \mathrm{d}z + \int_{\Gamma_r(y)} |w|^q \, \mathrm{d}x \right)^{1/q}, \quad q > 0,\ r > 0.$$

Now for some $\theta_o > 0$ let us define

$$\theta_m = \gamma^m \theta_o, \qquad r_m = 1 + 2^{-m}, \qquad m = 0, 1, 2, \ldots \ .$$

The choice of θ_o will be such that $\theta_m \neq 1$. Then, from (3.15) we get

$$\begin{aligned}
\phi(\theta_{m+1}, r_{m+1}) &\leq \left(C\gamma^{m+1}\theta_o (r_m - r_{m+1})^{-1} \right)^{2/(\gamma^m \theta_o)} \phi(\theta_m, r_m) \\
&\leq \left(C(2\gamma)^{m+1} \right)^{2\gamma^{-m}/\theta_o} \phi(\theta_m, r_m) \\
&= \left(C^{2/\theta_o} \right)^{\gamma^{-m}} \left((2\gamma)^{2/\theta_o} \right)^{(m+1)\gamma^{-m}} \phi(\theta_m, r_m) \\
&\leq \left(C^{2/\theta_o} \right)^{\sum \gamma^{-m}} \left((2\gamma)^{2/\theta_o} \right)^{\sum (m+1)\gamma^{-m}} \phi(\theta_o, 2).
\end{aligned} \tag{3.16}$$

Now, observing that $\gamma > 1$ and taking the limit in (3.16) we obtain

$$\max_{B_1^+} w + \max_{\Gamma_1} w = \phi(+\infty, 1) \leq C\phi(\theta_o, 2).$$

Taking $\theta_o > 1$, and making the change of variable $\overline{z} = \rho z$ with $z \in B_2^+$, and $\overline{z} = \rho x$ with $x \in \Gamma_2$, we conclude the proof. ■

Lemma 3.17. *If w is a nonnegative $H-$weak solution of (P), then it has polynomial decay in $\overline{\mathbb{R}^N_+}$, more precisely,*

$$w(z) = O(|z|^{2-N}) \quad as \quad |z| \to +\infty. \tag{3.18}$$

Proof. Consider $\varphi : \overline{\mathbb{R}^N_+} \to \mathbb{R}$ defined by $\varphi = (Aw - v)_+$ where

$$v(x,t) = \left(\frac{\mu}{(\mu+t)^2 + |x|^2}\right)^{(N-2)/2}, \quad \mu > 0 \quad \text{and} \quad z = (x,t) \in \mathbb{R}^N_+,$$

is a solution of problem

$$\begin{cases} -\Delta v = 0 & \text{in} \quad \mathbb{R}^N_+, \\ \dfrac{\partial v}{\partial \eta} = (N-2)v^{2_*-1} & \text{on} \quad \mathbb{R}^{N-1}. \end{cases}$$

Since $w(z) \to 0$ as $|z| \to \infty$ we can take $R,\ A > 0$ such that $w^{p-2}(x,0) < 1/2$ if $|x| \geq R$ and $\varphi \equiv 0$ if $|z| \leq R$. Now, using that

$$\begin{cases} -\Delta(Aw - v) = 0 & \text{in} \quad \mathbb{R}^N_+, \\ \dfrac{\partial(Aw - v)}{\partial \eta} = A(w^{p-1} - w) - (N-2)v^{2_*-1} & \text{on} \quad \mathbb{R}^{N-1}, \end{cases}$$

and choosing $\varphi = (Aw - v)_+$ as test function, we have

$$\int_{|z|\geq R} |\nabla\varphi|^2 \,\mathrm{d}z + (N-2)\int_{|x|\geq R} v^{2_*-1}\varphi \,\mathrm{d}x = A\int_{|x|\geq R} (w^{p-1} - w)\varphi \,\mathrm{d}x \leq -\int_{|x|\geq R} \frac{w}{2}\varphi \,\mathrm{d}x \leq 0.$$

Thus, $\varphi \equiv 0$ in $\overline{\mathbb{R}^N_+}$. Consequently $w \leq c_1 v$ in $\overline{\mathbb{R}^N_+}$. This then yields the desired conclusion. ■

In order to obtain the decay of Dw we need to establish some regularity result.

Lemma 3.19. *If w is a nonnegative $H-$weak solution of (P), then for each $i = 1, ..., N$ we have that $D^i w \in H^1(\mathbb{R}^N_+)$.*

Proof. Setting

$$(D_h w)(z) = \frac{w(x + he_i, t) - w(x,t)}{|h|}, \quad \text{for} \quad 1 \leq i < N-1 \quad \text{and} \quad h \in \mathbb{R}\backslash\{0\},$$

where $\{e_1, \ldots, e_{N-1}\}$ is the canonical base of $\mathbb{R}^{N-1}$. Taking $\varphi = D_{-h}(D_h w)$ in (1.4), we obtain

$$\int_{\mathbb{R}^N_+} |\nabla(D_h w)|^2 \,\mathrm{d}z + \int_{\mathbb{R}^{N-1}} |D_h w|^2 \,\mathrm{d}x = \int_{\mathbb{R}^{N-1}} D_h(w^{p-1}) D_h w \,\mathrm{d}x,$$

which implies that

$$\int_{\mathbb{R}^{N-1}} D_h(w^{p-1}) D_h w \,\mathrm{d}x \leq \int_{\mathbb{R}^{N-1}} \frac{|w^{p-1}(x + he_i, 0) - w^{p-1}(x,0)|}{|h|} |D_h w| \,\mathrm{d}x.$$

Using that for each $a, b \in (0, +\infty)$ fixed there exists $\theta \in (0,1)$ such that

$$|a^{p-1} - b^{p-1}| = (p-1)(\theta a + (1-\theta)b)^{p-2}|a-b|$$

we get

$$\|D_h w\|_\partial^2 \leq (p-1)\int_{\mathbb{R}^{N-1}} (\theta w(x+he_i,0) + (1-\theta)w(x,0))^{p-2}|D_h w|^2 \,\mathrm{d}x. \tag{3.20}$$

For fixed $\Gamma := \Gamma_R(0) \subset \mathbb{R}^{N-1}$ we have

$$\begin{aligned}
\int_{\mathbb{R}^{N-1}} &(\theta w(x+he_i,0)+(1-\theta)w(x,0))^{p-2}|D_h w|^2 \,\mathrm{d}x \\
&\leq 2^{p-2}\left[\|w\|_{L^\infty(\Gamma)}^{p-2}\int_\Gamma |D_h w|^2 \,\mathrm{d}x + \|w\|_{L^\infty(\mathbb{R}^{N-1}\setminus\Gamma)}^{p-2}\int_{\mathbb{R}^{N-1}\setminus\Gamma} |D_h w|^2 \,\mathrm{d}x\right] \\
&\leq 2^{p-2}\left[\|w\|_{L^\infty(\mathbb{R}^{N-1})}^{p-2}\int_\Gamma |D_h w|^2 \,\mathrm{d}x + \|w\|_{L^\infty(\mathbb{R}^{N-1}\setminus\Gamma)}^{p-2}\int_{\mathbb{R}^{N-1}} |D_h w|^2 \,\mathrm{d}x\right].
\end{aligned}$$

Now, by Lemma 3.7 we can choose Γ such that

$$\|w\|_{L^\infty(\mathbb{R}^{N-1}\setminus\Gamma)}^{p-2} < \frac{1}{(p-1)2^{p-1}}.$$

This, together with (3.20) implies that

$$\int_{\mathbb{R}^N_+} |\nabla(D_h w)|^2 \,\mathrm{d}z + \int_{\mathbb{R}^{N-1}} |D_h w|^2 \,\mathrm{d}x \leq C(p, \|w\|_{L^\infty(\Gamma)})\int_\Gamma |D_h w|^2 \,\mathrm{d}x.$$

Since $w \in C^{1,\alpha}(\Gamma)$ we obtain

$$\int_{\mathbb{R}^N_+} |\nabla(D_h w)|^2 \,\mathrm{d}z + \int_{\mathbb{R}^{N-1}} |D_h w|^2 \,\mathrm{d}x \leq C. \tag{3.21}$$

For $1 \leq j \leq N$ we denote $D^j = \partial/\partial z_j$. For each $\varphi \in C_0^\infty(\mathbb{R}^N_+)$, and the definition of weak derivative together with (3.21) we have

$$\begin{aligned}
\left|\int_{\mathbb{R}^N_+} wD_{-h}(D^j\varphi)\,\mathrm{d}z\right| &= \left|\int_{\mathbb{R}^N_+} D_h(D^j w)\varphi \,\mathrm{d}z\right| \\
&\leq \|D_h(D^j w)\|_{L^2(\mathbb{R}^N_+)}\|\varphi\|_{L^2(\mathbb{R}^N_+)} \leq C\|\varphi\|_{L^2(\mathbb{R}^N_+)}.
\end{aligned}$$

Taking the limit when $|h| \to 0$ we obtain

$$\left|\int_{\mathbb{R}^N_+} wD^{i,j}\varphi \,\mathrm{d}z\right| \leq C\|\varphi\|_{L^2(\mathbb{R}^N_+)}, \tag{3.22}$$

for all $1 \leq i \leq N-1$ and $1 \leq j \leq N$. To conclude, taking $\varphi \in C_0^\infty(\mathbb{R}^N_+)$ as a test function in (1.4) and using (3.22) we get

$$\begin{aligned}
\left|\int_{\mathbb{R}^N_+} wD^{N,N}\varphi \,\mathrm{d}z\right| &= \left|\int_{\mathbb{R}^N_+} D^N w \; D^N\varphi \,\mathrm{d}z\right| \\
&\leq \sum_{i=1}^{N-1}\left|\int_{\mathbb{R}^N_+} wD^{i,i}\varphi \,\mathrm{d}z\right| \leq C\|\varphi\|_{L^2(\mathbb{R}^N_+)}.
\end{aligned}$$

Therefore,

$$\left|\int_{\mathbb{R}^N_+} wD^{i,j}\varphi\,\mathrm{d}x\right| \leq C\|\varphi\|_{L^2(\mathbb{R}^N_+)},$$

for all $1 \leq i \leq N$ and $1 \leq j \leq N$. This, together with Hanh-Banach Theorem and Riesz representation theorem implies that $D^i w \in H^1(\mathbb{R}^N_+)$ for all $1 \leq i \leq N$. Using the trace embedding, we conclude the proof of the lemma. ■

Lemma 3.23. *If $w \in H$ is a nonnegative $H-$weak solution of (P), then for each $1 \leq i \leq N$, we have*

$$\lim_{|z|\to\infty} |D^i w(z)| = 0, \qquad z \in \overline{\mathbb{R}^N_+}.$$

Proof. To prove the lemma, first we consider $1 \leq i \leq N-1$. Then for each $\varphi \in C_0^\infty(\mathbb{R}^N)$, if we take $D^i\varphi$ as a test function in (1.4) we get

$$\int_{\mathbb{R}^N_+} \nabla w \nabla(D^i\varphi)\,\mathrm{d}z + \int_{\mathbb{R}^{N-1}} wD^i\varphi\,\mathrm{d}x = \int_{\mathbb{R}^{N-1}} w^{p-1}D^i\varphi\,\mathrm{d}x.$$

Thus,

$$\int_{\mathbb{R}^N_+} \nabla(D^i w)\nabla\varphi\,\mathrm{d}z + \int_{\mathbb{R}^{N-1}} D^i w\varphi\,\mathrm{d}x = \int_{\mathbb{R}^{N-1}} (p-1)w^{p-2}D^i w\varphi\,\mathrm{d}x,$$

that is, $v = D^i w$ is a weak solution of (3.2) with $q = 1$ and $a = (p-1)w^{p-2}$. By Lemma 3.1 we conclude that $D^i w \in L^\infty(\mathbb{R}^N_+)$ and its trace belongs to $L^\infty(\mathbb{R}^{N-1})$. Now, taking $\varphi = \eta(D^i w)^\beta_\pm$ as a test function in (1.4) (where $\eta \in C_0^\infty(\mathbb{R}^N)$, $\beta > 1$), and arguing as in the proof of Lemma 3.7, one can complete the case $1 \leq i \leq N-1$.

For the case $i = N$, it is sufficient to observe that $w_t = w - w^{p-1}$ on $\mathbb{R}^{N-1}$ and w_t is a harmonic function in $\mathbb{R}^N_+$. ■

Corollary 3.24. *If w is a nonnegative $H-$weak solution of (P), then $w \in C^\infty(\mathbb{R}^N_+) \cap C^{2,\alpha}_{loc}(\overline{\mathbb{R}^N_+})$.*

Proof. Since w is a harmonic function we have that $w \in C^\infty(\mathbb{R}^N_+)$. From Lemma 3.1 and regularity results proved in [7], we obtain that $H-$weak solutions of (P) belongs to $C^{1,\alpha}_{loc}(\overline{\mathbb{R}^N_+})$. By Lemma 3.19, $v = D^i w$ $i = 1, ..., N-1$ is a $H-$weak solution of (3.2) with $a = (p-1)w^{p-2} \in L^\infty(\mathbb{R}^{N-1})$ and $q = 1$. Thus, $D^i w \in L^\infty(\mathbb{R}^N_+)$ and its trace belongs to $L^\infty(\mathbb{R}^{N-1})$. The case $i = N$ follows as in the prove of Lemma 3.19. By results of Lieberman [7], we get that $D^i w \in C^\infty(\mathbb{R}^N_+)$ which concludes the proof. ■

4 Symmetry and exponential decay

Next, we will prove that nonnegative $H-$weak solutions of (P) are radially symmetric with respect to the first $N-1$ variables, by using the regularity and decay obtained in Section 3 (see [14] for a related result). The proof relies on the so-called moving planes technique due to Serrin [12], see also the celebrated paper [3] by Gidas-Ni-Nirenberg.

We point out that the next result will be used to prove the exponential decay in the first $N-1$ variables for nonnegative $H-$weak solutions of (P).

Proposition 4.1. *If w is a nonnegative $H-$weak solution of (P), then w is radially symmetric with respect to the variable x, that is, $w(x,t) = w(r,t)$ if $r = |x|$. Moreover, $w_r(r,t) < 0$ in $(0,+\infty) \times [0,+\infty)$.*

Proof. For $\lambda > 0$ we consider the reflection

$$z = (x_1, x_2, \dots, t) \mapsto z^\lambda = (2\lambda - x_1, x_2, \dots, t)$$

where $z \in E_\lambda = \{z \in \overline{\mathbb{R}^N_+} : x_1 > \lambda\}$, and we put

$$u^\lambda(z) = w(z^\lambda) - w(z).$$

Note that

$$w(z^\lambda) = w(z) \quad \text{for} \quad z \in T_\lambda = \{z \in \mathbb{R}^N_+ : x_1 = \lambda\}.$$

Step1. We claim that there exists $\lambda > 0$ such that

$$u^\lambda(z) > 0 \quad \text{for all} \quad z \in E_\lambda. \tag{4.2}$$

Indeed, since $w(z) \to 0$ as $|z| \to \infty$, we can choose λ sufficiently large such that

$$u^\lambda(2\lambda, x_2, ..., t) = w(0, x_2, ..., t) - w(2\lambda, x_2, ..., t) > 0. \tag{4.3}$$

Next, we prove that (4.2) holds for this choice of λ. Arguing by contradiction, let us assume that there exists $z_\lambda \in E_\lambda$ such that $u^\lambda(z_\lambda) \le 0$. In particular, we can take

$$u^\lambda(z_\lambda) = \inf \left\{u^\lambda(z) : z \in E_\lambda\right\} \le 0.$$

We claim that $z_\lambda \in \mathbb{R}^{N-1} \cap E_\lambda$. Otherwise, we have $z_\lambda \in \mathbb{R}^N_+ \cap E_\lambda$, and thus $B(z_\lambda, 2\delta) \subset \mathbb{R}^N_+ \cap E_\lambda$ for some $\delta > 0$ sufficiently small. Using that $v^\lambda(z) = u^\lambda(z) - u^\lambda(z_\lambda)$ we have $v^\lambda(z_\lambda) = 0$ and

$$\begin{cases} \Delta v^\lambda = 0 & \text{in} \quad B(z_\lambda, \delta) \\ \quad v^\lambda \ge 0 & \text{in} \quad B(z_\lambda, \delta). \end{cases}$$

In view of Harnack inequality and unique continuation methods for elliptic equations, we conclude that $v^\lambda \equiv 0$ in E_λ. Consequently, u^λ is a non positive constant in E_λ, which contradicts (4.3). Thus, we conclude that $u^\lambda(z) > 0$ for all $z \in E_\lambda \cap \mathbb{R}^N_+$, which implies that $u^\lambda(z) \ge 0$ for all $z \in E_\lambda \cap \mathbb{R}^{N-1}$. Hence, $z_\lambda \in E_\lambda \cap \mathbb{R}^{N-1}$ and $u^\lambda(z_\lambda) = \inf \left\{u^\lambda(z) : z \in E_\lambda\right\} = 0$. Taking a ball $B \subset E_\lambda \cap \mathbb{R}^N_+$ such that $z_\lambda \in \partial B$ we have

$$\begin{cases} \Delta u^\lambda = 0 & \text{in} \quad B \\ \quad u^\lambda > 0 & \text{in} \quad B, \end{cases}$$

which together with Hopf's lemma implies that $(\partial u^\lambda / \partial \nu)(z_\lambda) < 0$, in contradiction with

$$\frac{\partial u^\lambda}{\partial \nu}(z_\lambda) = \frac{\partial w^\lambda}{\partial \nu} - \frac{\partial w}{\partial \nu} = -w^\lambda_t + w_t = [(w^\lambda)^{p-1} - w^\lambda] + w - w^{p-1} = 0.$$

Step2. Set

$$\lambda_0 := \inf\{\lambda > 0 \text{ such that (4.2) holds}\}. \tag{4.4}$$

We will prove that $\lambda_0 = 0$. Assume instead that $\lambda_0 > 0$. Since $u^{\lambda_0} \equiv 0$ on T_{λ_0} and

$$\begin{cases} \Delta u^{\lambda_0} = 0 & \text{in} \quad E_{\lambda_0} \\ \quad u^{\lambda_0} > 0 & \text{in} \quad E_{\lambda_0}, \end{cases}$$

it follows by Hopf's lemma that

$$2 w_{x_1}(\lambda_0, \overline{x}) = -u^{\lambda_0}_{x_1}(\lambda_0, \overline{x}) < 0, \tag{4.5}$$

where $\overline{x} = (x_2, ..., t)$. Thus, there exist $\epsilon > 0$ such that $2(\lambda_0 - \epsilon) - x_1 < \lambda_0 - \epsilon < x_1 < \lambda_0$ and

$$u^{\lambda_0-\epsilon}(x_1, \overline{x}) = w(2(\lambda_0 - \epsilon) - x_1, \overline{x}) - w(x_1, \overline{x}) > 0. \tag{4.6}$$

Consequently, for each $(\lambda_0, \overline{x}) \in T^{\lambda_0}$ there exist $\delta > 0$ such that

$$u^{\lambda_0-\epsilon}(z) > 0 \quad \text{for all} \quad z \in B((\lambda_0, \overline{x}), \delta) \cap (\mathbb{R}^N_+ \setminus E_{\lambda_0}). \tag{4.7}$$

We claim that there exist $\epsilon > 0$ such that

$$u^{\lambda_0-\epsilon}(z) > 0 \quad \text{for all} \quad z \in E_{\lambda_0-\epsilon}. \tag{4.8}$$

Otherwise, there exists a sequence satisfying $\lambda_k \to \lambda_0$, $\lambda_k < \lambda_0$ and a sequence $(z_k) \subset E^{\lambda_k}$ such that $u^{\lambda_k}(z_k) < 0$ and $dist(z_k, T^{\lambda_0}) \to 0$. We have two cases to consider: either there exists a subsequence such that $z_{k_l} \to z_0 \in T^{\lambda_0}$, which is impossible, in view of (4.7), or else $\|z_k\| \to \infty$. In the latter case, using (4.7) we may assume without loss of generality that

$$u^{\lambda_k}(z_k) = \inf\{u^{\lambda_k}(z) : z \in E^{\lambda_k}\}.$$

Since $v^{\lambda_k}(z) := u^{\lambda_k}(z) - u^{\lambda_k}(z_k)$ we have $v^{\lambda_k}(z_k) = 0$ and

$$\begin{cases} \Delta v^{\lambda_k} = 0 & \text{in} \quad B_{\delta_k}(z_k) \\ v^{\lambda_k} > 0 & \text{in} \quad B_{\delta_k}(z_k). \end{cases}$$

Using Hanark inequality, we obtain that $v^{\lambda_k} \equiv 0$ in $B_{\delta_k}(z_k)$, which together with unique continuation methods for elliptic equations implies that u^{λ_k} is constant in E_{λ_k}, in contradiction with $u^{\lambda_k} \in H$. Thus, the assertion (4.8) contradicts our choice of λ_0, if $\lambda_0 > 0$.

Since $\lambda_0 = 0$, we see that $w(-x_1, ..., x_{N-1}, t) \geq w(x_1, ..., x_{N-1}, t)$ in $\overline{\mathbb{R}^N_+}$. A similar argument shows that $w(-x_1, ..., x_{N-1}, t) \leq w(x_1, ..., x_{N-1}, t)$. Thus w is symmetric in the plane T_0 and $w_{x_1} = 0$ on T_0. This argument applies as well after any rotation of coordinate axes in the variables $x_2, ..., x_{N-1}$.

Finally, setting $w(x, t) = v(r, t)$ where $r = |x|$, we will prove that $v_r(r, t) < 0$ for all $(r, t) \in (0, +\infty) \times [0, +\infty)$. For this, since that w is symmetric in $\mathbb{R}^{N-1}$, the same argument used to get (4.5) holds for $x_2, ..., x_{N-1}$ and all $\lambda > 0$. Therefore, it is sufficient to choose any point $x_0 \in \mathbb{R}^{N-1}$ such that $x_0 = (x_{1,0}, ..., x_{N-1})$ with $x_{i,0} > 0$ and note that

$$v_r(r_0, t) = \sum_{i=1}^{N-1} \frac{\partial w}{\partial x_i}(x_0, t) \cdot \frac{x_{i,0}}{|x_0|} < 0, \quad r_0 = |x_0|.$$

Again, by the symmetry of w we conclude $v_r(r, t) < 0$, for all $(r, t) \in (0, +\infty) \times (0, +\infty)$. To conclude, we need to prove that $v_r(r, 0) < 0$ for all $r > 0$. Arguing by contradiction suppose that $v_r(r_0, 0) = 0$ for some $r_0 > 0$. Since $w \in C^{2,\alpha}_{loc}(\overline{\mathbb{R}^N_+}) \cap C^\infty(\mathbb{R}^N_+)$ we get

$$\begin{cases} \Delta v_r = 0 & \text{in} \quad B^+(r_0) \\ v_r < 0 & \text{in} \quad B^+(r_0), \end{cases}$$

where $B^+(r_0) = B_\delta(r_0, 0) \cap \overline{\mathbb{R}^2_+}$ for some $\delta > 0$. By applying Hopf's lemma we conclude that

$$0 < \partial v_r(r_0, 0)/\partial \eta = -(v_r)_t(r_0, 0) = -(v_t)_r(r_0, 0) = v_r(r_0, 0)[(p-1)v^{p-2} - 1] = 0,$$

which is impossible. ■

In order to obtain the exponential decay of w we will use the follow result.

Lemma 4.9. *Let w be a nonnegative $H-$weak solution of (P). Then for each $\nu > 0$ there exists $c_i = c_i(\nu) > 0$ such that for each $i = 1, ..., N-1$ we have*

$$w(x_1, ..., x_i, ..., t) \leq c_i |D^i w(x_1, ..., x_i, ..., t)|, \quad |x_i| \geq \nu. \tag{4.10}$$

Proof. Fixed $i \in \{1, ..., N-1\}$ and $\nu > 0$, for each $x \in \overline{\mathbb{R}^N_+}$ define

$$D^i_\nu w(z) := \begin{cases} D^i w(x_1, ..., x_i + \nu, ..., t), & \text{if} \quad x_i > 0 \\ D^i w(x_1, ..., -x_i + \nu, ..., t), & \text{if} \quad x_i \leq 0. \end{cases}$$

Note that, by Proposition 4.1, $D^i w = w_r x_i / r < 0$ for all $x_i > 0$, which together with Lemma 3.17, implies that we may choose $R > 0$ and $A_{i_1} := A_{i_1}(R, \nu) > 0$ such that

$$\begin{array}{rlll} w^{p-2}(z) \leq \dfrac{1}{2(p-1)} & \text{for all} \quad z \in \overline{\mathbb{R}^N_+} & \text{with} & |z| \geq R \\ \varphi_i := (A_{i_1} w + D^i_\nu w)_+ \equiv 0 & \text{for all} \quad z \in \overline{\mathbb{R}^N_+} & \text{with} & |z| \leq R. \end{array}$$

Taking φ_i as a test function in the problem

$$\begin{cases} -\Delta(A_{i_1} w + D^i_\nu w) &= 0 & \text{in} & \mathbb{R}^N_+, \\ -\dfrac{\partial(A_{i_1} w + D^i_\nu w)}{\partial x_n} &= A_{i_1} w^{p-1} + (p-1) w^{p-2} D^i_\nu w - (A_{i_1} w + D^i_\nu w) & \text{on} & \mathbb{R}^{N-1}, \end{cases}$$

we obtain

$$\begin{aligned} \int_{\mathbb{R}^N_+} |\nabla \varphi_i|^2 \, \mathrm{d}z + \int_{\mathbb{R}^{N-1}} \varphi_i^2 \, \mathrm{d}x &= \int_{\mathbb{R}^{N-1}} \left(A_{i_1} w^{p-1} + (p-1) w^{p-2} D^i_\nu w \right) \varphi_i \, \mathrm{d}x \\ &\leq (p-1) \int_{\mathbb{R}^{N-1}} w^{p-2} \left(A_{i_1} w + D^i_\nu w \right) \varphi_i \, \mathrm{d}x \leq \frac{1}{2} \int_{\mathbb{R}^{N-1}} \varphi_i^2 \, \mathrm{d}x. \end{aligned}$$

Thus, $\varphi_i \equiv 0$ in $\overline{\mathbb{R}^N_+}$, which yields

$$w(x_1, ..., x_i, ..., t) \leq A_{i_1}^{-1} (-D^i w(x_1, ..., x_i + \nu, ..., t)).$$

Since $D^i w(x_1, ..., x_i, ..., t) < 0$ for $x_i > 0$ we obtain

$$w(x_1, ..., x_i + \nu, ..., t) < w(x_1, ..., x_i, ..., t) \leq A_{i_1}^{-1} (-D^i w(x_1, ..., x_i + \nu, ..., t)). \tag{4.11}$$

Now, define for all $z \in \overline{\mathbb{R}^N_+}$, the function

$$D^i_{-\nu} w(z) := \begin{cases} D^i w(x_1, ..., x_i - \nu, ..., t), & \text{if} \quad x_i < 0 \\ D^i w(x_1, ..., -x_i - \nu, ..., t), & \text{if} \quad x_i \geq 0. \end{cases}$$

Note that, by Proposition 4.1, $D^i w = w_r x_i / r > 0$ for all $x_i < 0$, which together with Lemma 3.17 implies that there exist $R > 0$ and $A_{i_2} := A_{i_2}(R, \nu) > 0$ such that

$$\begin{array}{rlll} w^{p-2}(z) \leq \dfrac{1}{2(p-1)} & \text{for all} \quad z \in \overline{\mathbb{R}^N_+} & \text{with} & |z| \geq R \\ \phi_i := (A_{i_2} w - D^i_{-\nu} w)_+ \equiv 0 & \text{for all} \quad z \in \overline{\mathbb{R}^N_+} & \text{with} & |z| \leq R. \end{array}$$

Taking ϕ_i as a test function in the problem

$$\begin{cases} -\Delta(A_{i_2} w - D^i_{-\nu} w) = 0 & \text{in} \quad \mathbb{R}^N_+, \\ -\dfrac{\partial(A_{i_2} w - D^i_{-\nu} w)}{\partial x_n} = A_{i_2} w^{p-1} - (p-1) w^{p-2} D^i_{-\nu} w - (A_{i_2} w - D^i_{-\nu} w) & \text{on} \quad \mathbb{R}^{N-1} \end{cases}$$

and arguing as above, we get $\phi_i \equiv 0$ in $\overline{\mathbb{R}^N_+}$, which yields

$$w(x_1, ..., x_i, ..., t) \leq A_{i_2}^{-1}(D^i_{-\nu} w) = A_{i_2}^{-1} D^i w(x_1, ..., x_i - \nu, ..., t).$$

Since $D^i w(x_1, ..., x_i, ..., t) > 0$ for $x_i < 0$ we have

$$w(x_1, ..., x_i - \nu, ..., t) < w(x_1, ..., x_i, ..., t) \leq A_{i_2}^{-1} D^i w(x_1, ..., x_i - \nu, ..., t), \quad x_i < 0. \tag{4.12}$$

The desired conclusion follows easily from (4.11)-(4.12). ■

Now, we summarize our results about the decay estimate from above.

Proposition 4.13. *Let w be a nonnegative $H-$weak solution of (P). Then, there exist $c_1, c_2 > 0$ such that*

$$w(x,t) \leq c_1 \exp(-c_2|x|) \frac{1}{(1+t^2)^{\frac{N-2}{2}}}, \qquad \forall\ z \in \overline{\mathbb{R}^N_+}.$$

Proof. If $x_i > 0$ $(1 \leq i \leq N-1)$, by inequality (4.10) we get

$$\frac{\partial}{\partial x_i}(ln(w(x_1, ..., x_i + \nu, ..., t))) = \frac{D^i w(x_1, ..., x_i + \nu, ..., t)}{w(x_1, ..., x_i + \nu, ..., t)} \leq -c_i^{-1}.$$

By integration, we get

$$ln(w(x_1, ..., x_i + \nu, ..., t)) - ln(w(x_1, ..., \nu, ..., t)) \leq -c_i^{-1} x_i,$$

that is,

$$w(x_1, ..., x_i + \nu, ..., t) \leq w(x_1, ..., \nu, ..., t) \exp(-c_i^{-1}|x_i|), \quad x_i > 0. \tag{4.14}$$

Using again (4.10), we obtain

$$c_i^{-1} \leq \frac{D^i w(x_1, ..., x_i - \nu, ..., t)}{w(x_1, ..., x_i - \nu, ..., t)} = \frac{\partial}{\partial x_i}(ln(w(x_1, ..., x_i - \nu, ..., t))).$$

Analogously if $x_i < 0$, using (4.10) we get

$$c_i^{-1}(0 - x_i) \leq ln(w(x_1, ..., -\nu, ..., t)) - ln(w(x_1, ..., x_i - \nu, ..., t)),$$

which implies that,

$$w(x_1, ..., x_i - \nu, ..., t) \leq w(x_1, ..., -\nu, ..., t) \exp(-(c_i^{-1})|x_i|), \quad x_i < 0. \tag{4.15}$$

It follows from (4.14)-(4.15) and Lemma 3.17 that

$$w(x_1, ..., x_i, ..., t) \leq c_1 \frac{1}{(1+t^2)^{(N-2)/2}} \exp(-c_2|x_i|), \quad |x_i| \geq \nu > 0,$$

which implies the desired result. ■

To complete the proof of Theorem 1.6 we only need to obtain the lower polynomial decay on the variable t. Using the mean value theorem for harmonic functions, we have

$$u(x,t) = \frac{1}{\omega_N R^N} \int_{B((x,t),R)} u(\overline{z})\, \mathrm{d}\overline{z}, \quad \forall\ \ B((x,t),R) \subset \mathbb{R}^N_+,$$

where ω_N denotes the volume of the unit ball in $\mathbb{R}^N$. We may assume that $t > 1$. Now, taking $R = t$ we get

$$u(x,t) = \frac{1}{\omega_N t^N} \int_{B((x,t),t)} u(\overline{z})\, \mathrm{d}\overline{z} \geq \frac{1}{\omega_N t^N} \int_{B((x,1),1)} u(\overline{z})\, \mathrm{d}\overline{z} = \frac{C(x)}{t^N}, \tag{4.16}$$

for all $t \geq 1$. The proof of Theorem1.6 is complete.

References

[1] E. Abreu, J. M. do Ó, E. S. Medeiros, *Least energy solutions for a semilinear elliptic problem with nonlinear boundary condition* (in preparation).

[2] Adimurthi; S. L. Yadava, *Positive solution for Neumann problem with critical nonlinearity on boundary*, Comm. Partial Differential Equations **16** (1991),1733–1760.

[3] G. Gidas, W. Ni and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Physics **68** (1979), 209-243.

[4] B. Hu, *Nonexistence of a positive solution of the Laplace equation with a nonlinear boundary condition*, Differential Integral Equations **7** (1994), 301-313.

[5] B. Hu and H. -M. Yin, *The profile near blowup time for solution of the heat equation with a nonlinear boundary condition*, Trans. Amer. Math. Soc. **346** (1994), 117-135.

[6] Y. Li, L. Zhang, *Liouville-type theorems and Harnack-type inequalities for semilinear elliptic equations*, Journal d'Analyse Mathmatique, **90** (2003), 27–87.

[7] G. M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. **12** (1988), 1203-1219.

[8] J. L. Lions, *Theorèmes de trace et d'interpolation (IV)*, Math. Ann. **151** (1963), 42-56.

[9] P. L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case, Part II,* Ann. Inst. H. Poincaré Anal. Non Linéare **1**, (1984) 223-283.

[10] W. M. Ni and I. Takagi, *On the shape of least-energy solutions to a semilinear Neumann problem*, Comm. Pure Appl. Math. **44** (1991), 819–851.

[11] J. Serrin, *Local Behavior of solutions of quasilinear equations*, Acta. Math., **111** (1964), 247-302.

[12] J. Serrin, *A symmetry theorem in potential theory*, Arch. Rational Mech. Anal. **43** (1971), 304-318.

[13] M. V. Steklov, *Sur les problèmes fondamentaux de la physique mathemátique*, Ann. Sci. Ecole Norm. Sup., **19** (1902), 455-490.

[14] S. Terracini, *Symmetry properties of positive solutions to some elliptic equations with nonlinear boundary conditions*, Differential Integral Equations **8** (1995), 1911-1922.

[15] P. Tolksdorff, *Regularity for a more general class of quasilinear elliptic equations*, J.Differential Equations, **51** (1984), 126-150.

[16] N. S. Trudinger, *On harnack type inequalities and their application to quasilinear elliptic equations*, Comm. Pure Appl. Math **20** (1967), 721-747.

[17] J. L. Vazquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. **12** (1984) 191-202.