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SEMI-CLASSICAL STATES FOR QUASILINEAR SCHRÖDINGER EQUATIONS ARISING IN PLASMA PHYSICS

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In this paper, the existence and qualitative properties of positive ground state solutions for the following class of Schrödinger equations $-\varepsilon^2\Delta u + V(x)u - \varepsilon^2[\Delta(u^2)]u = f(u)$ in the whole two-dimensional space are established. We develop a variational method based on a penalization technique and Trudinger–Moser inequality, in a nonstandard Orlicz space context, to build up a one parameter family of classical ground state solutions which concentrates, as the parameter approaches zero, around some point at which the solutions will be localized. The main feature of this paper is that the nonlinearity f is allowed to enjoy the critical exponential growth and also the presence of the second order nonhomogeneous term $-\varepsilon^2[\Delta(u^2)]u$ which prevents us from working in a classical Sobolev space. Our analysis shows the importance of the role played by the parameter ε for which is motivated by mathematical models in physics. Schrödinger equations of this type have been studied as models of several physical phenomena. The nonlinearity here corresponds to the superfluid film equation in plasma physics.

Keywords: Schrödinger equations; solitary waves; variational methods; Orlicz spaces; Trudinger–Moser inequality.

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1. Introduction

This paper deals with the study of positive ground state solutions of the equation

$$\begin{aligned} -\varepsilon^2\Delta u + V(z)u - \varepsilon^2[\Delta(u^2)]u &= f(u) \quad \text{in } \mathbb{R}^2 \\ u(z) &\to 0 \qquad \text{as } |z| \to \infty. \end{aligned} \tag{P_ε}$$

A basic motivation for the study of this equation comes from the fact that it is satisfied by standing-wave solutions of the quasilinear Schrödinger equations

$$i\varepsilon\frac{\partial\psi}{\partial t} = -\varepsilon^2\Delta\psi + W(z)\psi - l(|\psi|^2)\psi - \varepsilon^2\kappa[\Delta h(|\psi|^2)]h'(|\psi|^2)\psi \tag{1}$$

namely, solutions of the form $\psi(t,z) = e^{-i\xi t}u(z)$, where $\xi \in \mathbb{R}$ and $u > 0$ is a real function. With this ansatz, one obtains a corresponding equation of elliptic type like (P_ε) which has a formal variational structure whose amplitude $u(z)$ vanishes at infinity.

Quasilinear equations of the form (1) appear naturally in mathematical physics and have been derived as models of several physical phenomena corresponding to various types of nonlinear term h. The superfluid film equation in plasma physics has this structure for $h(s) = s$, see [23]. In the case $h(s) = (1+s)^{1/2}$, Eq. (1) models the self-channeling of a high-power ultra short laser in matter, see [37]. Equation (1) also appears in fluid mechanics [24], in the theory of Heisenberg ferromagnets and magnons [42], in dissipative quantum mechanics and in condensed matter theory [28].

Motivated by the afore-mentioned physical aspects, Eq. (1) has recently attracted a lot of attention and some existence results have been obtained. Direct variational methods by using constrained minimization arguments were used in [34] and then extended in [27] to provide existence of positive solutions up to an unknown Lagrange multiplier because of the mixed homogeneity in Eq. (1). A Nehari manifold approach was used in [26] to establish existence of a class of solutions, in a suitable weak sense, among which sign changing solutions are also included. In dimension one, the existence of positive solutions via perturbation methods are obtained in [3] and we refer to [11] for existence of multiple nodal bound states. In [12, 25, 29, 31] a reduction method was introduced which relies on a suitable change of variable which turns the problem into finding solutions of an auxiliary semilinear equation. In particular, in [25] a very interesting but somehow intricate Orlicz space framework was proposed to set up the problem. Existence results when the nonlinearity f exhibits critical exponential growth in dimension two are also established, under additional conditions, in [17, 31] while in [30] the fibering method is used to obtain multiplicity results for closely related problems.

An interesting class of solutions of (P_ε) are the so called semi-classical states, which are families of solutions u_ε which develop a spike shape around one or more distinguished points of the space, while vanishing asymptotically elsewhere as $\varepsilon \to 0$ see [2, 5, 9, 14, 15, 18, 21].

The prospect of exhibiting a unify variational framework of concentration of single spike solutions, associated to general topology of nontrivial critical points of the potential V for such a disparate class of equations with critical exponential growth in $\mathbb{R}^2$, is the main motivating factor to write this paper.

In recent years, the related semilinear equations for $\kappa = 0$ have been extensively studied. See, for example, [1, 2, 5–8, 14, 15, 35, 38–41] and references therein.

Throughout this paper the following hypotheses on the potential $V : \mathbb{R}^2 \to \mathbb{R}$ will be assumed:

(V_0) V is locally Hölder continuous and uniformly positive, that is,

$$V(z) \geq \beta_0 > 0 \quad \text{for all } z \in \mathbb{R}^2;$$

(V_1) There exists a bounded smooth domain $\Lambda \subset \mathbb{R}^2$ such that

$$\inf_{z \in \partial\Lambda} V(z) > \inf_{z \in \Lambda} V(z) =: \beta_1.$$

We are interested in the case that the nonlinear term $f(s)$ has the maximal growth which allows us to treat the problem (P_ε) variationally in a suitable function space. In fact the Trudinger–Moser inequality is one of the main ingredients of the present paper. We say that the function f has subcritical growth at infinity if for all $\alpha > 0$,

$$\lim_{s \to +\infty} \frac{f(s)}{e^{\alpha s^4}} = 0 \tag{2}$$

and f has critical growth at infinity if there exists $\alpha_0 > 0$ such that

$$\lim_{s \to +\infty} \frac{f(s)}{e^{\alpha s^4}} = \begin{cases} 0, & \text{if } \alpha > \alpha_0, \\ +\infty, & \text{if } \alpha < \alpha_0. \end{cases} \tag{3}$$

Note that such notion is motivated by Trudinger–Moser estimates in a bounded domain $\Omega \subset \mathbb{R}^2$ [32, 43] which provides for all $\alpha > 0$,

$$e^{\alpha|u|^2} \in L^1(\Omega), \quad u \in H_0^1(\Omega),$$

and for all $\alpha \leq 4\pi$,

$$\sup_{\|\nabla u\|_2 \leq 1} \int_\Omega e^{\alpha|u|^2} \mathrm{d}x \leq C,$$

as well as for the entire space $\mathbb{R}^2$ [8, 16] which provides for all $\alpha > 0$,

$$e^{\alpha|u|^2} - 1 \in L^1(\mathbb{R}^2), \quad u \in H^1(\mathbb{R}^2) \tag{4}$$

and also if $\alpha < 4\pi$ and $\|u\|_2 \leq C$, there exists a constant $C_1 = C_1(C, \alpha)$ such that

$$\sup_{\|\nabla u\|_2 \leq 1} \int_{\mathbb{R}^2} (e^{\alpha|u|^2} - 1) \mathrm{d}x \leq C_1. \tag{5}$$

We assume the following conditions on the nonlinearity f:

(f_0) $f : [0, \infty) \to \mathbb{R}$ is of class C^1 and $f(s) = o(s)$ at the origin.
(f_1) There exists $q > 3$ such that

$$f'(s)s \geq qf(s) \quad \text{for } s > 0.$$

As an immediate consequence of (f_1), the following version of the classical Ambrosetti–Rabinowitz condition holds:

$$0 < \theta F(s) \leq sf(s) \quad \text{for } s > 0, \tag{6}$$

where $\theta = q+1 > 4$ and $F(s) = \int_0^s f(t)\mathrm{d}t$. Also it follows from (f_1) that $f(s)/s$ is increasing for $s > 0$.

The main results of this paper are stated as follows.

Theorem 1 (The Subcritical Case). *Suppose* (V_0)–(V_1) *hold and* f *has subcritical growth and satisfies the conditions* (f_0) *and* (f_1)*. Then there exists* $\varepsilon_0 > 0$ *such that when* $0 < \varepsilon < \varepsilon_0$*, the problem* (P_ε) *possesses a positive ground state solution* $u_\varepsilon(z) \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^2)$ *for some* $\alpha \in (0,1)$ *with the following properties*:

(i) u_ε *has at most one local* (*hence global*) *maximum* z_ε *in* $\mathbb{R}^2$ *and* $z_\varepsilon \in \Lambda$;
(ii) $\lim_{\varepsilon\to 0^+} V(z_\varepsilon) = \beta_1 = \inf_\Lambda V$;
(iii) *There exist positive constants* C *and* ξ *such that*

$$u_\varepsilon(z) \le Ce^{-\xi|(z-z_\varepsilon)/\varepsilon|} \quad \textit{for } z \in \mathbb{R}^2.$$

Theorem 2 (The Critical Case). *Suppose* (V_0)–(V_1) *hold and* f *has critical growth and satisfies the conditions* (f_0) *and* (f_1) *as well as the following condition*

(f_2) *There exist* $p > 2$ *and* $C_p > 0$ *such that* $f(s) \ge C_p s^{p-1}$ *for all* $s \ge 0$ *where*

$$C_p > \left[\frac{\theta(p-2)}{p(\theta-4)}\right]^{(p-2)/2} (S_p^\infty)^p \quad \textit{and}$$

$$S_p^\infty := \inf_{u\in H_r^1(\mathbb{R}^2)\setminus\{0\}} \frac{\left[\int_{\mathbb{R}^2}(|\nabla u|^2+\beta_1 u^2)\mathrm{d}x + \left(\int_{\mathbb{R}^2} u^2|\nabla u|^2\mathrm{d}x\right)^{1/2}\right]^{1/2}}{\left(\int_{\mathbb{R}^2}|u|^p\mathrm{d}x\right)^{1/p}}.$$

Then there exists $\varepsilon_0 > 0$ *such that when* $0 < \varepsilon < \varepsilon_0$ *problem* (P_ε) *possesses a positive ground state solution* $u_\varepsilon(z) \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^2)$ *for some* $\alpha \in (0,1)$ *satisfying the properties* (i)–(iii) *of Theorem* 1.

Our premise here is that the assumptions in Theorems 1 and 2 are prevalent in the equations originating on the subject. Most of nonlinearities with critical growth verify (f_2) and also this condition is more general than the following one used in [17]

$$\lim_{u\to\infty} uf(u)e^{-\alpha u^4} \ge \beta > 0 \quad \text{for some constants } \alpha, \beta > 0. \tag{7}$$

Notice that the hypotheses of Theorems 1 and 2 are, for instance, satisfied by nonlinearities of the following two forms:

(a) Subcritical growth: $f(u) = 5u^4(e^{u^3}-1) + 3u^7e^{u^3}$.
(b) Critical growth:

$$f(u) = \begin{cases} 5u^4 + \cos(u)(e^{5u^4}-1) + 20(1+\sin(u))u^3(e^{5u^4}-1), & u \ge \dfrac{3\pi}{2}, \\ 5u^4, & 0 \le u \le \dfrac{3\pi}{2}. \end{cases}$$

Note that example (b) does not verify the condition (7) for which was used in [17] to ensure the existence of a positive solution.

1.1. *The underling idea for proving Theorems 1 and 2*

Motivated by the argument used in [25], we use a change of variable to reformulate the problem obtaining a semilinear equation which has an associated functional well defined and Gateaux differentiable in a suitable Orlicz space. Then we consider a reduction of the nonlinear term f outside Λ in such a way that the new functional verifies the geometric hypotheses of the mountain-pass theorem. We achieve the existence results by using a version of the mountain-pass theorem which is a consequence of the Ekeland Variational Principle. Finally we show that these local mountain-pass solutions indeed yield, as the parameter ε approaches zero, a solution of the original equation and they concentrate around the minimum of the potential V in Λ.

1.2. *The outline of the paper*

In the forthcoming section a reformulation of the problem and also some preliminary results including the Orlicz space setting suitable to study this class of problems are given. In Sec. 3, we use a penalization technique to obtain a one parameter family of mountain-pass critical points for a modified energy functional. Section 4 is devoted to obtain required estimates on the family of critical points of the modified energy functional. In Sec. 5, we show that these local mountain-pass solutions actually yield, as the parameter goes to zero, a solution of the original equation whose qualitative properties and in particular the developing of concentration around a point, which is localized by the critical points of the potential, are established in Sec. 6.

1.3. *Notation*

In this paper we make use of the following notation:

- $C, C_0, C_1, C_2, \ldots$ denote positive (possibly different) constants.
- B_R denotes the open ball centered at the origin and radius $R > 0$.
- For $1 \leq p \leq \infty$, $L^p(\mathbb{R}^N)$ denotes the usual Lebesgue spaces with norms

$$\|u\|_p = \left(\int_{\mathbb{R}^2} |u|^p \mathrm{d}x\right)^{1/p}, \quad 1 \leq p < \infty,$$

$$\|u\|_\infty = \inf\{C > 0 : |u(z)| \leq C \text{ almost everywhere in } \mathbb{R}^2\}.$$

- $H^1(\mathbb{R}^2)$ denotes the Sobolev spaces modeled in $L^2(\mathbb{R}^2)$ with norm

$$\|u\|_{H^1} = \left[\int_{\mathbb{R}^2} (|\nabla u|^2 + |u|^2)\mathrm{d}x\right]^{1/2}$$

  and $H_r^1(\mathbb{R}^2)$ is the space of radially symmetric functions in $H^1(\mathbb{R}^2)$.

- $\mathcal{C}_0^\infty(\mathbb{R}^2)$ denotes the space of infinitely differentiable functions with compact support.
- X^* is the topological dual of the Banach space X.
- By $\langle \cdot, \cdot \rangle$ we denote the duality pairing between X^* and X.

2. The Variational Framework

In this section we first have the reformulation of the problem. Then some preliminary results including a delicate Orlicz space setting suitable to deal with this class of problems involving the quasilinear term are proposed.

2.1. *Reformulation of the problem and preliminaries*

First, since we look for positive solutions of (P_ε) we assume that $f(s) = 0$ for all $s \in (-\infty, 0]$.

Observe that formally (P_ε) is the Euler–Lagrange equation associated to the following functional

$$J_\varepsilon(u) = \frac{\varepsilon^2}{2}\int_{\mathbb{R}^2}(1+u^2)|\nabla u|^2 \mathrm{d}z + \frac{1}{2}\int_{\mathbb{R}^2} V(z)u^2 \mathrm{d}z - \int_{\mathbb{R}^2} F(u)\mathrm{d}z.$$

From the variational point of view, the first difficulty that we have to deal with is to find an appropriate variational setting in order to apply minimax methods to study the existence of nontrivial solution of (P_ε). However, it should be pointed out that we may not apply directly such methods since the natural associated functional J_ε is not well defined in the usual Sobolev space. To overcome this difficult, we follow the idea introduced in [25] (see also [12]) and the approach used in [9] to reformulate the problem by means of the following change of variable:

$$\mathrm{d}v = \sqrt{1+u^2}\,\mathrm{d}u, \quad \text{thereby giving}$$
$$v = l(u) := \frac{1}{2}u\sqrt{1+u^2} + \frac{1}{2}\ln(u + \sqrt{1+u^2})$$

and since $l(0) = 0$ and l is strictly monotone on $\mathbb{R}_+$, the inverse function $g := l^{-1}$ is well defined on $\mathbb{R}_+$ and

$$g'(t) = \frac{1}{(1+g^2(t))^{1/2}} \quad \text{on } [0, +\infty),$$
$$g(t) = -g(-t) \qquad \text{on } (-\infty, 0].$$

We shall make frequent use of the following lemma in which we summarize some properties of the function g.

Proposition 3. *The following properties involving $g(t)$ and its derivative hold*:

(1) *g is uniquely defined C^∞ function and invertible.*
(2) $|g'(t)| \leq 1$ *for all* $t \in \mathbb{R}$.
(3) $|g(t)| \leq |t|$ *for all* $t \in \mathbb{R}$.

(4) $g(t)/t \to 1$ *as* $t \to 0$.
(5) $g(t)/\sqrt{t} \to \sqrt{2}$ *as* $t \to +\infty$.
(6) $g(t)/2 \le tg'(t) \le g(t)$ *for all* $t \ge 0$.
(7) $|g(t)| \le C|t|^{1/2}$ *for all* $t \in \mathbb{R}$.
(8) *the function* $g^2(t)$ *is a strictly convex.*
(9) *there exists a positive constant* C *such that*

$$|g(t)| \ge \begin{cases} C|t|, & |t| \le 1 \\ C|t|^{1/2}, & |t| \ge 1. \end{cases}$$

(10) $|t| \le C_1|g(t)| + C_2|g(t)|^2$ *for all* $t \in \mathbb{R}$.
(11) $|g(t)g'(t)| \le 1$ *for all* $t \in \mathbb{R}$.

Proof. It is elementary and will be omitted. □

Setting

$$G(t) := g^2(t)$$

we have that

$$G'(v) = \frac{2g(v)}{\sqrt{1+g^2(v)}}, \quad G''(v) = \frac{2}{(1+g^2(v))^2}$$

By exploiting this change of variable, we can rewrite the functional J_ε in the following form

$$\widetilde{I}_\varepsilon(v) := J_\varepsilon(g(v)) = \frac{\varepsilon^2}{2}\int_{\mathbb{R}^2} |\nabla v|^2 \mathrm{d}z + \frac{1}{2}\int_{\mathbb{R}^2} V(z)G(v)\mathrm{d}z - \int_{\mathbb{R}^2} F(g(v))\mathrm{d}z \tag{8}$$

which has finite energy provided that

$$\int_{\mathbb{R}^2} |\nabla v|^2 \mathrm{d}z < \infty \quad \text{and} \quad \int_{\mathbb{R}^2} V(z)G(v)\mathrm{d}z < \infty.$$

Observe that G is convex, $G(0) = 0$, $G(s)\nearrow\infty$ as $s \to \infty$ and G is even so that it is a Young function and one can consider the Orlicz class (see [36]), which we denote by $L^V_G(\mathbb{R}^2)$, of measurable functions $v : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\int_{\mathbb{R}^2} G(|v|)\mathrm{d}\mu < \infty, \quad \mathrm{d}\mu = V(z)\mathrm{d}z.$$

Remark 4. The Young function G satisfies the Δ_2-condition globally (see [36]), that is: there exists $K > 0$ such that $G(2s) \le KG(s)$ for all $s \ge 0$. As a consequence, one has that L^V_G is a linear space on which one can define the following norm

$$\|v\|_G := \sup\left\{\int_{\mathbb{R}^2} |vw|\mathrm{d}\mu : w \in L^V_{\widetilde{G}}(\mathbb{R}^2), \int_{\mathbb{R}^2} \widetilde{G}(|w|)\mathrm{d}\mu \le 1\right\} \tag{9}$$

where $(G, \widetilde{G})$ denotes a Young pair.

Thus, the new functional $\widetilde{I}_\varepsilon$ in (8) turns out to be well defined in a natural fashion on the Banach space

$$E := \left\{ v \in L_G^V(\mathbb{R}^2) : \int_{\mathbb{R}^2} |\nabla v|^2 \mathrm{d}z < \infty \right\}$$

which can be obtained as the completion of $\mathcal{C}_0^\infty(\mathbb{R}^2)$ with respect to the norm

$$\|v\| := \|\nabla v\|_2 + \|v\|_G.$$

At this stage, we also consider the closed subspace of $H^1(\mathbb{R}^2)$

$$H_V^1 := \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} V(z)u^2 \mathrm{d}z < \infty \right\}$$

equipped with the norm

$$\|u\|_V = \left(\int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}z + \int_{\mathbb{R}^2} V(z)u^2 \mathrm{d}z \right)^{1/2}.$$

Remark 5. Under the condition (V_0) for all $q \geq 2$,

$$H_V^1(\mathbb{R}^2) \hookrightarrow H^1(\mathbb{R}^2) \hookrightarrow L^q(\mathbb{R}^2)$$

with continuous embedding.

2.2. *Properties of the Orlicz space E*

In the following proposition we state some facts about the Banach space E and the nonlinear map $v \to g(v)$ which are useful in the sequel.

Proposition 6. *The space E enjoys the following properties:*

(1) *Let $u = g(v)$ and $v \in E$. Then the following estimate holds*:

$$\|u\|_V \leq \|\nabla v\|_2 + \|v\|_G^{1/4} + 2^{K_0/2}\|v\|_G^{K_0/2}$$

where K_0 is a positive constant independent of v and u.

(2) *If $q \geq 2$, then the map $v \to g(v)$ from E to $L^q(\mathbb{R}^2)$ is continuous.*

(3) *If $q \geq 2$, then E is continuously embedded into $L^q(\mathbb{R}^2)$.*

(4) *$E \hookrightarrow H^1(\mathbb{R}^2)$ with continuous embedding.*

Proof. We proceed the proof of (1) in several steps:

Step 1. First we prove that for all $k > 0$,

$$\|v\|_G \leq \frac{1}{k} \left(1 + \int_{\mathbb{R}^2} G(kv) \mathrm{d}\mu \right). \tag{10}$$

Indeed, by (9) and using the Young inequality $xy \leq G(x) + \widetilde{G}(y)$ one has

$$\begin{aligned}\|v\|_G &= \frac{1}{k}\sup\left\{\int_{\mathbb{R}^2}|kvw|\mathrm{d}\mu : \int_{\mathbb{R}^2}\widetilde{G}(|w|)\mathrm{d}\mu \leq 1\right\}\\ &\leq \frac{1}{k}\sup\left\{\int_{\mathbb{R}^2}[G(kv)+\widetilde{G}(|w|)]\mathrm{d}\mu : \int_{\mathbb{R}^2}\widetilde{G}(|w|)\mathrm{d}\mu \leq 1\right\}\\ &\leq \frac{1}{k}\left(\int_{\mathbb{R}^2}G(kv)\mathrm{d}\mu + 1\right).\end{aligned}$$

Step 2. We next show that there exists a constant $K_0 > 0$ such that

$$\int_{\mathbb{R}^2} G(v)\mathrm{d}\mu \leq \begin{cases}\|v\|_G, & \|v\|_G \leq 1\\ 2^{K_0}\|v\|_G^{K_0}, & \|v\|_G > 1\end{cases} \quad \forall\, v \in L_G^V(\mathbb{R}^2). \tag{11}$$

We recall from [36, Proposition 3, p. 60] that if $v \in L_G^V(\mathbb{R}^2)$, $v \neq 0$, one has

$$\int_{\mathbb{R}^2} G\left(\frac{v}{\|v\|_G}\right)\mathrm{d}\mu \leq 1$$

and in particular (11) follows if $\|v\|_G = 1$. Otherwise we distinguish when $\|v\|_G < 1$ and $\|v\|_G > 1$. In the first case, $v < v/\|v\|_G$ and since G is increasing, we get

$$\int_{\mathbb{R}^2} G(v)\mathrm{d}\mu \leq \int_{\mathbb{R}^2} G\left(\frac{v}{\|v\|_G}\right)\mathrm{d}\mu \leq 1.$$

Moreover, since G is strictly convex, we have

$$\begin{aligned}G(v\|v\|_G) &= G(v\|v\|_G + (1-\|v\|_G)0)\\ &\leq G(v)\|v\|_G + G(0)(1-\|v\|_G) = G(v)\|v\|_G\end{aligned}$$

and thus

$$\int_{\mathbb{R}^2} G(v\|v\|_G)\mathrm{d}\mu \leq \|v\|_G\int_{\mathbb{R}^2} G(v)\mathrm{d}\mu \leq \|v\|_G.$$

Now, we set $w = v\|v\|_G$ to get for all $\|w\|_G \leq 1$,

$$\int_{\mathbb{R}^2} G(w)\mathrm{d}\mu = \int_{\mathbb{R}^2} G(v\|v\|_G)\mathrm{d}\mu \leq \|v\|_G = \|w\|_G^{1/2}.$$

If $\|v\|_G > 1$, let $\eta := 1/\|v\|_G$ and $\bar{v} := \eta v$. Since $0 < \eta < 1$ we can find $n = n(v) \in \mathbb{N}$, such that $1/2^n < \eta < 1/2^{n-1}$ and since G is increasing we have

$$G\left(\frac{v}{2^n}\right) \leq G(\eta v) = G(\bar{v}). \tag{12}$$

By exploiting Δ_2-condition in Remark 4 with a constant $K > 1$, we obtain

$$G(v) = G\left(2^n\frac{v}{2^n}\right) \leq K^n G\left(\frac{v}{2^n}\right) \tag{13}$$

and then joining (12) and (13) we obtain

$$\int_{\mathbb{R}^2} G(v)d\mu \leq K^n\int_{\mathbb{R}^2} G(\bar{v})\mathrm{d}\mu \leq K^n \leq K^{1+\log_2\|v\|_G} \leq 2^{K_0}\|v\|_G^{K_0}$$

for a constant K_0 such that $2^{K_0} \geq K$. We complete the proof of the lemma by evaluating for $u = g(v)$

$$\|u\|_V \leq \left(\int_{\mathbb{R}^2} \frac{1}{1+G(v)}|\nabla v|^2 \mathrm{d}z\right)^{1/2} + \left(\int_{\mathbb{R}^2} G(v)\mathrm{d}\mu\right)^{1/2}$$
$$\leq \|\nabla v\|_2 + \|v\|_G^{1/4} + 2^{K_0/2}\|v\|_G^{K_0/2}.$$

This proves part (1).

Part (2) follows from part (1), together with Remark 5. Let $v_n \to v$ in E. Using the mean value theorem and property (2) in Proposition 3,

$$\int_{\mathbb{R}^2} |g(v_n) - g(v)|^q \mathrm{d}z \leq \int_{\mathbb{R}^2} |v_n - v|^q \mathrm{d}z$$

which together with property (10) in Proposition 3 we obtain

$$\int_{\mathbb{R}^2} |g(v_n) - g(v)|^q \mathrm{d}z \leq C_1 \int_{\mathbb{R}^2} |g(v_n - v)|^q \mathrm{d}z + C_2 \int_{\mathbb{R}^2} |g(v_n - v)|^{2q} \mathrm{d}z$$
$$\leq C_1 \|g(v_n - v)\|_V^q + C_2 \|g(v_n - v)\|_V^{2q},$$

where in the last inequality we have used Remark 5. Finally, using part (1) we have the desired conclusion.

Now, we prove (3). Let $(v_n) \subset E$ such that $v_n \to 0$ in E. Using part (1) we have that $g(v_n) \to 0$ in H_V^1 and by property (10) in Proposition 3 we obtain

$$\int_{\mathbb{R}^2} |v_n|^q \mathrm{d}z \leq C_1 \int_{\mathbb{R}^2} |g(v_n)|^q \mathrm{d}z + C_2 \int_{\mathbb{R}^2} |g(v_n)|^{2q} \mathrm{d}z$$

which together with the continuous embedding $H_V^1(\mathbb{R}^2) \hookrightarrow L^q(\mathbb{R}^2)$ for $q \geq 2$ completes the proof of part (3). Finally, from part (3) it follows that

$$\|v\|_{1,2}^2 = \|\nabla v\|_2^2 + \|v\|_2^2 \leq \|v\|^2 + C\|v\|^2 = (1+C)\|v\|^2$$

and the proof of Proposition 6 is complete. □

3. Modified Problem

As in [14, 15, 21] (see also [2, 18]) in this section, we make a suitable modification on the nonlinear term $f(u)$ outside the domain Λ such that the associated energy functional satisfies the hypotheses of the following version of the mountain-pass theorem which is a consequence of the Ekeland Variational Principle as developed in [4] (see also [10, 44] for related results) in the Orlicz space E.

Theorem 7. *Let E be a Banach space and $\Phi \in C(E; \mathbb{R})$, Gateaux differentiable for all $v \in E$, with G-derivative $\Phi'(v) \in E^*$ continuous from the norm topology of E to the weak-* topology of E^* and $\Phi(0) = 0$. Let $\mathcal{S}$ be a closed subset of E which disconnects (archwise) E. Let $v_0 = 0$ and $v_1 \in E$ be points belonging to distinct* *connected components of* $E\backslash\mathcal{S}$*. Suppose that*

$$\inf_{\mathcal{S}} \Phi \geq \alpha > 0 \quad \textit{and} \quad \Phi(v_1) \leq 0$$

and let

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0 \textit{ and } \gamma(1) = v_1\}.$$

Then

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi(\gamma(t)) \geq \alpha$$

and there exists a Palais–Smale sequence for Φ *at the mountain-pass level* c*.*

We recall that $(v_n) \subset E$ is a Palais–Smale sequence for $\mathcal{F}$ at level C (denoted in the sequel as $(\text{P.-S.})_C$ sequence), if $\mathcal{F}(v_n) \to C$ and $\mathcal{F}'(v_n) \to 0$ in E^* as $n \to \infty$.

We define the Carathéodory function

$$h(z,s) = \chi_\Lambda(z) f(s) + (1 - \chi_\Lambda(z)) \overline{f}(s)$$

where χ_Λ is the characteristic function of Λ and

$$\overline{f}(s) = \begin{cases} f(s), & \text{if } s \leq a \\ \dfrac{\beta_0}{\tau} s, & \text{if } s > a \end{cases}$$

with $\tau > 2\theta/(\theta - 4) > 2$ and $a > 0$ is such that $f(a) = a\beta_0/\tau$.

It is not difficult to check that the function $h(z,s)$ enjoys the following properties:

(h_1) $h(z,s)$ is piecewise C^1 in s for any fixed z and $h(z,s) = 0$ for $s \leq 0$;

$(h_2)_s$ **(subcritical case)** for each $\delta > 0$, $\alpha > 0$ and $q \geq 0$ there is a constant $C = C(\delta, \alpha, q) > 0$ such that for all $s \geq 0$ and $z \in \mathbb{R}^2$, we have

$$h(z,s) \leq \delta s + C s^q [\exp(\alpha s^4) - 1] \quad \text{or}$$

$(h_2)_c$ **(critical case)** for each $\delta > 0$, $\beta > \alpha_0$ and $q \geq 0$ there is a constant $C = C(\delta, \beta, q) > 0$ such that for all $s \geq 0$ and $z \in \mathbb{R}^2$, we have

$$h(z,s) \leq \delta s + C s^q [\exp(\beta s^4) - 1];$$

(h_3) $0 < \theta H(z,s) \leq h(z,s)s$, $(z,s) \in [\Lambda \times (0,+\infty)] \cup [(\mathbb{R}^2 - \Lambda) \times (0,a)]$ and

$$0 \leq 2H(z,s) \leq h(z,s)s \leq \frac{1}{\tau} V(z) s^2, \quad (z,s) \in [(\mathbb{R}^2 - \Lambda) \times [0,+\infty)]$$

where $H(z,s) = \int_0^s h(z,t)\mathrm{d}t$;

(h_4) For each $z \in \mathbb{R}^2$, the function $s \to h(z,s)s^{-1}$ is nondecreasing for $s > 0$.

Now, we consider the modified problem

$$-\varepsilon^2 \Delta v = g'(v)[h(z, g(v)) - V(z) g(v)] \quad \text{in } \mathbb{R}^2. \tag{14}$$

The energy functional $\mathcal{F}_\varepsilon : E \to \mathbb{R}$ associated to (14) is given by

$$\mathcal{F}_\varepsilon(v) = \frac{\varepsilon^2}{2}\int_{\mathbb{R}^2} |\nabla v|^2 \mathrm{d}z + \frac{1}{2}\int_{\mathbb{R}^2} V(z) g^2(v)\mathrm{d}z - \int_{\mathbb{R}^2} H(z, g(v))\mathrm{d}z.$$

3.1. *Properties of the functional $\mathcal{F}_\varepsilon$*

In what follows, without loss of generality, we may assume that $\varepsilon = 1$ and $\mathcal{F} = \mathcal{F}_\varepsilon$.

Proposition 8. *The functional $\mathcal{F}$ is well defined on E. Moreover,*

(a) *$\mathcal{F}$ is continuous on E;*
(b) *$\mathcal{F}$ is Gateaux differentiable on E with G-derivative given by*

$$\langle \mathcal{F}'(v), \varphi\rangle = \int_{\mathbb{R}^N} \nabla v \nabla \varphi \mathrm{d}z + \int_{\mathbb{R}^N} V(z) g(v) g'(v)\varphi \mathrm{d}z - \int_{\mathbb{R}^N} h(z, g(v)) g'(v)\varphi \mathrm{d}z,$$

for $v, \varphi \in E$;
(c) *for $v \in E$ we have that $\mathcal{F}'(v) \in E^*$ and if $v_n \to v$ in E then*

$$\langle \mathcal{F}'(v_n), \varphi\rangle \to \langle \mathcal{F}'(v), \varphi\rangle$$

for each $\varphi \in E$.

Proof. By $(h_2)_s$ (or $(h_2)_c$) and (h_3) we have that

$$\left|\int_{\mathbb{R}^2} H(z, g(v))\mathrm{d}z\right| \le \frac{1}{2}\int_{\mathbb{R}^2} |h(z, g(v)) g(v)|\mathrm{d}z \le C_1 \int_{\mathbb{R}^2} |g(v)|^2 \mathrm{d}z + C_2 \int_{\mathbb{R}^2} |g(v)|[e^{\alpha(g(v))^4} - 1]\mathrm{d}z. \quad (15)$$

Using (7) in Proposition 3, Hölder inequality, Trudinger–Moser inequality and the Lemma 10, it follows that

$$\int_{\mathbb{R}^2} |g(v)|[e^{\alpha(g(v))^4} - 1]\mathrm{d}z \le \left(\int_{\mathbb{R}^2} |g(v)|^2 \mathrm{d}z\right)^{1/2} \left(\int_{\mathbb{R}^2} (e^{Cv^2} - 1)\mathrm{d}z\right)^{1/2} \quad (16)$$

which together with the definition of E and (15) shows that the functional $\mathcal{F}$ is well defined on E. Now, suppose that $v_n \to v$ in E. By Proposition 6 we can conclude that

$$\int_{\mathbb{R}^2} |\nabla v_n|^2 \mathrm{d}z \to \int_{\mathbb{R}^2} |\nabla v|^2 \mathrm{d}z$$

$$\int_{\mathbb{R}^2} V(z) g^2(v_n)\mathrm{d}z \to \int_{\mathbb{R}^2} V(z) g^2(v)\mathrm{d}z.$$

From (3) in Proposition 6, $v_n \to v$ in $L^2(\mathbb{R}^2)$ and this implies that $v_n \to v$ in $H^1(\mathbb{R}^2)$. Thus, up to subsequence, we know that $|v_n| \le \hat{v}$ almost everywhere in $\mathbb{R}^2$

for some $\hat{v} \in H^1(\mathbb{R}^2)$. From this, using the same previous arguments, the fact that g is increasing and Lebesgue dominated convergence theorem we obtain

$$\int_{\mathbb{R}^2} H(z, g(v_n))\mathrm{d}z \to \int_{\mathbb{R}^2} H(z, g(v))\mathrm{d}z.$$

Consequently, $\mathcal{F}(v_n) \to \mathcal{F}(v)$ and the continuity is proved.

Next, let $v, \varphi \in E$. We have that

$$\frac{1}{2}\int_{\mathbb{R}^2} \frac{V(z)(g^2(v+t\varphi) - g^2(v))}{t}\mathrm{d}z = \int_{\mathbb{R}^2} V(z)g(\xi)g'(\xi)\varphi\mathrm{d}z$$

where

$$\min\{v, v+t\varphi\} \leq \xi \leq \max\{v, v+t\varphi\}.$$

If $|t| \leq 1$ it is clear that $|\xi| \leq |v| + |\varphi|$ and using (2), (9)–(10) in Proposition 3 and the fact that g is increasing we get

$$\begin{aligned}
|V(z)g(\xi)g'(\xi)\varphi| &\leq V(z)|g(\xi)g'(\xi)||g(\varphi)| + V(z)|g(\xi)g'(\xi)|g^2(\varphi) \\
&\leq V(z)g(|v| + |\varphi|)|g(\varphi)| + V(z)g^2(\varphi) \\
&\leq V(z)g^2(|v| + |\varphi|) + V(z)g^2(\varphi)
\end{aligned}$$

and

$$V(z)g^2(|v| + |\varphi|) + V(z)g^2(\varphi) \in L^1(\mathbb{R}^2).$$

As $V(z)g(\xi)g'(\xi)\varphi \to V(z)g(v)g'(v)\varphi$ almost everywhere as $t \to 0$, by the Lebesgue dominated convergence theorem we conclude that

$$\lim_{t\to 0}\frac{1}{2}\int_{\mathbb{R}^2} \frac{V(z)(g^2(v+t\varphi) - g^2(v))}{t}\mathrm{d}z = \int_{\mathbb{R}^2} V(z)g(v)g'(v)\varphi\mathrm{d}z.$$

Similarly, using arguments as in (15)–(16), the fact that g is increasing and one more time the Lebesgue dominated convergence theorem we achieve

$$\lim_{t\to 0}\int_{\mathbb{R}^2} \frac{H(z, g(v+t\varphi)) - H(z, g(v))}{t}\mathrm{d}z = \int_{\mathbb{R}^2} h(z, g(v))g'(v)\varphi\mathrm{d}z.$$

Thus, $\mathcal{F}$ is Gateaux-differentiable in E.

To see that $\mathcal{F}'(v) \in E^*$ for each $v \in E$, the main difficulty comes from the term $\int_{\mathbb{R}^N} V(z)g(v)g'(v)\varphi\mathrm{d}z$. Suppose that $\varphi_n \to 0$ in E. It follows from Proposition 6 that

$$\int_{\mathbb{R}^N} V(z)g^2(\varphi_n)\mathrm{d}z \to 0.$$

Now, by (2) and (9)–(10) in Proposition 3 we have

$$\begin{aligned}
\left|\int_{\mathbb{R}^2} V(z)g(v)g'(v)\varphi_n\mathrm{d}z\right| &\leq \int_{\mathbb{R}^2} V(z)|g(v)g'(v)||g(\varphi_n)|\mathrm{d}z \\
&\quad + \int_{\mathbb{R}^2} V(z)|g(v)g'(v)|g^2(\varphi_n)\mathrm{d}z
\end{aligned}$$

$$\leq \left(\int_{\mathbb{R}^2} V(z)g^2(v)\mathrm{d}z\right)^{1/2} \left(\int_{\mathbb{R}^N} V(z)g^2(\varphi_n)\mathrm{d}z\right)^{1/2} + \int_{\mathbb{R}^2} V(z)g^2(\varphi_n)\mathrm{d}z.$$

which implies that

$$\int_{\mathbb{R}^N} V(z)f(v)f'(v)\varphi_n \mathrm{d}z \to 0.$$

Thus, $\mathcal{F}'(v) \in E^*$ and by similar arguments it is not difficult to check that if $v_n \to v$ in E, then $\langle \mathcal{F}'(v_n), \varphi\rangle \to \langle \mathcal{F}'(v), \varphi\rangle$, for each $\varphi \in E$. □

Proposition 9. *If v is a critical point of $\mathcal{F}$ then $v \in \mathcal{C}^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$. Moreover, $v > 0$ provided that v is nontrivial.*

Proof. It is standard that critical points of the functional $\mathcal{F}$ are weak solutions of the corresponding Euler–Lagrange equation. Indeed, we have

$$-\Delta v = w \quad \text{in } \mathbb{R}^2$$

in the weak sense, where

$$w(z) := g'(v(z))[h(z, g(v(z))) - V(z)g(v(z))].$$

According to $(h_2)_s$ (or $(h_2)_c$), we obtain

$$|w| \leq g'(v)[C_1|g(v)| + C_2|g(v)|(e^{\alpha(g(v))^4} - 1)] \leq C_3 + C_4(e^{C_5 v^2} - 1)$$

in any ball B_R, where we have used (6) and (10) in Proposition 3. Using Lemma 10 and Trudinger–Moser inequality, it follows that $w \in L^q(B_R)$ for all $q \geq 2$. Thus, by elliptic regularity theory we obtain that $v \in W^{2,q}(B_R)$ for all $q \geq 2$. Hence, $v \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$ and this implies that w is locally Hölder continuous. Consequently, by Schauder regularity theory $v \in C^{2,\gamma}_{\mathrm{loc}}(\mathbb{R}^N)$ for some $\gamma \in (0, 1)$.

Furthermore, $v > 0$ in $\mathbb{R}^2$. In fact, suppose otherwise, that there exists $z_0 \in \mathbb{R}^2$ such that $v(z_0) = 0$. Equation (14) can be written of the form

$$-\Delta v + c(z)v = V(z)g'(v)(v - g(v)) + h(z, g(v))g'(v) \geq 0,$$

where $c(z) = V(z)g'(v(z)) > 0$ for all $z \in \mathbb{R}^2$. Applying the strong maximum principle for an arbitrary ball centered in z_0 we can conclude that $v \equiv 0$ and this is impossible. □

3.2. *Mountain-pass geometry*

In order to show that the functional $\mathcal{F}$ has the mountain-pass geometry, we shall use the following result:

Lemma 10. *Let $\beta > 0$ and $r > 1$. Then for each $\alpha > r$ there exists a positive constant $C = C(\alpha)$ such that for all $s \in \mathbb{R}$*

$$(e^{\beta s^2} - 1)^r \leq C(e^{\alpha\beta s^2} - 1).$$

Proof. We have that

$$\lim_{s\to 0}\frac{(e^{\beta s^2}-1)^r}{e^{\alpha\beta s^2}-1}=\lim_{s\to 0}\frac{r(e^{\beta s^2}-1)^{r-1}e^{\beta s^2}}{\alpha e^{\alpha\beta s^2}}=0.$$

Moreover,

$$\lim_{|s|\to\infty}\frac{(e^{\beta s^2}-1)^r}{e^{\alpha\beta s^2}-1}=\lim_{|s|\to\infty}\frac{e^{r\beta s^2}(1-e^{-\beta s^2})^r}{e^{\alpha\beta s^2}(1-e^{-\alpha\beta s^2})}=0$$

and the result follows. □

For $\rho>0$ we define

$$\mathcal{S}_\rho \doteq \left\{v\in E:\int_{\mathbb{R}^2}[|\nabla v|^2+V(z)g^2(v)]\mathrm{d}z=\rho^2\right\}.$$

Since $\mathcal{Q}:E\to\mathbb{R}$ given by

$$\mathcal{Q}(v)=\int_{\mathbb{R}^2}[|\nabla v|^2+V(z)g^2(v)]\mathrm{d}z$$

is continuous then $\mathcal{S}_\rho$ is a closed subset and disconnects the space E.

The next two lemmas are crucial to show that the functional $\mathcal{F}$ possesses the mountain-pass geometry.

Lemma 11. *There exist $\rho,\alpha>0$ such that*

$$\mathcal{F}(v)\geq\alpha\quad\text{for all } v\in\mathcal{S}_\rho.$$

Proof. Note first that for $2\beta\rho^2<\pi$, by Lemma 10 we have

$$\begin{aligned}
&\int_{\mathbb{R}^2}(e^{\beta g^4(v)}-1)|g(v)|^q\mathrm{d}z\\
&\quad\leq C_1\left[\int_{\mathbb{R}^2}(e^{\xi\beta g^4(v)}-1)\mathrm{d}z\right]^{1/2}\|g(v)\|_{2q}^q\\
&\quad\leq C_1\left[\int_{\mathbb{R}^2}\left(e^{\xi\beta\|\nabla g^2(v)\|_2^2\left(\frac{g^2(v)}{\|\nabla g^2(v)\|_2}\right)^2}-1\right)\mathrm{d}z\right]^{1/2}\|g(v)\|_{2q}^q\\
&\quad\leq C_1\left[\int_{\mathbb{R}^2}\left(e^{4\xi\beta\rho^2\left(\frac{g^2(v)}{\|\nabla g^2(v)\|_2}\right)^2}-1\right)\mathrm{d}z\right]^{1/2}\|g(v)\|_{2q}^q\\
&\quad\leq C_2\|g(v)\|_{2q}^q
\end{aligned}$$

where $\xi>2$ is such that $\xi\beta\rho^2<\pi$. Also note that for C small but independent of v we have

$$\begin{aligned}
C\|g(v)\|_{2q}^q\leq\|g(v)\|_V^q&=\left(\int_{\mathbb{R}^2}|\nabla g(v)|^2\mathrm{d}z+\int_{\mathbb{R}^2}V(z)g^2(v)\mathrm{d}z\right)^{q/2}\\
&\leq\left(\int_{\mathbb{R}^2}|g'(v)\nabla v|^2\mathrm{d}z+\int_{\mathbb{R}^2}V(z)g^2(v)\mathrm{d}z\right)^{q/2}\leq\rho^q.
\end{aligned}$$

Therefore it follows from $(h_2)_c$ (or $(h_2)_s$) and (h_3) and the above inequalities that for $v \in \mathcal{S}_\rho$

$$\begin{aligned}\mathcal{F}(v) &\geq \frac{1}{2}\rho^2 - \frac{\beta_0}{4}\int_{\mathbb{R}^2} g^2(v)\mathrm{d}z - C\int_{\mathbb{R}^2}(e^{\beta|g(v)|^4}-1)|g(v)|^q\mathrm{d}z \\ &\geq \frac{1}{2}\rho^2 - \frac{1}{4}\int_{\mathbb{R}^2} V(z)g^2(v)\mathrm{d}z - C_1\|g(v)\|_V^q \geq \frac{1}{4}\rho^2 - C_1\rho^q\end{aligned}$$

with $q > 2$ and $\|v\|$ small. Therefore, if $\rho > 0$ is sufficiently small we obtain for $v \in S_\rho$ that

$$\mathcal{F}(v) \geq \alpha = \frac{1}{4}\rho^2 - C\rho^q > 0. \qquad \square$$

Lemma 12. *There exists $v \in E$ such that $\mathcal{Q}(v) > \rho^2$ and $\mathcal{F}(v) < 0$.*

Proof. We are going to prove that there exists $\varphi \in E$ such that $\mathcal{F}(t\varphi) \to -\infty$ as $t \to +\infty$, which proves our thesis if we take $v = t\varphi$ with t large enough.

Note that by (h_3) there exist positive constants C_1, C_2 such that

$$H(z,s) \geq C_1 s^\theta - C_2 \tag{17}$$

for all $(z,s) \in \overline{\Lambda}\times[0,+\infty)$. Choosing any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N,[0,1])\backslash\{0\}$ such that $\operatorname{supp}\varphi \subseteq \overline{\Lambda}$, it follows from (17) that

$$\mathcal{F}(t\varphi) \leq \frac{t^2}{2}\int_\Lambda(|\nabla\varphi|^2 + V(z)\varphi^2)\mathrm{d}z - C_1\int_\Lambda |g(t\varphi)|^\theta\mathrm{d}z + C_2|\Lambda|$$

where $|\Lambda|$ denotes the Lebesgue measure of Λ in $\mathbb{R}^2$.

Using property (6) in Proposition 3, it follows that $g(s)/s$ is decreasing for $s > 0$. Since $0 \leq t\varphi(z) \leq t$ for $z \in \overline{\Lambda}$ and $t > 0$, we obtain $g(t\varphi(z)) \geq g(t)\varphi(z)$, which implies that

$$\begin{aligned}\mathcal{F}(t\varphi) &\leq \frac{t^2}{2}\left[\int_\Lambda(|\nabla\varphi|^2 + V(z)\varphi^2)\mathrm{d}z - C_1 g(t)^\theta\int_\Lambda \varphi^\theta\mathrm{d}z + C_2|\Lambda|\right] \\ &\to -\infty \text{ as } t \to +\infty,\end{aligned}$$

where we have used that

$$\lim_{t\to+\infty}\frac{g(t)^\theta}{t^2} = +\infty,$$

which is a consequence of $\theta > 4$ and property (5) in Proposition 3. $\square$

3.3. *Palais–Smale sequences*

In this subsection, we establish some properties of the Palais-Smale sequences of $\mathcal{F}$.

Proposition 13. *Any Palais–Smale sequence for $\mathcal{F}$ is bounded in E.*

Proof. Let $(v_n) \subset E$ be a (P.-S.)$_C$ sequence. Thus,

$$\mathcal{F}(v_n) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla v_n|^2 \mathrm{d}z + \frac{1}{2}\int_{\mathbb{R}^2} V(z) g(v_n)^2 \mathrm{d}z - \int_{\mathbb{R}^2} H(z, g(v_n)) \mathrm{d}z$$
$$= C + \delta_n, \tag{18}$$

and

$$|\langle \mathcal{F}'(v_n), \phi\rangle| = \left| \int_{\mathbb{R}^2} \nabla v_n \cdot \nabla \phi \mathrm{d}z + \int_{\mathbb{R}^2} V(z) g(v_n) g'(v_n) \phi \mathrm{d}z - \int_{\mathbb{R}^2} h(z, g(v_n)) g'(v_n) \phi \mathrm{d}z \right| \le \varepsilon_n \|\phi\| \tag{19}$$

where $\delta_n, \varepsilon_n \to 0$ as $n \to \infty$. Next, we pick

$$\phi = \frac{g(v_n)}{g'(v_n)} = \sqrt{1 + g(v_n)^2} g(v_n)$$

as a test function in (19). One can easily deduce that

$$\|\phi\|_G \le C_1 \|v_n\|_G \quad \text{and} \quad |\nabla \phi| = \left[1 + \frac{g(v_n)^2}{1 + g(v_n)^2}\right] |\nabla v_n| \le 2|\nabla v_n|,$$

which implies $\|\phi\| \le C_0 \|v_n\|$. Substituting ϕ in (19), gives

$$|\langle \mathcal{F}'(v_n), \phi\rangle| = \left| \int_{\mathbb{R}^2} \left[1 + \frac{g(v_n)^2}{1 + g(v_n)^2}\right] |\nabla v_n|^2 \mathrm{d}z + \int_{\mathbb{R}^2} V(z) g(v_n)^2 \mathrm{d}z - \int_{\mathbb{R}^2} h(z, g(v_n)) g(v_n) \mathrm{d}z \right| \le \varepsilon_n \|v_n\|. \tag{20}$$

Taking into account property (6) and (18)–(20) we have

$$\begin{aligned} C + \delta_n + \varepsilon_n \|v_n\| &\ge \frac{1}{2}\int_{\mathbb{R}^2} |\nabla v_n|^2 \mathrm{d}z + \frac{1}{2}\int_{\mathbb{R}^2} V(z) g(v_n)^2 \mathrm{d}z \\ &\quad - \frac{1}{\theta}\int_{\mathbb{R}^2} \left[1 + \frac{g(v_n)^2}{1 + g(v_n)^2}\right] |\nabla v_n|^2 \mathrm{d}z - \frac{1}{\theta}\int_{\mathbb{R}^2} V(z) g(v_n)^2 \mathrm{d}z \\ &\quad + \int_{\mathbb{R}^2} \left[\frac{1}{\theta} h(z, g(v_n)) g(v_n) - H(z, g(v_n))\right] \mathrm{d}z \\ &\ge \int_{\mathbb{R}^2} \left[\frac{1}{2} - \frac{1}{\theta}\left(1 + \frac{g(v_n)^2}{1 + g(v_n)^2}\right)\right] |\nabla v_n|^2 \mathrm{d}z \\ &\quad + \left(\frac{1}{2} - \frac{1}{\theta}\right) \int_{\mathbb{R}^2} V(z) g(v_n)^2 \mathrm{d}z. \end{aligned}$$

Now, by considering (10) with $k = 1$ we have

$$\int_{\mathbb{R}^2} V(z) g(v_n)^2 \mathrm{d}z \ge \|v_n\|_G - 1$$

and therefore we obtain

$$\begin{aligned} C+\delta_n+\varepsilon_n\|v_n\| &\geq \left(\frac{1}{2}-\frac{2}{\theta}\right)\int_{\mathbb{R}^2}|\nabla v_n|^2\mathrm{d}z+\left(\frac{1}{2}-\frac{1}{\theta}\right)\int_{\mathbb{R}^2}V(z)g(v_n)^2\mathrm{d}z \\ &\geq \frac{\theta-4}{2\theta}\int_{\mathbb{R}^2}[|\nabla v_n|^2+V(z)g(v_n)^2]\mathrm{d}z \\ &\geq \frac{\theta-4}{2\theta}\left(\int_{\mathbb{R}^2}|\nabla v_n|^2\mathrm{d}z+\|v_n\|_G-1\right) \\ &\geq \frac{\theta-4}{2\theta}(\|\nabla v_n\|_2+\|v_n\|_G-2) \\ &= \frac{\theta-4}{2\theta}(\|v_n\|-2). \end{aligned} \tag{21}$$

Since $\theta > 4$, it follows from the above estimate that

$$C+\delta_n+\varepsilon_n\|v_n\| \geq C_1\|v_n\|,$$

which implies that (v_n) is bounded in E. □

Remark 14. From (21) we can conclude that

$$\int_{\mathbb{R}^2}|\nabla v_n|^2\mathrm{d}z \leq \frac{2\theta}{\theta-4}C+o_n(1).$$

Lemma 15. *Let (v_n) be a* (P.-S.)$_C$ *sequence for $\mathcal{F}$. Then,*

(i) *given $\delta > 0$ there exists $R > 0$ such that*

$$\limsup_{n\to\infty}\int_{|z|\geq R}(|\nabla v_n|^2+V(z)g^2(v_n))\mathrm{d}z < \delta.$$

(ii) *Up to a subsequence, $V(x)g^2(v_n)$ converges to $V(x)g^2(v)$ in $L^1(\mathbb{R}^2)$ and consequently $g(v_n)\to g(v)$ converges in $L^2(\mathbb{R}^2)$.*

Proof. Consider the test function $\varphi_R v_n$, where $\varphi_R \in \mathcal{C}^\infty(\mathbb{R}^2,[0,1])$, $\varphi_R(z)=0$ if $|z|\leq R/2$, $\varphi_R(z)=1$ if $|z|\geq R$ and $|\nabla\varphi_R(z)|\leq C/R$ for all $z\in\mathbb{R}^2$. By Proposition 13, $(\varphi_R v_n)$ is bounded in E. Thus, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^2}|\nabla v_n|^2\varphi_R\mathrm{d}z+\int_{\mathbb{R}^2}V(z)g(v_n)g'(v_n)v_n\varphi_R\mathrm{d}z+\int_{\mathbb{R}^2}v_n\nabla v_n\nabla\varphi_R\mathrm{d}z \\ &\quad= \int_{\mathbb{R}^2}h(z,g(v_n))g'(v_n)v_n\varphi_R\mathrm{d}z+o_n(1). \end{aligned}$$

From (h_3) and properties of g,

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^2}(|\nabla v_n|^2+V(z)g^2(v_n))\varphi_R\mathrm{d}z+\int_{\mathbb{R}^2}v_n\nabla v_n\nabla\varphi_R\mathrm{d}z \\ &\quad\leq \frac{1}{\tau}\int_{\mathbb{R}^2}V(z)g^2(v_n)\varphi_R\mathrm{d}z+o_n(1) \end{aligned}$$

for $R > 0$ suitably large, which implies that

$$\int_{|z|\geq R} (|\nabla v_n|^2 + V(z)g^2(v_n))\mathrm{d}z \leq \frac{C}{R}\|v_n\|_2\|\nabla v_n\|_2 + o_n(1)$$

and this proves part (i) of this lemma. Part (ii) is a consequence of (i) and Lebesgue dominated convergence theorem since $g(v_n) \to g(v)$ in $L^2_{\mathrm{loc}}(\mathbb{R}^2)$. □

For the next result, we will use the following result of convergence, whose proof can be found in [13].

Lemma 16. *Suppose $\mathcal{O}$ is a bounded domain in $\mathbb{R}^2$. Let (u_n) in $L^1(\mathcal{O})$ such that $u_n \to u$ in $L^1(\mathcal{O})$ and let $g(x,s)$ be a continuous function. Then $g(x,u_n) \to g(x,u)$ in $L^1(\mathcal{O})$ provided that $g(x,u_n) \in L^1(\mathcal{O})$ for all n and $\int_{\mathcal{O}} |g(x,u_n)u_n|\mathrm{d}x \leq C$.*

Lemma 17. *Suppose that (v_n) is a (P.-S.)$_C$ sequence for $\mathcal{F}$. If either of the following conditions hold:*

(1) *The function f has critical growth and* $0 < C < (\theta - 4)/8\theta$;
(2) *The function f has subcritical growth,*

then, up to a subsequence, we have:

(i) $\int_{\mathbb{R}^2} h(z, g(v_n))g'(v_n)v_n\mathrm{d}z \to \int_{\mathbb{R}^2} h(z, g(v))g'(v)v\mathrm{d}z$;
(ii) $\int_{\mathbb{R}^2} H(z, g(v_n))\mathrm{d}z \to \int_{\mathbb{R}^2} H(z, g(v))\mathrm{d}z$,

for some $v \in E$ which is indeed a critical point of $\mathcal{F}$.

Proof. We shall prove this lemma only in the critical case. The subcritical case can be proceeded similarly. By Lemma 13, the sequence (v_n) is bounded in E and, consequently, from (4) in Proposition 6 it is also bounded in $H^1(\mathbb{R}^2)$. Thus, up to a subsequence, $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^2)$ and $v_n \to v$ almost everywhere in $\mathbb{R}^2$. By using (1) in Proposition 6, we conclude that $\int_{\mathbb{R}^2} V(z)g^2(v_n)\mathrm{d}z$ is bounded and by Fatou's Lemma

$$\int_{\mathbb{R}^2} V(z)g^2(v)\mathrm{d}z \leq \liminf_{n\to\infty} \int_{\mathbb{R}^2} V(z)g^2(v_n)\mathrm{d}z$$

which implies that $v \in E$.

First, we prove that $h(z, g(v_n))g(v_n) \to h(z, g(v))g(v)$ in $L^1(\mathbb{R}^2)$. Given $\delta > 0$, we consider $R > 0$ such that $\Lambda \subset B_R$ and

$$\int_{B_R^c} h(z, g(v_n))g(v_n)\mathrm{d}z \leq \frac{1}{\tau}\int_{B_R^c} V(z)g^2(v_n)\mathrm{d}z < \delta.$$

From Fatou's Lemma, we also have $\int_{B_R^c} h(z, g(v))g(v)\mathrm{d}z < \delta$. Hence,

$$\begin{aligned}&\int_{\mathbb{R}^2} |h(z, g(v_n))g(v_n) - h(z, g(v))g(v)|\mathrm{d}z \\ &\qquad \leq \int_{B_R} |h(z, g(v_n))g(v_n) - h(z, g(v))g(v)|\mathrm{d}z + 2\delta\end{aligned}$$

and therefore we just need to prove that $h(z,g(v_n))g(v_n) \to h(z,g(v))g(v)$ in $L^1(B_R)$. We claim that v is a critical point of $\mathcal{F}$. In fact, let ϕ in $C_0^\infty(\mathbb{R}^2)$ and $Q = \operatorname{supp}(\phi)$. Since $\int_{\mathbb{R}^2} h(z,g(v_n))g(v_n)\mathrm{d}z$ is bounded by Lemma 16 we conclude that $h(z,g(v_n)) \to h(z,g(v))$ in $L^1(Q)$. Hence, up to subsequences, there exists $\varphi \in L^1(Q)$ such that

$$|h(z,g(v_n))| \le \varphi \quad \text{almost everywhere in } Q.$$

Thus,

$$|h(z,g(v_n))g'(v_n)\phi| \le \sup_Q |\phi|\varphi \quad \text{almost everywhere in } Q.$$

Therefore, as a consequence of the Lebesgue dominated convergence theorem

$$\int_{\mathbb{R}^2} h(z,g(v_n))g'(v_n)\phi\mathrm{d}z \to \int_{\mathbb{R}^2} h(z,g(v))g'(v)\phi\mathrm{d}z.$$

Similarly,

$$\int_{\mathbb{R}^2} V(z)g(v_n)g'(v_n)\phi\mathrm{d}z \to \int_{\mathbb{R}^2} V(z)g(v)g'(v)\phi\mathrm{d}z,$$

and since

$$\langle \mathcal{F}'(v_n),\phi\rangle = \int_{\mathbb{R}^2} \nabla v_n \nabla\phi\mathrm{d}z + \int_{\mathbb{R}^2} V(z)g(v_n)g'(v_n)\phi\mathrm{d}z - \int_{\mathbb{R}^2} h(z,g(v_n))g'(v_n)\phi\mathrm{d}z \to 0$$

it follows that, for all $\phi \in C_0^\infty(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} \nabla v\nabla\phi\mathrm{d}z + \int_{\mathbb{R}^2} V(z)g(v)g'(v)\phi\mathrm{d}z = \int_{\mathbb{R}^2} h(z,g(v))g'(v)\phi\mathrm{d}z$$

which shows that v is a critical point of $\mathcal{F}$. It also follows from Proposition 9 that $v \in C^2(\mathbb{R}^2)$. Next, we have

$$\begin{aligned}\int_{B_R} &|h(z,g(v_n))g(v_n) - h(z,g(v))g(v)|\mathrm{d}z \\ &\le \int_{B_R} |[h(z,g(v_n)) - h(z,g(v))]g(v)|\mathrm{d}z \\ &\quad + \int_{B_R} |h(z,g(v_n))||g(v_n) - g(v)|\mathrm{d}z\end{aligned}$$

and since

$$\int_{B_R} |h(z,g(v_n))g(v_n)g(v)|\mathrm{d}z \le \max_{B_R}|g(v)| \int_{B_R} |h(z,g(v_n))g(v_n)|\mathrm{d}z \le C_R,$$

using Lemma 16 we obtain

$$\lim_{n\to\infty} \int_{B_R} |[h(z,g(v_n)) - h(z,g(v))]g(v)|\mathrm{d}z = 0.$$

As $0 < C < (\theta - 4)/8\theta$, by Remark 14 we have that $\|\nabla v_n\|_2^2 \le K < 1/4$ for n sufficiently large. Taking $q > 1$ and $\varepsilon > 0$ such that $qK(\alpha_0 + \varepsilon) < \pi$ and using the growth properties of nonlinear term $h(z,s)$, we get

$$\begin{aligned}\int_{B_R} &|h(z, g(v_n))||g(v_n) - g(v)|\mathrm{d}z \\ &\le \int_{B_R} |g(v_n)||g(v_n) - g(v)|\mathrm{d}z + C\int_{B_R} |g(v_n) - g(v)|(e^{(\alpha_0+\varepsilon)g^4(v_n)} - 1)\mathrm{d}z \\ &\le C_1 \int_{B_R} |g(v_n) - g(v)|^2\mathrm{d}z + C\int_{B_R} |g(v_n) - g(v)|(e^{(\alpha_0+\varepsilon)g^4(v_n)} - 1)\mathrm{d}z.\end{aligned}$$

Moreover,

$$\begin{aligned}\int_{B_R} &|g(v_n) - g(v)|(e^{(\alpha_0+\varepsilon)g^4(v_n)} - 1)\mathrm{d}z \\ &\le C\left(\int_{B_R} e^{q(\alpha_0+\varepsilon)\|\nabla g^2(v_n)\|_2^2\left(\frac{g^2(v_n)}{\|\nabla g^2(v_n)\|_2}\right)^2}\mathrm{d}z\right)^{1/q}\left(\int_{B_R} |g(v_n) - g(v)|^{q'}\mathrm{d}z\right)^{1/q'} \\ &\le C_1\left(\int_{B_R} |g(v_n) - g(v)|^{q'}\mathrm{d}z\right)^{1/q'} \to 0\end{aligned}$$

by virtue of the following fact

$$q(\alpha_0 + \varepsilon)\|\nabla g^2(v_n)\|_2^2 = q(\alpha_0 + \varepsilon)\|2g(v_n)g'(v_n)\nabla v_n\|_2^2 \le q(\alpha_0 + \varepsilon)4K < \alpha_0.$$

Thus, $h(z, g(v_n))g(v_n) \to h(z, g(v))g(v)$ in $L^1(B_R)$ and therefore

$$h(z, g(v_n))g(v_n) \to h(z, g(v))g(v) \quad \text{in } L^1(\mathbb{R}^2).$$

From this and the following inequalities

$$h(z, g(v_n))g'(v_n)v_n \le h(z, g(v_n))g(v_n) \quad \text{and} \quad 2H(z, g(v_n)) \le h(z, g(v_n))g(v_n),$$

parts (i) and (ii) follow from the Lebesgue dominated convergence theorem. □

In view of the previous results, we can conclude that for all $\varepsilon > 0$ the functional $\mathcal{F}_\varepsilon : E \to \mathbb{R}$ given by

$$\mathcal{F}_\varepsilon(v) = \frac{\varepsilon^2}{2}\int_{\mathbb{R}^2} |\nabla v|^2\mathrm{d}z + \frac{1}{2}\int_{\mathbb{R}^2} V(z)g^2(v)\mathrm{d}z - \int_{\mathbb{R}^2} H(z, g(v))\mathrm{d}z$$

possesses the mountain-pass geometry, the Palais–Smale sequences are bounded and the mountain-pass level C_ε has the following characterization

$$C_\varepsilon = \inf_{v\in E\backslash\{0\}} \max_{t\ge 0} \mathcal{F}_\varepsilon(tv) > 0. \tag{22}$$

Furthermore, by condition (h_4) we can see that

$$C_\varepsilon = \inf_{v\in\mathcal{N}} \mathcal{F}_\varepsilon(v) \tag{23}$$

where $\mathcal{N} := \{v \in E\backslash\{0\} : \langle\mathcal{F}'_\varepsilon(v), v\rangle = 0\}$ (see, for example, [44]).

At this stage it is more convenient to work with stretched variables. Thus we change the variables as $z = \varepsilon x$. We denote $V_\varepsilon(x) = V(\varepsilon x)$ and we consider the following energy functional

$$\mathcal{I}_\varepsilon(v) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla v|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^2} V_\varepsilon(x) g^2(v)\mathrm{d}x - \int_{\mathbb{R}^2} H(\varepsilon x, g(v))\mathrm{d}x,$$

associated to the equation

$$-\Delta v = g'(v)[h(\varepsilon x, g(v)) - V_\varepsilon(x) g(v)] \quad \text{in } \mathbb{R}^2, \tag{24}$$

and defined on the Banach space

$$E_\varepsilon := \left\{ v \in L_G^{V_\varepsilon}(\mathbb{R}^2) : \int_{\mathbb{R}^2} |\nabla v|^2 \mathrm{d}z < \infty \right\}.$$

Proceeding similarly as for $\mathcal{F}_\varepsilon$, the functional $\mathcal{I}_\varepsilon$ has the mountain-pass geometry with the mountain-pass level given by

$$b_\varepsilon = \inf_{v \in E_\varepsilon \setminus \{0\}} \max_{t \geq 0} \mathcal{I}_\varepsilon(tv) > 0.$$

Next, we obtain an estimate, as $\varepsilon \to 0$, of the level b_ε by considering the following functional

$$\mathcal{F}_0(v) = \frac{1}{2}\int_{\mathbb{R}^2} [|\nabla v|^2 + \beta_1 G(v)]\mathrm{d}x - \int_{\mathbb{R}^2} F(g(v))\mathrm{d}x. \tag{25}$$

We may suppose, without loss of generality by the translation invariance of the problem, that $0 \in \Lambda$ and $\beta_1 = V(0)$. Roughly speaking, the idea which motivates a comparison argument is that we expect $\mathcal{F}_\varepsilon(v_\varepsilon) \to \mathcal{F}_0(v)$ to hold for a suitable v.

Critical points of $\mathcal{F}_0$ are classical solutions of the following autonomous *limit problem*

$$-\Delta v = h_1(v) \doteq g'(v)[f(g(v)) - \beta_1 g(v)] \quad \text{in } \mathbb{R}^2. \tag{26}$$

We recall the following result established in [31]:

Theorem 18. *Suppose the nonlinearity f has subcritical growth and satisfies the conditions (f_0) and (f_1) or it has critical growth and satisfies (f_0), (f_1) and (f_2). Then the following statements hold*:

(i) *There exists $\omega \in H^1(\mathbb{R}^2)$ such that $\mathcal{F}_0(\omega) = C_1$ and $\mathcal{F}_0'(\omega) = 0$ where C_1 is the mountain-pass level*

$$C_1 = \inf_{v \in H^1(\mathbb{R}^2) \setminus \{0\}} \max_{t \geq 0} \mathcal{F}_0(tv) > 0;$$

(ii) *C_1 is bounded from above by $(\theta - 4)/8\theta$ in the critical case*;

(iii) *ω is a nonnegative solution of* (26) *and moreover $\omega(x) \to 0$ as $|x| \to \infty$.*

Furthermore, since

$$h_1'(s) = g''(s)f(g(s)) + (g'(s))^2 f'(g(s)) - \beta_1 g''(s)g(s) - \beta_1 (g'(s))^2$$

we obtain $h_1'(0) = -\beta_1 < 0$. Thus, using a result of Gidas–Ni–Nirenberg [19] we conclude that ω is spherically symmetric about some point in $\mathbb{R}^2$ and $\partial\omega/\partial r < 0$ for all $r > 0$, where r is the radial coordinate about that point.

3.4. *Estimate of the mountain-pass level* b_ε

Lemma 19. $\limsup_{\varepsilon\to 0} b_\varepsilon \le C_1$.

Proof. Define $\omega_\varepsilon(x) := \varphi(\varepsilon x)\omega(x)$, where $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ is a standard cut-off function, such that $\varphi \equiv 1$ on B_ρ and $\varphi \equiv 0$ on $B_{2\rho}^c$, with $\rho > 0$ such that $B_{2\rho} \subset \Lambda$. In particular, supp $\omega_\varepsilon \subset \Lambda_\varepsilon := \{x \in \mathbb{R}^2 | \varepsilon x \in \Lambda\}$ and $\omega_\varepsilon \to \omega$ in $H^1(\mathbb{R}^2)$. By definition,

$$b_\varepsilon \le \max_{t\ge 0} \mathcal{I}_\varepsilon(t\omega_\varepsilon) = \mathcal{I}_\varepsilon(t_\varepsilon\omega_\varepsilon) \tag{27}$$

and

$$\langle I_\varepsilon'(t_\varepsilon\omega_\varepsilon), t_\varepsilon\omega_\varepsilon\rangle = 0,$$

that is,

$$\begin{aligned}\int_{\mathbb{R}^2} [t_\varepsilon^2|\nabla\omega_\varepsilon|^2 &+ V(\varepsilon x)g(t_\varepsilon\omega_\varepsilon)g'(t_\varepsilon\omega_\varepsilon)t_\varepsilon\omega_\varepsilon]\mathrm{d}x \\ &= \int_{\mathbb{R}^2} f(g(t_\varepsilon\omega_\varepsilon))g'(t_\varepsilon\omega_\varepsilon)t_\varepsilon\omega_\varepsilon \mathrm{d}x. \end{aligned}\tag{28}$$

Thus, from (28) we have

$$\begin{aligned} C_1 t_\varepsilon^2 &\ge \int_{\mathbb{R}^2} [t_\varepsilon^2|\nabla\omega_\varepsilon|^2 + V(\varepsilon x)g^2(t_\varepsilon\omega_\varepsilon)]\mathrm{d}x \\ &\ge \int_{\mathbb{R}^2} f(g(t_\varepsilon\omega_\varepsilon))g'(t_\varepsilon\omega_\varepsilon)t_\varepsilon\omega_\varepsilon\mathrm{d}x \\ &\ge \frac{1}{2}\int_{\mathbb{R}^2} f(g(t_\varepsilon\omega_\varepsilon))g(t_\varepsilon\omega_\varepsilon)\mathrm{d}x. \end{aligned}\tag{29}$$

Note that since $\omega \ge 0$ and $\omega \not\equiv 0$ there exists $n_0 \in \mathbb{N}$ such that $A_{n_0} := \{x \in \mathbb{R}^2; 1/n_0 \le \omega(x) \le n_0\}$ has positive Lebesgue measure. Define $A_{n_0}^\varepsilon := \{x \in \mathbb{R}^2; 1/(n_0+\varepsilon) \le \omega_\varepsilon(x) \le n_0 + \varepsilon\}$. Since $\omega_\varepsilon \to \omega$ converges in $L^\theta(\mathbb{R}^2)$, we obtain

$$\int_{A_{n_0}^\varepsilon} \omega_\varepsilon^\theta(x)\mathrm{d}x \to \int_{A_{n_0}} \omega^\theta(x)\mathrm{d}x \ne 0.$$

Also note that $g(t)/t$ is decreasing from which we obtain

$$g(t_\varepsilon\omega_\varepsilon) \ge \frac{g(t_\varepsilon(n_0+\varepsilon))}{n_0+\varepsilon}\omega_\varepsilon \quad \text{on } A_{n_0}^\varepsilon.$$

Furthermore, from (6) we have that there exist $C_3, C_4 > 0$ such that $F(s) \geq C_3 s^\theta - C_4$ for all $s \geq 0$. Thus,

$$\begin{aligned}\int_{\mathbb{R}^2} f(g(t_\varepsilon\omega_\varepsilon))g(t_\varepsilon\omega_\varepsilon)\mathrm{d}x &\geq \theta\int_{A^\varepsilon_{n_0}} F(g(t_\varepsilon\omega_\varepsilon))\mathrm{d}x \\ &= \theta\int_{A^\varepsilon_{n_0}} [C_3 g^\theta(t_\varepsilon\omega_\varepsilon) - C_4]\mathrm{d}x \\ &\geq C_5\frac{g^\theta(t_\varepsilon(n_0+\varepsilon))}{(n_0+\varepsilon)^\theta}\int_{A^\varepsilon_{n_0}} \omega_\varepsilon^\theta \mathrm{d}x - C_6|A^1_{n_0}|. \qquad (30)\end{aligned}$$

If $t_\varepsilon \to +\infty$ as $\varepsilon \to 0$ we have that

$$\lim_{\varepsilon\to 0}\frac{g^\theta(t_\varepsilon(n_0+\varepsilon))}{t_\varepsilon^2} = +\infty$$

where we obtain a contradiction in view of (29) and (30). Therefore, $\{t_\varepsilon\}_{\varepsilon>0}$ is bounded. Hence, up to a subsequence, $t_\varepsilon \to t_0$ as $\varepsilon \to 0$. We *claim* that $t_0 = 1$. Suppose for the moment that the claim holds true, in order to get as $\varepsilon \to 0$

$$\begin{aligned}\mathcal{I}_\varepsilon(t_\varepsilon\omega_\varepsilon) &= \mathcal{F}_0(t_\varepsilon\omega_\varepsilon) + \frac{1}{2}\int_{\Lambda_\varepsilon}[V(\varepsilon x) - \beta_0]g^2(t_\varepsilon\omega_\varepsilon)\mathrm{d}x \\ &\leq \mathcal{F}_0(t_\varepsilon\omega_\varepsilon) + C\int_{\mathbb{R}^2}[V(\varepsilon x) - \beta_1]\omega_\varepsilon^2\mathrm{d}x = \mathcal{F}_0(t_\varepsilon\omega_\varepsilon) + o_\varepsilon(1) \qquad (31)\end{aligned}$$

by the Lebesgue dominated convergence theorem, since

$$\sup_{x\in\Lambda_\varepsilon} V(\varepsilon x) \leq C, \quad \text{for all } \varepsilon > 0$$

for a positive constant C. Hence, from (27) the lemma follows.

Proof of the claim. From one side ω satisfies the limit equation

$$\int_{\mathbb{R}^2}|\nabla\omega|^2\mathrm{d}x + \int_{\mathbb{R}^2}\beta_1 g(\omega)g'(\omega)\omega\mathrm{d}x = \int_{\mathbb{R}^2} f(g(\omega))g'(\omega)\omega\mathrm{d}x \qquad (32)$$

whence from the other side, taking the limit in (28), as $\varepsilon \to 0$, we obtain

$$\int_{\mathbb{R}^2} t_0^2|\nabla\omega|^2\mathrm{d}x + \int_{\mathbb{R}^2}\beta_1 g(t_0\omega)g'(t_0\omega)t_0\omega\mathrm{d}x = \int_{\mathbb{R}^2} f(g(t_0\omega))g'(t_0\omega)t_0\omega\mathrm{d}x$$

and hence

$$\int_{\mathbb{R}^2}|\nabla\omega|^2\mathrm{d}x + \int_{\mathbb{R}^2}\beta_1 g(t_0\omega)\frac{g'(t_0\omega)}{t_0\omega}\omega^2\mathrm{d}x = \int_{\mathbb{R}^2} f(g(t_0\omega))\frac{g'(t_0\omega)}{t_0\omega}\omega^2\mathrm{d}x. \qquad (33)$$

Subtracting (33) from (32) we get

$$\int_{\mathbb{R}^2}[L(t_0\omega) - L(\omega)]\omega^2\mathrm{d}x = 0 \qquad (34)$$

where

$$L(u) := \beta_1 g(u)\frac{g'(u)}{u} - f(g(u))\frac{g'(u)}{u}.$$

It follows from assumptions (f_1) and straightforward calculations that $L(u)$ is monotone and therefore from (34) necessarily $t_0 = 1$. ∎

3.5. *Existence result via mountain-pass theorem*

In this subsection, we are going to prove that there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, Eq. (14) possesses a positive ground state solution. The theorem below together with Proposition 9 provide this result.

Theorem 20. *Suppose that V satisfies (V_0)–(V_1) and either of the following conditions hold*:

(i) *The nonlinear term f is subcritical and enjoys (f_0) and (f_1);*
(ii) *The nonlinear term f is critical and enjoys (f_0), (f_1) and (f_2).*

Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, the functional $\mathcal{F}_\varepsilon$ has a critical point $v_\varepsilon \in E$ at the mountain-pass level C_ε given in (22).

Proof. First of all, notice that $C_\varepsilon = \varepsilon^2 b_\varepsilon$. Thus, from Lemma 19 there exists $\varepsilon_0 > 0$ such that $0 < C_\varepsilon < (\theta - 4)/8\theta$. In order to simplify the notation, without loss of generality, we fix ε and we denote $\mathcal{F}_\varepsilon \equiv \mathcal{F}$ and $C_\varepsilon \equiv C_0$. It follows from Lemmas 11 and 12 that the functional $\mathcal{F}$ has the geometry of the mountain-pass theorem. Therefore applying Theorem 7 we obtain a bounded $(\text{P.–S.})_{C_0}$ sequence (v_n) in E (cf. Proposition 13), that is,

$$\mathcal{F}(v_n) \to C_0 \quad \text{and} \quad \mathcal{F}'(v_n) \to 0.$$

Since $0 < C_0 < (\theta - 4)/8\theta$, by Lemmas 17 and 15 there exists a critical point v of $\mathcal{F}$ satisfying

$$\int_{\mathbb{R}^2} h(z, g(v_n))g'(v_n)v_n \mathrm{d}z \to \int_{\mathbb{R}^2} h(z, g(v))g'(v)v \mathrm{d}z, \tag{35}$$

$$\int_{\mathbb{R}^2} H(z, g(v_n))\mathrm{d}z \to \int_{\mathbb{R}^2} H(z, g(v))\mathrm{d}z, \tag{36}$$

$$\int_{\mathbb{R}^2} V(z)g^2(v_n)\mathrm{d}z \to \int_{\mathbb{R}^2} V(z)g^2(v)\mathrm{d}z. \tag{37}$$

Now, we claim that $v \not\equiv 0$. Indeed, if $v = 0$, using that $\langle \mathcal{F}(v_n), v_n \rangle \to 0$ and $g(s)g'(s)s \leq g^2(s)$ for all $s \in \mathbb{R}$ together with (35), we get

$$\int_{\mathbb{R}^2} |\nabla v_n|^2 \mathrm{d}z + \int_{\mathbb{R}^2} V(z)g^2(v_n)\mathrm{d}z \to 0.$$

From this and (36) we conclude that

$$\mathcal{F}(v_n) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla v_n|^2 \mathrm{d}z + \frac{1}{2}\int_{\mathbb{R}^2} V(z)g^2(v_n)\mathrm{d}z - \int_{\mathbb{R}^2} H(z, g(v_n))\mathrm{d}z \to 0$$

which is a contradiction. Therefore, v is a nontrivial critical of $\mathcal{F}$. Next, by the characterization (23) we must have $\mathcal{F}(v) \geq C_0$. Moreover, as $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^2)$ by the semi-continuity of norm and (36), (37) we achieve, up to a subsequence, that

$$\begin{aligned}\mathcal{F}(v) &\leq \lim_{n\to\infty}\left[\frac{1}{2}\int_{\mathbb{R}^2}|\nabla v_n|^2\mathrm{d}z + \frac{1}{2}\int_{\mathbb{R}^2}V(z)g^2(v_n)\mathrm{d}z - \int_{\mathbb{R}^2}H(z,g(v_n))\mathrm{d}z\right]\\ &= \lim_{n\to\infty}\mathcal{F}(v_n) = C_0.\end{aligned}$$

Hence, $\mathcal{F}(v) = C_0$ and the proof of the theorem is complete. □

By performing the scaling $x \mapsto \varepsilon x$, Theorem 20 also yields a one parameter family of critical points for the functional $\mathcal{I}_\varepsilon$, namely

$$\vartheta_\varepsilon(x) := v_\varepsilon(\varepsilon x), \quad \varepsilon > 0.$$

4. L^∞-Estimate and the Behavior of ϑ_ε as $\varepsilon \to 0$

In this section, we shall prove that the family $(\vartheta_\varepsilon)_{\{0<\varepsilon<\varepsilon_0\}}$ decays uniformly to zero. To do this, we first prove that this family is uniformly bounded in L^∞.

Proposition 21. *There exist $\varepsilon_0 > 0$ and $C > 0$ such that $\|\vartheta_\varepsilon\|_{E_\varepsilon} \leq C$ for all $0 < \varepsilon < \varepsilon_0$.*

Proof. Since ϑ_ε is a positive critical point of $\mathcal{I}_\varepsilon$ at the level b_ε, that is,

$$\mathcal{I}_\varepsilon(\vartheta_\varepsilon) = b_\varepsilon = \inf_{v\in E_\varepsilon\setminus\{0\}}\max_{t\geq 0}\mathcal{F}_\varepsilon(tv) > 0,$$

by Lemma 19, we have $\mathcal{I}_\varepsilon(\vartheta_\varepsilon) \leq C_1 + o_\varepsilon(1)$, where $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$. Hence

$$\frac{\theta}{2}\left(\int_{\mathbb{R}^2}|\nabla\vartheta_\varepsilon|^2\mathrm{d}x + \int_{\mathbb{R}^2}V(\varepsilon x)g^2(\vartheta_\varepsilon)\mathrm{d}x\right) \leq \int_{\mathbb{R}^2}\theta H(\varepsilon x, g(\vartheta_\varepsilon))\mathrm{d}x + \theta C_1 + 1 \tag{38}$$

for all $\varepsilon \in (0, \varepsilon_0)$. On the other hand, notice that

$$\begin{aligned}\int_{\mathbb{R}^2}|\nabla\vartheta_\varepsilon|^2\mathrm{d}x + \int_{\mathbb{R}^2}V(\varepsilon x)g^2(\vartheta_\varepsilon)\mathrm{d}x &\geq \int_{\mathbb{R}^2}|\nabla\vartheta_\varepsilon|^2\mathrm{d}x + \int_{\mathbb{R}^2}V(\varepsilon x)g(\vartheta_\varepsilon)g'(\vartheta_\varepsilon)\vartheta_\varepsilon\mathrm{d}x\\ &= \int_{\mathbb{R}^2}h(\varepsilon x, g(\vartheta_\varepsilon))g'(\vartheta_\varepsilon)\vartheta_\varepsilon\mathrm{d}x\\ &\geq \frac{1}{2}\int_{\mathbb{R}^2}h(\varepsilon x, g(\vartheta_\varepsilon))g(\vartheta_\varepsilon)\mathrm{d}x,\end{aligned}$$

which together with (h_3) and (38) implies that

$$\begin{aligned}&\left(\frac{\theta-4}{2}\right)\left[\int_{\mathbb{R}^2}(|\nabla\vartheta_\varepsilon|^2 + V(\varepsilon x)g^2(\vartheta_\varepsilon))\mathrm{d}x\right]\\ &\quad\leq \int_{\mathbb{R}^2}[\theta H(\varepsilon x, g(\vartheta_\varepsilon)) - h(\varepsilon x, g(\vartheta_\varepsilon))g(\vartheta_\varepsilon)]\mathrm{d}x + \theta C_1 + 1\\ &\quad\leq \int_{\mathbb{R}^2\setminus\Lambda_\varepsilon}[\theta H(\varepsilon x, g(\vartheta_\varepsilon)) - h(\varepsilon x, g(\vartheta_\varepsilon))g(\vartheta_\varepsilon)]\mathrm{d}x + \theta C_1 + 1\\ &\quad\leq \frac{\theta-2}{\tau}\int_{\mathbb{R}^2\setminus\Lambda_\varepsilon}V(\varepsilon x)f^2(\vartheta_\varepsilon)\mathrm{d}x + 2\theta C_1 + 1\end{aligned}$$

where $\Lambda_\varepsilon = \{x \in \mathbb{R}^2 : \varepsilon x \in \Lambda\}$. Therefore,

$$\int_{\mathbb{R}^2} [|\nabla \vartheta_\varepsilon|^2 + V(\varepsilon x) g^2(\vartheta_\varepsilon)] \mathrm{d}x \leq C \tag{39}$$

which implies that $\|\vartheta_\varepsilon\|_\varepsilon \leq C$ for all $\varepsilon \in (0, \varepsilon_0)$. □

The next result concerns the regularity of the family (ϑ_ε) and it is essential for the proof of Theorems 1 and 2. We will use the Gagliardo–Nirenberg inequality (see [22, p. 31]), which asserts

$$\|u\|_q \leq C(\theta) \|u\|_r^{1-\theta} \|\nabla u\|_2^\theta \tag{40}$$

for all $u \in H^1(\mathbb{R}^2) \cap L^r(\mathbb{R}^2)$, where $1 \leq r < \infty$, $0 < \theta \leq 1$ and

$$\frac{1}{q} \doteq \frac{1-\theta}{r}. \tag{41}$$

Proposition 22. *The functions ϑ_ε belongs to $L^\infty(\mathbb{R}^2)$. Furthermore, there exist $\varepsilon_0 > 0$ and $C > 0$ such that $\|\vartheta_\varepsilon\|_\infty \leq C$ for all $0 < \varepsilon < \varepsilon_0$.*

Proof. Taking $\theta = 1/2$ in (41), it follows that $q = 2r$ and (40) implies

$$\|u\|_{2r} \leq C \|u\|_r^{1/2} \|\nabla u\|_2^{1/2}.$$

Now, setting $u = g(\vartheta_\varepsilon)$, $r = \sigma_n \doteq 2^n$, $n \geq 1$, we have

$$\|g(\vartheta_\varepsilon)\|_{\sigma_{n+1}} \leq C \|g(\vartheta_\varepsilon)\|_{\sigma_n}^{1/2}$$

because $\|\nabla \vartheta_\varepsilon\|_2 \leq C$ and $g'(\vartheta_\varepsilon) \leq 1$. Hence, by iteration, we see that

$$\|g(\vartheta_\varepsilon)\|_{\sigma_{n+1}} \leq C^{1+1/2+\cdots+1/2^{n-1}} \|g(\vartheta_\varepsilon)\|_2^{1/2^n}.$$

Using that

$$\|g(\vartheta_\varepsilon)\|_2 \leq \beta_0^{-1/2} \left[\int_{\mathbb{R}^2} V(\varepsilon x) g^2(\vartheta_\varepsilon) \mathrm{d}x\right]^{1/2} \leq C$$

and since the series $1 + 1/2 + \cdots + 1/2^{n-1} + \cdots$ is convergent, we conclude that

$$\|g(\vartheta_\varepsilon)\|_{L^\infty(B_\rho(x))} \leq \lim_{n\to\infty} \|g(\vartheta_\varepsilon)\|_{L^{\sigma_{n+1}}(B_\rho(x))} \leq \lim_{n\to\infty} \|g(\vartheta_\varepsilon)\|_{\sigma_{n+1}} \leq C$$

where $\rho > 0$ and $x \in \mathbb{R}^2$ are arbitrary. Thus, since g^{-1} is continuous, it follows that

$$\|\vartheta_\varepsilon\|_\infty \leq C \quad \text{for all } 0 < \varepsilon < \varepsilon_0. \tag{42}$$

□

Corollary 23. *There exists $C_0 > 0$ such that $\|\vartheta_\varepsilon\|_{H^1} \leq C_0$ for all $0 < \varepsilon < \varepsilon_0$.*

Proof. Since $\|\vartheta_\varepsilon\|_\infty \leq C$ for all $0 < \varepsilon < \varepsilon_0$, using property (9) in Proposition 3, we have that

$$g(\vartheta_\varepsilon) \geq C_2 \vartheta_\varepsilon \quad \text{for some } C_2 > 0. \tag{43}$$

Thus, in view of (39) and (V_0) the result follows. □

Lemma 24. *There exist* $\varepsilon_0 > 0$, *a family* $(y_\varepsilon)_{\{0<\varepsilon<\varepsilon_0\}}$ *in* $\mathbb{R}^2$ *and positive constants* R *and* β *such that*

$$\int_{B_R(y_\varepsilon)} g^2(\vartheta_\varepsilon)\mathrm{d}x \geq \beta \quad \textit{for all } 0 < \varepsilon < \varepsilon_0.$$

Proof. We assume, for the sake of contradiction, that there exists a sequence $\varepsilon_n \to 0$ as $n \to \infty$ such that for all $R > 0$

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^2} \int_{B_R(x)} g^2(\vartheta_{\varepsilon_n})\mathrm{d}x = 0.$$

Using a result by Lions (see [44]), we conclude that $g(\vartheta_{\varepsilon_n}) \to 0$ in $L^s(\mathbb{R}^2)$ for all $s > 2$. Hence, using $(h_2)_c$ (or $(h_2)_s$), for $\delta > 0$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} h(\varepsilon_n x, g(\vartheta_{\varepsilon_n}))g(\vartheta_{\varepsilon_n})\mathrm{d}x &\leq \delta \int_{\mathbb{R}^2} g^2(\vartheta_{\varepsilon_n})\mathrm{d}x \\ &\quad + C\int_{\mathbb{R}^2} g^2(\vartheta_{\varepsilon_n})e^{[\beta g^4(\vartheta_{\varepsilon_n})-1]}g(\vartheta_{\varepsilon_n})\mathrm{d}x \\ &\leq C_1\delta + C_2\int_{\mathbb{R}^2}(g(\vartheta_{\varepsilon_n}))^3\mathrm{d}x\end{aligned}$$

because $\|\vartheta_{\varepsilon_n}\|_\infty \leq C$ and this shows that

$$\int_{\mathbb{R}^2} h(\varepsilon_n x, g(\vartheta_{\varepsilon_n}))g(\vartheta_{\varepsilon_n})\mathrm{d}x \to 0.$$

Consequently, we also have

$$\int_{\mathbb{R}^2} H(\varepsilon_n x, g(\vartheta_{\varepsilon_n}))\mathrm{d}x \to 0.$$

Since $\langle \mathcal{I}'_{\varepsilon_n}(\vartheta_{\varepsilon_n}), \vartheta_{\varepsilon_n}\rangle = 0$, we get

$$\int_{\mathbb{R}^2} |\nabla\vartheta_{\varepsilon_n}|^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^2} V(\varepsilon_n x)g^2(\vartheta_{\varepsilon_n})\mathrm{d}x \leq \int_{\mathbb{R}^2} h(\varepsilon_n x, g(\vartheta_{\varepsilon_n}))g(\vartheta_{\varepsilon_n})\mathrm{d}x$$

which implies that

$$\int_{\mathbb{R}^2} |\nabla\vartheta_{\varepsilon_n}|^2\mathrm{d}x + \int_{\mathbb{R}^2} V(\varepsilon_n x)g^2(\vartheta_{\varepsilon_n})\mathrm{d}x \to 0.$$

Thus

$$b_{\varepsilon_n} = \mathcal{I}_{\varepsilon_n}(\vartheta_{\varepsilon_n}) \to 0$$

which is a contradiction, because $b_{\varepsilon_n} \geq c_0 > 0$ for all n, where c_0 is the mountain-pass level of the functional

$$I_0(v) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla v|^2\mathrm{d}x + \frac{\beta_0}{2}\int_{\mathbb{R}^2} g^2(v)\mathrm{d}x - \int_{\mathbb{R}^2} F(g(v))\mathrm{d}x$$

and the result is proved. ∎

Lemma 25. *The family* $(\varepsilon y_\varepsilon)_{\{0<\varepsilon<\varepsilon_0\}}$ *has the following property*

$$\operatorname{dist}(\varepsilon y_\varepsilon, \Lambda) \leq \varepsilon R.$$

Proof. For every $\delta > 0$, we define $\mathcal{K}_\delta \doteq \{x \in \mathbb{R}^2 : \operatorname{dist}(x, \Lambda) \leq \delta\}$ and $\phi_\varepsilon(x) \doteq \phi(\varepsilon x)$, where $\phi \in \mathcal{C}^\infty(\mathbb{R}^2, [0,1])$ is such that $\phi(x) = 1$ if $x \notin \mathcal{K}_\delta$, $\phi(x) = 0$ if $x \in \Lambda$ and $|\nabla \phi| \leq C/\delta$. Note that $|\nabla \phi_\varepsilon| \leq C\varepsilon/\delta$. By condition (V_0), we have

$$\beta_0 \left(\frac{1}{2} - \frac{1}{\tau}\right) \int_{\mathbb{R}^2} g^2(\vartheta_\varepsilon)\phi_\varepsilon \mathrm{d}x \leq \int_{\mathbb{R}^2} |\nabla \vartheta_\varepsilon|^2 \phi_\varepsilon \mathrm{d}x + \left(\frac{1}{2} - \frac{1}{\tau}\right) \int_{\mathbb{R}^2} V(\varepsilon x) g^2(\vartheta_\varepsilon)\phi_\varepsilon \mathrm{d}x.$$

On the other hand, since $\langle \mathcal{I}'_\varepsilon(\vartheta_\varepsilon), \vartheta_\varepsilon \phi_\varepsilon \rangle = 0$ and using (h_3) and the fact that the support of ϕ_ε does not intercept Λ_ε, we obtain

$$\int_{\mathbb{R}^2} |\nabla \vartheta_\varepsilon|^2 \phi_\varepsilon \mathrm{d}x + \left(\frac{1}{2} - \frac{1}{\tau}\right) \int_{\mathbb{R}^2} V(\varepsilon x) g^2(\vartheta_\varepsilon)\phi_\varepsilon \mathrm{d}x \leq -\int_{\mathbb{R}^2} \vartheta_\varepsilon \nabla \vartheta_\varepsilon \nabla \phi_\varepsilon \mathrm{d}x.$$

Thereby giving,

$$\begin{aligned} \beta_0 \left(\frac{1}{2} - \frac{1}{\tau}\right) \int_{\mathbb{R}^2} g^2(\vartheta_\varepsilon)\phi_\varepsilon \mathrm{d}x &\leq -\int_{\mathbb{R}^2} \vartheta_\varepsilon \nabla \vartheta_\varepsilon \nabla \phi_\varepsilon \mathrm{d}x \\ &\leq \frac{C\varepsilon}{\delta} \left(\int_{\mathbb{R}^2} |\nabla \vartheta_\varepsilon|^2 \mathrm{d}x\right)^{1/2} \left(\int_{\mathbb{R}^2} \vartheta_\varepsilon^2 \mathrm{d}x\right)^{1/2} \\ &\leq \frac{C_1 \varepsilon}{\delta}. \end{aligned}$$

From this inequality, if for some sequence $\varepsilon_n \searrow 0$ and

$$B_R(y_{\varepsilon_n}) \cap \{x \in \mathbb{R}^2 : \varepsilon_n x \in \mathcal{K}_\delta\} = \emptyset$$

we conclude that

$$\left(\frac{1}{2} - \frac{1}{\tau}\right) \int_{B_R(y_{\varepsilon_n})} g^2(\vartheta_{\varepsilon_n}) \mathrm{d}x \leq \frac{C_1 \varepsilon_n}{\beta_0 \delta}$$

which contradicts Lemma 24. Thus, for all $\varepsilon \in (0, \varepsilon_0)$, there exists an x such that $\varepsilon x \in \mathcal{K}_\delta$ and $|x - y_\varepsilon| \leq R$, showing that $\operatorname{dist}(\varepsilon y_\varepsilon, \Lambda) \leq \varepsilon R + \delta$ and from this we conclude the proof. □

Remark 26. It follows from the previous lemma that the family $(\varepsilon y_\varepsilon)_{\{0<\varepsilon<\varepsilon_0\}}$ can be taken in such a way that $\varepsilon y_\varepsilon \in \Lambda$ for all $0 < \varepsilon < \varepsilon_0$. Indeed, since $\operatorname{dist}(\varepsilon y_\varepsilon, \Lambda) < 2\varepsilon R$ for each $\varepsilon \in (0, \varepsilon_0)$, there exists $x_\varepsilon \in \Lambda$ satisfying $|y_\varepsilon - \varepsilon^{-1} x_\varepsilon| < 2R$. Thus,

$$0 < \beta \leq \int_{B_R(y_\varepsilon)} g^2(\vartheta_\varepsilon) \mathrm{d}x \leq \int_{B_{3R}(\varepsilon^{-1} x_\varepsilon)} g^2(\vartheta_\varepsilon) \mathrm{d}x.$$

Replacing R by $3R$ in Lemma 24, we can replace y_ε by $\varepsilon^{-1} x_\varepsilon$.

Lemma 27. *There exists* $\varepsilon_0 > 0$ *sufficiently small such that the family* $(\vartheta_\varepsilon)_{\{0<\varepsilon<\varepsilon_0\}}$ *decays to zero as* $|x| \to \infty$ *uniformly with respect to* $\varepsilon \in (0, \varepsilon_0)$.

Proof. We know that for all $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} \nabla\vartheta_\varepsilon \nabla\phi \mathrm{d}x + \int_{\mathbb{R}^2} V(\varepsilon x) g(\vartheta_\varepsilon) g'(\vartheta_\varepsilon)\phi \mathrm{d}x = \int_{\mathbb{R}^2} h(\varepsilon x, g(\vartheta_\varepsilon)) g'(\vartheta_\varepsilon)\phi \mathrm{d}x. \tag{44}$$

This, together with Propositions 3 and 22, implies that

$$\int_{\mathbb{R}^2} \nabla\vartheta_\varepsilon \nabla\phi \mathrm{d}x \leq C \int_{\mathbb{R}^2} \vartheta_\varepsilon \phi \mathrm{d}x,$$

for all nonnegative functions $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$. By standard local behavior result [20, Theorem 8.17], for any ball $B_{2r}(x)$ centered at any $x \in \mathbb{R}^2$,

$$\sup_{y \in B_r(x)} \vartheta_\varepsilon(y) \leq C\|\vartheta_\varepsilon\|_{L^2(B_{2r}(x))} \quad \text{for all } 0 < \varepsilon < \varepsilon_0.$$

Therefore, the assertion of the lemma will be a consequence of the following result.

Claim 28. *There exists $\varepsilon_0 > 0$ sufficiently small such that the following holds*:

$$\lim_{R \to \infty} \int_{|x| \geq R} \vartheta_\varepsilon^2 \mathrm{d}x = 0$$

uniformly with respect to $\varepsilon \in (0, \varepsilon_0)$.

For the proof of this claim we use the Radial Lemma (see [6, Lemma A.IV]) which asserts that for all $x \neq 0$ and $u \in H^1(\mathbb{R}^2)$ holds

$$|u^*(x)| \leq \frac{1}{\sqrt{\pi}|x|}\|u^*\|_{H^1}, \tag{45}$$

where u^* denotes the Schwarz symmetrization of u.

For $R > 0$, let ψ_R be in $\mathcal{C}^\infty(\mathbb{R}^2, [0,1])$ such that

$$\psi_R(x) = \begin{cases} 0, & \text{if } |x| \leq R \\ 1, & \text{if } |x| \geq 2R \end{cases}$$

and satisfying $|\nabla\psi_R| \leq C/R$ for some $C > 0$. Taking $\phi = \vartheta_\varepsilon \psi_R$ in (44), we obtain

$$\int_{\mathbb{R}^2} |\nabla\vartheta_\varepsilon|^2 \psi_R \mathrm{d}x + \int_{\mathbb{R}^2} \vartheta_\varepsilon \nabla\vartheta_\varepsilon \nabla\psi_R \mathrm{d}x + \int_{\mathbb{R}^2} V(\varepsilon x) g(\vartheta_\varepsilon) g'(\vartheta_\varepsilon)\vartheta_\varepsilon \psi_R \mathrm{d}x$$
$$= \int_{\mathbb{R}^2} h(z, g(\vartheta_\varepsilon)) g'(\vartheta_\varepsilon)\vartheta_\varepsilon \psi_R \mathrm{d}x. \tag{46}$$

Using property $(h_2)_s$ or $(h_2)_c$, we have that

$$h(\varepsilon_n x, g(\vartheta_\varepsilon)) \leq \frac{\beta_0}{4} g(\vartheta_\varepsilon) + C g(\vartheta_\varepsilon) e^{[\beta g^4(\vartheta_\varepsilon) - 1]}$$

which together with (V_0), Proposition 3 and (46) imply that

$$\frac{\beta_0}{2} \int_{\mathbb{R}^2} g^2(\vartheta_\varepsilon)\psi_R \mathrm{d}x \leq -\int_{\mathbb{R}^2} \vartheta_\varepsilon \nabla\vartheta_\varepsilon \nabla\psi_R \mathrm{d}x + \frac{\beta_0}{4} \int_{\mathbb{R}^2} g^2(\vartheta_\varepsilon)\psi_R \mathrm{d}x$$
$$+ \int_{\mathbb{R}^2} e^{[\beta g^4(\vartheta_\varepsilon) - 1]} g(\vartheta_\varepsilon) g'(\vartheta_\varepsilon)\vartheta_\varepsilon \psi_R \mathrm{d}x.$$

From Proposition 22 and (45) it follows that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\int_{|x|\geq 2R} g^2(\vartheta_\varepsilon)\mathrm{d}x \leq \frac{C}{R} \tag{47}$$

where we have used the following estimate

$$\int_{|x|\geq R} e^{[2\beta\vartheta_\varepsilon^2 - 1]}\vartheta_\varepsilon^2 \mathrm{d}x \leq \frac{C}{R}. \tag{48}$$

Indeed, by the properties of Schwarz symmetrization,

$$\begin{aligned}\int_{|x|\geq R} e^{[2\beta\vartheta_\varepsilon^2 - 1]}\vartheta_\varepsilon^2 \mathrm{d}x &= \int_{|x|\geq R} e^{[2\beta(\vartheta_\varepsilon^*)^2 - 1]}(\vartheta_\varepsilon^*)^2 \mathrm{d}x \\ &= \sum_{k=1}^{\infty} \frac{(2\beta)^k}{k!} \int_{|x|\geq R} (\vartheta_\varepsilon^*)^{2k+2} \mathrm{d}x. \end{aligned} \tag{49}$$

Since $\|\vartheta_\varepsilon^*\|_{H^1} \leq \|\vartheta_\varepsilon\|_{H^1} \leq C$ for all $0 < \varepsilon < \varepsilon_0$, it follows from (45) that for all $k \geq 1$

$$\begin{aligned}\int_{|x|\geq R} (\vartheta_\varepsilon^*)^{2k+2}\mathrm{d}x &\leq \left(\frac{C}{\sqrt{\pi}}\right)^{2k+2} \int_{|x|\geq R} \frac{1}{|x|^{2k+2}}\mathrm{d}x \\ &= \pi \left(\frac{C}{\sqrt{\pi}}\right)^{2k+2} \frac{1}{kR^{2k}} \\ &\leq C^2 \left(\frac{C^2}{\pi}\right)^k \frac{1}{R}, \end{aligned}$$

where we have assumed that $R > 1$. Thus, from this estimate and (49) we obtain (48). Finally, using Proposition 3 and (47), the claim is proved. □

5. The Concentration Behavior

The critical points v_ε of the modified functional $\mathcal{F}_\varepsilon$ actually yield, as $\varepsilon \to 0$, critical points of the reduced functional $\widetilde{I}_\varepsilon$ which, by means of the change of variable $u_\varepsilon = g(v_\varepsilon)$, are eventually solutions of the original problem (P_ε). Furthermore, we are going to show that such solutions inherit the shape of the solutions of the *limit problem* (26) and how this fact forces them, as $\varepsilon \to 0$, to concentrate around a point which is localized by the potential V.

Lemma 29. *The following limit holds*

$$\lim_{\varepsilon \to 0} V(\varepsilon y_\varepsilon) = \beta_1$$

and $w_\varepsilon(x) \doteq \vartheta_\varepsilon(x + y_\varepsilon)$ *converges uniformly to a nontrivial solution of problem* (26) *over compacts subsets of* $\mathbb{R}^2$.

Proof. Let ε_n be a sequence such that $\varepsilon_n \to 0$ and $y_n \in \mathbb{R}^2$ verifying $\varepsilon_n y_n \in \Lambda$. As $\varepsilon_n y_n \in \overline{\Lambda}$, up to subsequences, we get $\varepsilon_n y_n \to x_0 \in \overline{\Lambda}$. To simplify the notation, set

$\vartheta_n = \vartheta_{\varepsilon_n}$ and $w_n(x) = \vartheta_n(x + y_n)$. Since $\|w_n\|_{H^1} = \|\vartheta_n\|_{H^1}$ is bounded, we may assume that there exists $w \in H^1(\mathbb{R}^2)$ such that

$$w_n \rightharpoonup w \text{ in } H^1(\mathbb{R}^2) \quad \text{and} \quad w_n \to w \text{ almost everywhere in } \mathbb{R}^2.$$

By Lemma 24, we have $w \neq 0$. We define

$$\chi(x) \doteq \lim_{n\to\infty} \chi_\Lambda(\varepsilon_n x + \varepsilon_n y_n) \quad \text{almost everywhere in } \mathbb{R}^2$$

and

$$\widetilde{h}(x, s) \doteq \chi(x) f(s) + (1 - \chi(x))\overline{f}(s).$$

We have that

$$\int_{\mathbb{R}^2} [\nabla w_n \nabla \phi + V(\varepsilon_n x + \varepsilon_n y_n) g(w_n) g'(w_n)\phi] \mathrm{d}x$$
$$= \int_{\mathbb{R}^2} h(\varepsilon_n x + \varepsilon_n y_n, g(w_n)) g'(w_n)\phi \mathrm{d}x, \tag{50}$$

for all $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$. Since $\|\vartheta_n\|_\infty \leq C$ for all n, by the Lebesgue dominated convergence theorem it follows that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} h(\varepsilon_n x + \varepsilon_n y_n, g(w_n)) g'(w_n)\phi \mathrm{d}x = \int_{\mathbb{R}^2} \widetilde{h}(x, g(w)) g'(w)\phi \mathrm{d}x,$$

for all $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$. Taking the limit in (50) we achieve that w satisfies

$$\int_{\mathbb{R}^2} [\nabla w \nabla \phi + V(x_0) g(w) g'(w)\phi] \mathrm{d}x = \int_{\mathbb{R}^2} \widetilde{h}(x, g(w)) g'(w)\phi \mathrm{d}x,$$

for all $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$. Therefore, w is a critical point of the functional given by

$$\widetilde{\mathcal{F}}(v) = \frac{1}{2} \int_{\mathbb{R}^2} [|\nabla v|^2 + V(x_0) g^2(v)] \mathrm{d}x - \int_{\mathbb{R}^2} \widetilde{H}(x, g(v)) \mathrm{d}x,$$

where $\widetilde{H}$ is the primitive of $\widetilde{h}$. If $x_0 \in \Lambda$ we have $\varepsilon_n x + \varepsilon_n y_n \in \Lambda$ for n sufficiently large. Hence, $\chi(x) = 1$ for all $x \in \mathbb{R}^2$ and so w is a critical point of the following functional

$$I_{x_0}(v) = \frac{1}{2} \int_{\mathbb{R}^2} [|\nabla v|^2 + V(x_0) g^2(v)] \mathrm{d}x - \int_{\mathbb{R}^2} F(g(v)) \mathrm{d}x.$$

Denoting by C_{x_0} the mountain-pass level associated to the functional I_{x_0} and by $\widetilde{C}$ the mountain-pass level associated to the functional $\widetilde{\mathcal{F}}$, we claim that $C_{x_0} \leq \widetilde{C}$. In fact, since $\widetilde{H}(x, s) \leq F(s)$ for all $x \in \mathbb{R}^2$ and $s \in \mathbb{R}$, we obtain $I_{x_0}(v) \leq \widetilde{\mathcal{F}}(v)$ for all $v \in H^1(\mathbb{R}^2)$ and this implies that $C_{x_0} \leq \widetilde{C}$. Let us define the set

$$A_n = \{x \in \mathbb{R}^2 : \varepsilon_n x + \varepsilon_n y_n \in \Lambda\}.$$

If $x \in A_n$, using (h_3) we have

$$\begin{aligned}
&\frac{\theta}{4}V(\varepsilon_n x+\varepsilon_n y_n)g^2(w_n(x)) - V(\varepsilon_n x+\varepsilon_n y_n)g(w_n(x))g'(w_n(x))w_n(x)\\
&\qquad + h(\varepsilon_n x+\varepsilon_n y_n, g(w_n(x)))g'(w_n(x))w_n(x) - \frac{\theta}{2}H(\varepsilon_n x+\varepsilon_n y_n, g(w_n(x)))\\
&\quad\geq \left(\frac{\theta}{4}-1\right)V(\varepsilon_n x+\varepsilon_n y_n)g^2(w_n(x))\\
&\qquad + \frac{1}{2}[f(g(w_n(x)))g(w_n(x)) - \theta F(g(w_n(x)))] \geq 0
\end{aligned}$$

and if $x \notin A_n$

$$\begin{aligned}
&\frac{\theta}{4}V(\varepsilon_n x+\varepsilon_n y_n)g^2(w_n(x)) - V(\varepsilon_n x+\varepsilon_n y_n)g(w_n(x))g'(w_n(x))w_n(x)\\
&\qquad + h(\varepsilon_n x+\varepsilon_n y_n, g(w_n(x)))g'(w_n(x))w_n(x) - \frac{\theta}{2}H(\varepsilon_n x+\varepsilon_n y_n, g(w_n(x)))\\
&\quad\geq \left(\frac{\theta}{4}-1-\frac{\theta}{4\tau}\right)V(\varepsilon_n x+\varepsilon_n y_n)g^2(w_n(x)) \geq 0
\end{aligned}$$

because $\theta/4 - 1 - \theta/4\tau > 0$. Since $C_1 \leq C_{x_0}$ and $\widetilde{C} \leq \widetilde{\mathcal{F}}(w)$, we have

$$\frac{\theta}{2}C_1 \leq \frac{\theta}{2}C_{x_0} \leq \frac{\theta}{2}\widetilde{C} \leq \frac{\theta}{2}\widetilde{\mathcal{F}}(w) = \frac{\theta}{2}\widetilde{\mathcal{F}}(w) - \langle \widetilde{\mathcal{F}}'(w), w\rangle,$$

from which we obtain

$$\begin{aligned}
\frac{\theta}{2}C_1 \leq \left(\frac{\theta}{4}-1\right)\int_{\mathbb{R}^2}|\nabla w|^2 \mathrm{d}x + \int_{\mathbb{R}^2}\Big[&\frac{\theta}{4}V(x_0)g^2(w) - V(x_0)g(w)g'(w)w\\
&+ \widetilde{h}(x, g(w))g'(w)w - \frac{\theta}{2}\widetilde{H}(x, g(w))\Big]\,\mathrm{d}x.
\end{aligned}$$

It follows from the above inequality, Fatou's Lemma and semicontinuity of the norm that

$$\begin{aligned}
\frac{\theta}{2}C_1 &\leq \liminf_{n\to\infty}\left(\frac{\theta}{4}-1\right)\int_{\mathbb{R}^2}|\nabla w_n|^2\mathrm{d}x\\
&\quad + \liminf_{n\to\infty}\int_{\mathbb{R}^2}\Big[\frac{\theta}{4}V(\varepsilon_n x+\varepsilon_n y_n)g^2(w_n) - V(\varepsilon_n x+\varepsilon_n y_n)g(w_n)g'(w_n)w_n\\
&\quad + h(\varepsilon_n x+\varepsilon_n y_n, g(w_n))g'(w_n)w_n - \frac{\theta}{2}H(\varepsilon_n x+\varepsilon_n y_n, g(w_n))\Big]\,\mathrm{d}x\\
&= \liminf_{n\to\infty}\left(\frac{\theta}{4}-1\right)\int_{\mathbb{R}^2}|\nabla \vartheta_n|^2\mathrm{d}x\\
&\quad + \liminf_{n\to\infty}\int_{\mathbb{R}^2}\Big[\frac{\theta}{4}V(\varepsilon_n x)g^2(\vartheta_n) - V(\varepsilon_n x)g(\vartheta_n)g'(\vartheta_n)\vartheta_n\\
&\quad + h(\varepsilon_n x, g(\vartheta_n))g'(\vartheta_n)\vartheta_n - \frac{\theta}{2}H(\varepsilon_n x, g(\vartheta_n))\Big]\,\mathrm{d}x
\end{aligned}$$

$$= \liminf_{n\to\infty}\left[\frac{\theta}{2}\mathcal{F}_{\varepsilon_n}(\vartheta_n) - \langle \mathcal{F}'_{\varepsilon_n}(\vartheta_n), \vartheta_n\rangle\right]$$
$$= \frac{\theta}{2}\liminf_{n\to\infty} b_{\varepsilon_n} \le \frac{\theta}{2}C_1.$$

Thus, $\widetilde{\mathcal{F}}(w) = C_1$ and $\lim_{\varepsilon\to 0} b_\varepsilon = C_1$. Moreover, if $V(x_0) > \beta_1$ we obtain, by the fact that the mountain-pass level C_1 on the constant potential β_1 is continuous and increasing (see, for example, [35]), that $C_1 < C_{x_0} \le \widetilde{C} \le \widetilde{F}(w) = C_1$ which is a contradiction. Therefore, $V(x_0) = \beta_1$ and this implies that $x_0 \in \Lambda$ and $\widetilde{\mathcal{F}} = I_{x_0} = \mathcal{F}_0$. Therefore, w is a solution of (26). Also we have

$$-\Delta(w_n - w) = G_n \quad \text{in } \mathbb{R}^2$$

where

$$G_n(x) \doteq \beta_1 g(w(x))g'(w(x)) - V(\varepsilon_n x + \varepsilon_n y_n)g(w_n(x))g'(w_n(x))$$
$$+ h(\varepsilon_n x + \varepsilon_n y_n, g(w_n(x)))g'(w_n(x)) - f(g(w(x)))g'(w(x)).$$

As $w_n \to w$ almost everywhere in $\mathbb{R}^2$ this implies that $G_n \to 0$ almost everywhere in $\mathbb{R}^2$. Notice that for each compact subset $\mathcal{B}$ of $\mathbb{R}^2$ we have $|G_n|, |w| \le C_{\mathcal{B}}$ since $\|w_n\|_\infty \le C$ and $|\varepsilon_n x + \varepsilon_n y_n| \le C_1$ for all n and $x \in \mathcal{B}$. Thus, by the Lebesgue dominated convergence theorem it follows that $G_n \to 0$ in $L^s_{\text{loc}}(\mathbb{R}^2)$ for all $s \ge 1$. Using [20, Theorem 9.11] we can conclude that $w_n \to w$ in $W^{2,s}_{\text{loc}}(\mathbb{R}^2)$ for all $s \ge 1$ and from this $w_n \to w$ in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^2)$ for some $\alpha \in (0,1)$. Now, by [20, Theorem 6.2], $w_n \to w$ in $C^{2,\alpha}_{\text{loc}}(\mathbb{R}^2)$ for some $\alpha \in (0,1)$ and the lemma is proved. □

6. Proof of Theorems 1 and 2

Since ϑ_ε decays uniformly to zero, there exists $R > 0$ such that $\vartheta_\varepsilon(x) \le a$ for all $|x| \ge R$. Choosing $\varepsilon_0 > 0$ sufficiently small such that $B_R \subset \Lambda_{\varepsilon_0}$, we conclude that for all $\varepsilon \in (0, \varepsilon_0)$

$$-\Delta\vartheta_\varepsilon + V(\varepsilon x)g(\vartheta_\varepsilon)g'(\vartheta_\varepsilon) = f(g(\vartheta_\varepsilon))g'(\vartheta_\varepsilon) \quad \text{in } \mathbb{R}^2.$$

Thus,

$$-\varepsilon^2\Delta v_\varepsilon + V(z)g(v_\varepsilon)g'(v_\varepsilon) = f(g(v_\varepsilon))g'(\vartheta_\varepsilon) \quad \text{in } \mathbb{R}^2$$

and this implies that $u_\varepsilon = g(v_\varepsilon)$ is a positive solution of problem (P_ε) for all $\varepsilon \in (0, \varepsilon_0)$.

By Proposition 22, we have that, for all $\varepsilon \in (0, \varepsilon_0)$, w_ε possesses a global maximum point $x_\varepsilon \in B_\rho$ for some $\rho > 0$. Considering the translation $\widetilde{w}_\varepsilon(x) = w_\varepsilon(x + x_\varepsilon)$, we may assume that the function w_ε achieve its global maximum at the origin of $\mathbb{R}^2$. Using the fact that w is spherically symmetric, $\partial w/\partial r < 0$ for all $r > 0$ and w_ε converges to w in $C^{2,\alpha}_{\text{loc}}(\mathbb{R}^2)$, by Lemma 4.2 in [33] we can conclude that w_ε possesses no critical point other than the origin for all $\varepsilon \in (0, \varepsilon_0)$.

Notice that the maximum value of $v_\varepsilon(z) = v_\varepsilon(\varepsilon x) = \vartheta_\varepsilon(x) = w_\varepsilon(x - y_\varepsilon)$ is achieved at the point $z_\varepsilon = \varepsilon y_\varepsilon \in \Lambda$. As the function g is strictly increasing, the maximum value of $u_\varepsilon(z) = g(v_\varepsilon(z))$ is also achieved at the point $z_\varepsilon = \varepsilon y_\varepsilon \in \Lambda$. As $\nabla u_\varepsilon = g'(v_\varepsilon)\nabla v_\varepsilon$, u_ε possesses no critical point other than z_ε and the item (i) in Theorems 1 and 2 is proved. The item (ii) is a consequence of Lemma 29.

6.1. *Exponential decay of the solutions*

To finalize, we are going to prove the exponential decay of the solutions u_ε. Using that $\lim_{s\to 0} g(s)g'(s)/s = 1$ and (f_0), we can choose $R_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and $|x| \geq R_0$

$$g(w_\varepsilon(x))g'(w_\varepsilon(x)) \geq \frac{3}{4}w_\varepsilon(x) \quad \text{and} \quad f(g(w_\varepsilon(x))) \leq \frac{\beta_0}{2}g(w_\varepsilon(x)). \tag{51}$$

We define $\psi(x) := Me^{-\xi|x|}$ where ξ and M are such that $4\xi^2 < \beta_0$ and $Me^{-\xi R_0} \geq w_\varepsilon(x)$ for all $|x| = R_0$. It is not difficult to check that

$$\Delta\psi \leq \xi^2\psi, \quad \forall\, x \neq 0. \tag{52}$$

We consider the function $\psi_\varepsilon = \psi - w_\varepsilon$. Thus, using (51), (52) and the following equation

$$-\Delta w_\varepsilon + V(\varepsilon x + \varepsilon y_\varepsilon)g(w_\varepsilon)g'(w_\varepsilon) = f(g(w_\varepsilon))g'(w_\varepsilon) \quad \text{in } \mathbb{R}^2,$$

we obtain

$$\begin{aligned} -\Delta\psi_\varepsilon + \frac{\beta_0}{4}\psi_\varepsilon &\geq 0 \quad \text{in } |x| \geq R_0, \\ \psi_\varepsilon &\geq 0 \quad \text{on } |x| = R_0, \\ \lim_{|x|\to\infty} \psi_\varepsilon(x) &= 0. \end{aligned}$$

By the maximum principle, we have that $\psi_\varepsilon(x) \geq 0$ for all $|x| \geq R_0$. Hence, $\psi_\varepsilon(x) \leq Me^{-\xi|x|}$ for all $|x| \geq R_0$ and $\varepsilon \in (0, \varepsilon_0)$. This implies that

$$u_\varepsilon(z) = g(v_\varepsilon(z)) \leq v_\varepsilon(z) = \vartheta_\varepsilon\left(\frac{z}{\varepsilon}\right) = w_\varepsilon\left(\frac{z - z_\varepsilon}{\varepsilon}\right) \leq Ce^{-\xi\left|\frac{z-z_\varepsilon}{\varepsilon}\right|}$$

and the item (iii) of Theorems 1 and 2 is proved.

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