Solitary waves for quasilinear Schrödinger equations arising in plasma physics*

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Abstract

We study the quasilinear Schrödinger equation

$$iz_t = -\Delta z + W(x)z - \eta(|z|^2)z - \kappa\left[\Delta\rho(|z|^2)\right]\rho'(|z|^2)z \quad \text{in} \quad \mathbb{R}^2,$$

where $W : \mathbb{R}^2 \to \mathbb{R}$ is a positive potential and the nonlinearity $\eta : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ has critical or sub-critical exponential growth. Quasilinear Schrödinger equations of this type have been studied as models of several physical phenomena such as superfluid film equation, in the theory of Heisenberg ferromagnets and magnons, in dissipative quantum mechanics and in condensed matter theory. In a suitable Orlicz space together with Trudinger-Moser inequality we establish an existence of standing wave solutions for this problem. The second order nonlinearity considered in this paper corresponds to the superfluid equation in plasma physics.

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1 Introduction

We study *quasilinear Schrödinger equations* of the form

$$iz_t = -\Delta z + W(x)z - \eta(|z|^2)z - \kappa\left[\Delta\rho(|z|^2)\right]\rho'(|z|^2)z \quad \text{in } \mathbb{R}^2, \tag{1.1}$$

where $z : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{C}$, W is a given potential, κ is a real constant, and η and ρ are real functions.

Such equations arise in various branches of mathematical physics and have been a subject of extensive studies in the past few years corresponding to various types of ρ. The superfluid film equation in plasma physics has this structure for $\rho(s) = s$, see [20]. In the case $\rho(s) = (1+s)^{1/2}$, equation (1.1) models the self-channeling of a high-power ultra short laser in matter, see [35]. Equation (1.1) also appears in fluid mechanics [21], in the theory of Heisenberg ferromagnets and magnons [22], in dissipative quantum mechanics and in condensed matter theory [27].

We consider the case where $\rho(s) = s$. However, our approach is quite general and goes beyond this nonlinearity. Here our special interest is in the existence of *standing wave solutions*, that is, solutions of type

$$z(t,x) = \exp(-iEt)u(x),$$

where $E \in \mathbb{R}$ and $u > 0$ is a real function. It is well known that z satisfies (1.1) if and only if the function $u(x)$ solves the following equation of elliptic type with the formal variational structure

$$-\Delta u + V(x)u - \kappa\left[\Delta\left(u^2\right)\right]u = \eta(u) \quad \text{in} \quad \mathbb{R}^2,$$

where $V(x) := W(x) - E$ is the new potential, $\kappa > 0$, η is the new nonlinearity and without loss of generality we assume $\kappa = 1$. Indeed, we intend to consider a more general situation involving non-autonomous nonlinearities. More precisely, our purpose is to study the following quasilinear equation of elliptic type

$$-\Delta u + V(x)u - \left[\Delta(|u|^2)\right]u = h(x,u) \quad \text{in} \quad \mathbb{R}^2. \tag{1.2}$$

We establish the existence of positive solutions for the above quasilinear elliptic equation when $V : \mathbb{R}^2 \to \mathbb{R}$ is a positive potential bounded away from zero and it can be large at infinity, and the nonlinearity $h(x,u)$ has the maximal growth which allows us to treat problem (1.2) variationally in a suitable function space. In fact the subcritical and also the critical case will be considered. We say that h has subcritical growth at $+\infty$ if for all $\alpha > 0$

$$\lim_{t\to+\infty} \frac{h(x,t)}{e^{\alpha t^4}} = 0$$

and h has critical growth at $+\infty$ if there exists $\alpha_0 > 0$, such that

$$\lim_{t\to+\infty} \frac{h(x,t)}{e^{\alpha t^4}} = \begin{cases} 0 & \text{if} \quad \alpha > \alpha_0, \\ +\infty & \text{if} \quad \alpha < \alpha_0. \end{cases}$$

In the case of critical growth, we say that α_0 is the critical exponent of h. We believe that the exponential growth above is the critical growth for this problem when $N = 2$, as the counterpart of the case $N \geq 3$ in which the critical exponent is $2(2^*) = 4N/(N-2)$ (see [24] and [9]).

We note that such notion is motivated by Trudinger-Moser estimates [31, 38] which provide

$$e^{\alpha|u|^2} \in L^1(\Omega) \quad \text{for all} \quad u \in H_0^1(\Omega) \quad \text{and} \quad \alpha > 0,$$

and

$$\sup_{\|u\|_{H_0^1} \leq 1} \int_\Omega e^{\alpha|u|^2} \, \mathrm{d}x \leq C \quad \text{for all} \quad \alpha \leq 4\pi,$$

where $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain. Subsequently Cao [5] proved a version of Trudinger-Moser inequality in the whole space, which was improved in [13, 37], namely,

$$e^{\alpha|u|^2} - 1 \in L^1(\mathbb{R}^2) \quad \text{for all} \quad u \in H^1(\mathbb{R}^2) \quad \text{and} \quad \alpha > 0.$$

Moreover, if $\alpha < 4\pi$ and $|u|_{L^2(\mathbb{R}^2)} \leq C$, there exists a constant $C_2 = C_2(C, \alpha)$ such that

$$\sup_{\|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1} \int_{\mathbb{R}^2} \left(e^{\alpha|u|^2} - 1\right) \mathrm{d}x \leq C_2. \tag{1.3}$$

Recent mathematical studies on the subject have been focused on the existence of solutions to quasilinear Schrödinger equations of the form

$$-\Delta u + V(x)u - \left[\Delta(|u|^2)\right] u = h(x, u) \quad \text{in } \mathbb{R}^N.$$

In dimension one see [32] (indeed the first paper on the subject), and [1, 7] and for $N = 2$ see [16, 28] and finally for $N \geq 3$ see for example [9, 17, 24, 26, 25, 29, 30, 32]. For existence and concentration of solitary waves for this quasilinear Schrödinger equation we refer to [6] and [18]. In recent years, the related semilinear equations for $\kappa = 0$ have been extensively studied. See e.g. [4, 33, 36], and references therein.

Quasilinear Schrödinger equations in dimension two and involving sub-critical and critical exponential growth have been considered recently in [16] and [28] where the potential V is bounded. In [16] by using a change of variable, the quasilinear equations are reduced to semilinear equations, whose respective associated functionals are well defined in the usual Sobolev space $H^1(\mathbb{R}^2)$ and satisfy the geometric hypotheses of the mountain-pass theorem. Using this fact, it was obtained a Cerami sequence converging weakly to a solution v. In the proof that v is nontrivial, the main tool is the concentration-compactness principle due to P. L. Lions [23], combined with test functions connected with optimal Trudinger-Moser inequality. In [28] the case of periodic potential was considered and the existence of at least one weak solution was proved. Indeed, the mountain-pass theorem in a suitable Orlicz space together with the Trudinger-Moser inequality were employed to establish this result.

The main motivation for the present paper is to deal with equations with more general class of potentials and nonlinearities involving critical growth. As a direct consequence we extend and complement the results in [16] and [28].

This paper contains a delicate Orlicz space approach introduced in [24] and [6] together with the ingredients from several recent papers on elliptic problems involving critical growth in the Trudinger-Moser case, see [5, 11, 12, 13, 16, 28] and references therein.

Throughout the paper, we assume the following basic hypothesis on the potential:

(V_0) V is a continuous function and

$$V(x) \geq V_0 > 0 \text{ for all } x \in \mathbb{R}^2.$$

We consider the situation in which the potential $V(x)$ is unbounded from above. Indeed, we prove the existence under either of the following assumptions on the potential.

(V_1) $V(x) \to \infty$ as $|x| \to \infty$; or more generally, for every $M > 0$, the set

$$\{x \in \mathbb{R}^2 : V(x) \leq M\}$$

has finite Lebesgue measure.

(V_2) The function $[V(x)]^{-1}$ belongs to $L^1(\mathbb{R}^2)$, that is,

$$\int_{\mathbb{R}^2} \frac{1}{V(x)}\,\mathrm{d}x < \infty.$$

We now introduce the following assumptions on the nonlinear term $h(x, u)$.

(H_0) $h : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is continuous, $h(x, 0) \equiv 0$ and

$$|h(x, u)| \leq b_1|u| + b_2\left(e^{\alpha_0|u|^4} - 1\right) \text{ for all } (x, u) \in \mathbb{R}^2 \times [0, +\infty),$$

for some constants α_0, b_1, $b_2 > 0$.

This assumption is motivated by the Trudinger-Moser inequality together with the presence of the term $[\Delta(u^2)]u$ in order to study the problem with maximum exponential growth of $h(x, u)$ in the whole two-dimensional space.

Let

$$H(x, s) := \int_0^s h(x, t)\,\mathrm{d}t \qquad \text{and}$$

$$\lambda_1 := \inf_{u \in H^1(\mathbb{R}^2)\setminus\{0\}} \frac{\int_{\mathbb{R}^2} \left[(1 + u^2)|\nabla u|^2 + V(x)u^2\right] \mathrm{d}x}{\int_{\mathbb{R}^2} u^2\,\mathrm{d}x}.$$

It is easy to see from our assumptions on the potential V that $\lambda_1 > 0$.

(H_1) $\limsup_{u\to 0^+} \dfrac{2H(x,u)}{u^2} < \lambda_1$ *uniformly in* $x \in \mathbb{R}^2$.

(H_2) There exists $\mu > 4$ such that

$$0 < \mu H(x,s) \le h(x,s)s \text{ for all } (x,s) \in \mathbb{R}^2 \times (0,\infty).$$

The main results are the following.

Theorem 1.1 (The subcritical case) *Suppose* (V_0) *and* (V_1) *(or* (V_2)*), if* h *has subcritical growth and* $(H_0) - (H_2)$ *are satisfied, then (1.2) possesses a positive solution.*

Theorem 1.2 (The critical case) *Suppose* (V_0) *and* (V_1) *(or* (V_2)*) are satisfied and that* h *has critical growth. If* $(H_0) - (H_2)$ *and also the following condition hold*

(H_3) *There exists* $\lambda > 0$ *such that*

$$h(x,s) \ge \lambda s^3 \text{ for all } (x,s) \in \mathbb{R}^2 \times [0,+\infty),$$

Then, there exists λ_∞ *such that for* $\lambda \ge \lambda_\infty$*, problem (1.2) possesses a positive solution.*

Note that (H_1) weakens the following standard condition used in the literature,

$$\limsup_{u\to 0^+} \frac{2H(x,u)}{u^2} = 0 \quad \text{uniformly in } \; x \in \mathbb{R}^2.$$

Condition (H_2), used for this class of quasilinear Schrödinger equations, is the counter part to the classical Ambrosetti-Rabinowitz condition and it is already used in [9], [16], [24] and [30]. Assumption (H_3) is technical and leaves room for improvement, although it is more general than the following one used in [16]

$$\lim_{u\to\infty} uh(u)e^{-\alpha u^4} \ge \beta > 0 \text{ for some constants } \alpha, \beta > 0. \tag{1.4}$$

Notice that the hypotheses of Theorems 1.1 and 1.2 are, for instance, satisfied by nonlinearities of the following forms:

(a) Subcritical growth: $h(u) = 5u^4(e^{u^3} - 1) + 3u^7 e^{u^3}$.

(b) Critical growth with $V(x) \le C(1 + |x|)$:

$$h(u) = \begin{cases} 5u^4 + \cos(u)(e^{5u^4} - 1) + 20(1+\sin(u))u^3(e^{5u^4} - 1), & u \ge \frac{3\pi}{2}, \\ 5u^4, & 0 \le u \le \frac{3\pi}{2}. \end{cases}$$

Note that Example (b) does not verify the condition (1.4).

A main difficulty in treating this class of quasilinear Schrödinger equations in $\mathbb{R}^2$ involving critical growth is the possible lack of compactness besides the quasilinear term. Moreover, there is no natural functions spaces for the associated energy functional to be well defined and this is due to the super-critical growth condition on the nonlinearity.

Remark 1.1 *Using elliptic regularity theory (see [19]) one can see that the solutions of (1.2) are of C^2 class and decay to zero at infinity, for details see [16].*

Remark 1.2 *It is readily seen that our method applies to other potentials, for example, radially symmetric potentials, namely $V(x) = V(|x|)$, for all $x \in \mathbb{R}^2$ (see [4], [23], [36]).*

Outline of the paper: Motivated by the argument used in [24] the forthcoming section contains a reformulation of the problem and some preliminary results on the function space setting. In Section 3, by using a version of the mountain-pass theorem, which is a consequence of the Ekeland Variational Principle we prove Theorems 1.1 and 1.2.

Notation.– *In this paper we make use of the following notation:*

- C, C_0, C_1, C_2, ... denote positive (possibly different) constants.
- For $1 \le p \le \infty$, $L^p(\mathbb{R}^2)$ denotes Lebesgue spaces with the norm $\|u\|_p$.
- $H^1(\mathbb{R}^2)$ denotes the Sobolev spaces modeled in $L^2(\mathbb{R}^2)$ with its usual norm
$$\|u\|_{1,2} := \left(\|\nabla u\|_2^2 + \|u\|_2^2\right)^{1/2}.$$
- $C_0^\infty(\mathbb{R}^2)$ denotes the functions infinitely differentiable with compact support in $\mathbb{R}^2$.

2 Preliminaries

First, since we look for positive solutions of (1.2) we assume $h(x,s) = 0$ for all $(x,s) \in \mathbb{R}^2 \times (-\infty, 0]$.

We observe that, formally (1.2) is the Euler-Lagrange equation associated to the following functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^2}(1+u^2)|\nabla u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^2} V(x)u^2\,dx - \int_{\mathbb{R}^2} H(x,u)\,dx$$

where $H(x,s) := \int_0^s h(x,t)\,dt$. From the variational point of view, the first difficulty we have to deal with, is to find an appropriate function space where the above functional is well defined. Following the idea introduced in [24] (see also [8] [9]), we recall the function space setting of [6]. We reformulate the problem by means of the following change of variable

$$dv = \sqrt{1+u^2}du,$$

which can be rewritten as

$$v = f(u) := \frac{1}{2}u\sqrt{1+u^2} + \frac{1}{2}\ln\left(u+\sqrt{1+u^2}\right)$$

and since f is strictly monotone it is well defined and so is the inverse function $g := f^{-1}$ with

$$g'(v) = \frac{1}{\sqrt{1+g^2(v)}}.$$

The following asymptotic behaviors will be fundamental in the sequel

$$f(s) \sim \begin{cases} s, & |s| \to 0 \\ \frac{1}{2}s|s|, & |s| \to \infty \end{cases}, \qquad g(s) \sim \begin{cases} s & |s| \to 0 \\ \sqrt{\frac{2}{|s|}}s, & |s| \to \infty \end{cases}$$

Moreover,

$$G(s) := g^2(s) \sim \begin{cases} s^2, & |s| \to 0 \\ 2|s|, & |s| \to \infty \end{cases}$$

and note that

$$G'(v) = \frac{2g(v)}{\sqrt{1+g^2(v)}}, \qquad G''(v) = \frac{2}{(1+g^2(v))^2}. \tag{2.1}$$

By exploiting this change of variable, we can rewrite the functional J in the following form

$$I(v) := J(g(v)) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla v|^2\, dx + \frac{1}{2}\int_{\mathbb{R}^2} V(x)G(v)\, dx - \int_{\mathbb{R}^2} H(x, g(v))\, dx \tag{2.2}$$

which has finite energy provided that

$$\int_{\mathbb{R}^2} |\nabla v|^2\, dx < \infty \quad \text{and} \quad \int_{\mathbb{R}^2} V(x)G(v)\, dx < \infty.$$

Observe that G is convex, $G(0) = 0$, $G(s) \nearrow \infty$, as $s \to \infty$ so that (up to extending G on $(-\infty, 0)$ by $G(-s) = G(s)$) it is a Young function and one can consider the Orlicz class (see [34]), which we denote by $L^V_G(\mathbb{R}^2)$, of measurable functions $v : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\int_{\mathbb{R}^2} G(|v|)\, d\mu < \infty, \quad d\mu = V(x)dx.$$

Remark 2.1 *The Young function G satisfies the Δ_2-condition globally (see [34]), that is: there exists $K > 0$ such that $G(2s) \leq KG(s)$ for all $s \geq 0$. As a consequence, one has that L_G^V is a linear space on which one can define the following norm*

$$\|v\|_G := \sup\left\{\int_{\mathbb{R}^2} |vz|\, d\mu \,:\, z \in L_{\widetilde{G}}^V(\mathbb{R}^2)\,,\, \int_{\mathbb{R}^2} \widetilde{G}(|z|)\, d\mu \leq 1\right\} \tag{2.3}$$

where $(G, \widetilde{G})$ denotes a Young pair.

Thus, the new functional I in (2.2) turns out to be well defined in a natural fashion on the Banach space

$$E := \left\{v \in L_G^V(\mathbb{R}^2) \,:\, \int_{\mathbb{R}^2} |\nabla v|^2\, dx < \infty\right\}$$

which can be obtained as the completion of $C_0^\infty(\mathbb{R}^2)$ with respect to the norm

$$\|v\| := \|\nabla v\|_2 + \|v\|_G.$$

We also consider the closed subspace of $H^1(\mathbb{R}^2)$

$$H_V^1 := \left\{u \in H^1(\mathbb{R}^2) \,:\, \int_{\mathbb{R}^2} V(x)u^2\, dx < \infty\right\}$$

equipped with the norm

$$\|u\|_V = \left(\int_{\mathbb{R}^2} |\nabla u|^2\, dx + \int_{\mathbb{R}^2} V(x)u^2\, dx\right)^{1/2}.$$

Remark 2.2 *Under the condition (V_0) for all $q \geq 2$,*

$$H_V^1(\mathbb{R}^2) \hookrightarrow H^1(\mathbb{R}^2) \hookrightarrow L^q(\mathbb{R}^2)$$

with continuous embedding and with compact embedding if V satisfies condition (V_1) or (V_2) (see [10] and [33]).

In the following proposition we state some facts about the Banach space E and the nonlinear map $v \to g(v)$ which are useful in the sequel.

Proposition 2.1 (1) *$(E, \|\cdot\|) \hookrightarrow L^q(\mathbb{R}^2)$ for all $q \geq 2$.*

(2) *Let $u = g(v)$ and $v \in E$. Then the following estimate holds:*

$$\|u\|_V \leq \|\nabla v\|_2 + \|v\|_G^{1/4} + 2^{K_0/2}\|v\|_G^{K_0/2}$$

where K_0 is a positive constant which does not depend on v nor on u.

(3) *The map $v \to g(v)$ from E to $L^q(\mathbb{R}^2)$ is weak to strong continuous.*

Proof. For part (1) by exploiting the asymptotic behavior of the strictly increasing and strictly convex function G, from (2.1) we have

$$G(t) \geq \begin{cases} At^2, & t \in [0, \tau) \\ A\tau t, & t \in [\tau, +\infty) \end{cases}$$

for a positive constant A and $\tau > 0$ sufficiently small. Hence for $v \in E$ we get

$$\int_{\mathbb{R}^2} G(|v|)\, d\mu \geq C \left(\int_{\Omega^1 := \{x \in \mathbb{R}^2 \,:\, |v| < \tau\}} v^2\, dx + \int_{\Omega^2 := \{x \in \mathbb{R}^2 \,:\, |v| \geq \tau\}} |v|\, dx \right)$$

for a positive constant C which does not depend on v. It follows that $v \in L^2(\mathbb{R}^2)$ from which together with the fact that $\|\nabla v\|_2 \leq \|v\|$ we obtain $v \in H^1(\mathbb{R}^2)$. Now the result follows from the Sobolev embedding $H^1(\mathbb{R}^2) \subseteq L^q(\mathbb{R}^2)$ for all $q \geq 2$.

We proceed the proof of part (2) in several steps:
Step 1. First we prove that

$$\|v\|_G \leq \frac{1}{k} \left(1 + \int_{\mathbb{R}^2} G(kv)\, d\mu \right), \quad \forall k > 0. \tag{2.4}$$

Indeed, by (2.3) and using the Young inequality $xy \leq G(x) + \widetilde{G}(y)$ one has

$$\begin{aligned} \|v\|_G &= \frac{1}{k} \sup \left\{ \int_{\mathbb{R}^2} |kvz|\, d\mu \,:\, \int_{\mathbb{R}^2} \widetilde{G}(|z|)\, d\mu \leq 1 \right\} \\ &\leq \frac{1}{k} \sup \left\{ \int_{\mathbb{R}^2} G(kv) + \widetilde{G}(|g|)\, d\mu \,:\, \int_{\mathbb{R}^2} \widetilde{G}(|z|)\, d\mu \leq 1 \right\} \\ &\leq \frac{1}{k} \left(\int_{\mathbb{R}^2} G(kv)\, d\mu + 1 \right). \end{aligned}$$

Step 2. We next show that there exists a constant $K_0 >$ such that

$$\int_{\mathbb{R}^2} G(v)\, d\mu \leq \begin{cases} \|v\|_G, & \|v\|_G \leq 1 \\ 2^{K_0} \|v\|_G^{K_0}, & \|v\|_G > 1 \end{cases} \quad \forall v \in L_G^V(\mathbb{R}^2). \tag{2.5}$$

We recall from [34, Proposition 3, p. 60] that if $v \in L_G^V(\mathbb{R}^2)$, $v \neq 0$, one has

$$\int_{\mathbb{R}^2} G\left(\frac{v}{\|v\|_G} \right) d\mu \leq 1$$

and in particular (2.5) follows if $\|v\|_G = 1$. Otherwise we distinguish when $\|v\|_G < 1$ and $\|v\|_G > 1$. In the first case, $v < v/\|v\|_G$ and since G is increasing, we get

$$\int_{\mathbb{R}^2} G(v)\, d\mu \leq \int_{\mathbb{R}^2} G\left(\frac{v}{\|v\|_G} \right) d\mu \leq 1.$$

Moreover, since G is strictly convex, we have

$$\begin{aligned} G\left(v\|v\|_G\right) =& G\left(v\|v\|_G + (1-\|v\|_G)0\right) \\ \leq& G(v)\|v\|_G + G(0)(1-\|v\|_G) = G(v)\|v\|_G \end{aligned}$$

thus

$$\int_{\mathbb{R}^2} G\left(v\|v\|_G\right)\, d\mu \leq \|v\|_G \int_{\mathbb{R}^2} G(v)\, d\mu \leq \|v\|_G$$

Now we set $w = v\|v\|_G$ to get

$$\int_{\mathbb{R}^2} G(w)\, d\mu = \int_{\mathbb{R}^2} G(v\|v\|_G)\, d\mu \leq \|v\|_G = \|w\|_G^{1/2}, \quad \forall\, \|w\|_G \leq 1.$$

If $\|v\|_G > 1$, let $\eta := 1/\|v\|_G$ and $\bar{v} := \eta v$. Since $0 < \eta < 1$ we can find $n = n(v) \in \mathbf{N}$, such that $1/2^n < \eta < 1/2^{n-1}$ and since G is increasing we have

$$G\left(\frac{v}{2^n}\right) \leq G(\eta v) = G(\bar{v}) \tag{2.6}$$

By exploiting Δ_2-condition in Remark 2.1 with a constant $K > 1$, we obtain

$$G(v) = G\left(2^n \frac{v}{2^n}\right) \leq K^n G\left(\frac{v}{2^n}\right) \tag{2.7}$$

and then joining (2.6) and (2.7) we obtain

$$\int_{\mathbb{R}^2} G(v)\, d\mu \leq K^n \int_{\mathbb{R}^2} G(\bar{v})\, d\mu \leq K^n \leq K^{1+\log_2 \|v\|_G} \leq 2^{K_0}\|v\|_G^{K_0}$$

for a constant K_0 such that $2^{K_0} \geq K$. We complete the proof of the lemma by evaluating for $u = g(v)$

$$\begin{aligned} \|u\|_V \leq& \left(\int_{\mathbb{R}^2} \frac{1}{1+G(v)}|\nabla v|^2\, dx\right)^{1/2} + \left(\int_{\mathbb{R}^2} G(v)\, d\mu\right)^{1/2} \\ \leq& \|\nabla v\|_2 + \|v\|_G^{1/4} + 2^{K_0/2}\|v\|_G^{K_0/2}. \end{aligned}$$

This proves part (2).
Part (3) follows from the inclusion $E \subseteq H_1^V(\mathbb{R}^2)$, due to part (2), together with Remark 2.2.

The following proposition states some properties of the functional I. See [6] and [28] for the proof.

Proposition 2.2 *The functional I is well defined on E. Moreover,*

(i) I is continuous on E.

(ii) I is Gâteaux differentiable on E and for each $v \in E$, $I'(v) \in E^*$.

(iii) If $v \in E$ is a critical point for I, then $v \in \mathcal{C}^2_{loc}(\mathbb{R}^N)$ and $u = g(v)$ is a classical solution of equation (1.2).

3 Existence results via mountain-pass

We will achieve the existence result by using the following version of the mountain–pass theorem which is a consequence of the Ekeland Variational Principle as developed in [2].

Theorem 3.1 *Let E be a Banach space and $\Phi \in \mathcal{C}(E;\mathbb{R})$, Gâteaux differentiable for all $v \in E$, with G-derivative $\Phi'(v) \in E^*$ continuous from the norm topology of E to the weak$^*-$topology of E^* and $\Phi(0) = 0$. Let $\mathcal{S}$ be a closed subset of E which disconnects (archwise) E. Let $v_0 = 0$ and $v_1 \in E$ be points belonging to distinct connected components of $E \setminus \mathcal{S}$. Suppose that*

$$\inf_{\mathcal{S}} \Phi \geq \alpha > 0 \quad \textit{and} \quad \Phi(v_1) \leq 0$$

and let

$$\Gamma = \{\gamma \in \mathcal{C}([0,1];E) \mid \gamma(0) = 0 \textit{ and } \gamma(1) = v_1\}.$$

Then

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi(\gamma(t)) \geq \alpha$$

and there exists a Palais-Smale sequence for Φ at level c (Denoted by(P.-S.)$_c$ sequence), that is: $(u_n) \subset E$ such that, as $n \to \infty$,

$$\Phi(u_n) \to c \quad \textit{and} \quad \Phi'(u_n) \to 0 \quad \textit{in} \quad E^*.$$

3.1 Mountain-pass geometry

Lemma 3.1 *Under conditions (V_0), (H_0) and (H_2), there exists $v \in E \setminus \{0\}$ such that $I(v) < 0$.*

Proof. Let $u \in C_0^\infty(\mathbb{R}^2) \setminus \{0\}$, $u \geq 0$, K is the support of u and $|K|$ denotes the Lebesgue measure of K. We are going to prove that $\lim_{t\to\infty} J(tu) = -\infty$, consequently $I(v) < 0$ for $v = f(tu)$ and t large enough. Notice that by (H_2), it is easy to see that there exist positive constants c, d such that

$$H(x,s) \geq cs^\mu - d, \quad \forall\ (x,s) \in K \times [0,+\infty).$$

Thus for the large values of t,

$$\begin{aligned} J(tu) &= \frac{1}{2}\int_{\mathbb{R}^2} \left(1+t^2u^2\right) t^2|\nabla u|^2\,dx + \frac{t^2}{2}\int_{\mathbb{R}^2} V(x)u^2\,dx - \int_{\mathbb{R}^2} H(x,tu)\,dx \\ &\leq \frac{t^4}{2}\int_{\mathbb{R}^2} \left(1+u^2\right)|\nabla u|^2\,dx + \frac{t^2}{2}\int_{\mathbb{R}^2} V(x)u^2\,dx - t^\mu \int_{\mathbb{R}^2} u^\mu\,dx + d|K|, \end{aligned}$$

which together with $\mu > 4$ implies that $\lim_{t\to\infty} J(tu) = -\infty$.

Lemma 3.2 *Assume* (V_0), (H_0) *and* (H_1). *Let* $\rho > 0$ *and define the following closed subset of* E

$$\mathcal{S}(\rho) := \left\{ v \in E \;\middle|\; \int_{\mathbb{R}^2} |\nabla v|^2\,dx + \int_{\mathbb{R}^2} V(x) g^2(v)\,dx = \rho^2 \right\}.$$

Then, there exist constants ρ, $\alpha > 0$, *such that*

$$I(v) \geq \alpha \text{ for all } v \in \mathcal{S}(\rho).$$

Proof. It is easy to see that using the change of variable we have the following equivalent formulation for λ_1 in our Orlicz space setting:

$$\lambda_1 := \inf_{v \in E\setminus\{0\}} \frac{\int_{\mathbb{R}^2} [|\nabla v|^2 + V(x) g^2(v)]\,dx}{\int_{\mathbb{R}^2} g^2(v)\,dx}.$$

Thus

$$\begin{aligned} \lambda_1 \int_{\mathbb{R}^2} g^2(v)\,dx &\leq \int_{\mathbb{R}^2} [|\nabla v|^2 + V(x) g^2(v)]\,dx \\ &= \rho^2. \end{aligned}$$

Combining conditions (H_0) and (H_1), for $\eta < \lambda_1$ and $q > 2$ we obtain

$$H(x,t) \leq \frac{1}{2}\eta t^2 + C t^q (e^{\beta t^4} - 1)$$

Notice that

$$\|\nabla g^2(v)\|_2 = \|2 g(v) g'(v) \nabla v\|_2 = \left\| \frac{2g(v)}{\sqrt{1 + g(v)^2}} \nabla v \right\|_2 \leq 2 \|\nabla v\|_2.$$

Thus, taking $\rho > 0$ sufficiently small, we have

$$\|\nabla g^2(v)\|_2 \leq 2\|\nabla v\|_2 \leq 2\rho^2 < 1.$$

Thus, using Trudinger-Moser inequality (1.3) and proceeding as in the proof of [13, Lemma 3], we obtain

$$\int_{\mathbb{R}^2} |g(v)|^q \left(e^{g(v)^4} - 1 \right)\,dx \leq C \rho^q.$$

Hence, for all $v \in \mathcal{S}(\rho)$,

$$I(v) \geq \frac{1}{2}\left(1 - \frac{\eta}{\lambda_1}\right)\rho^2 - C\rho^q.$$

Now, choosing $\rho > 0$ sufficiently small we have

$$I(v) \geq \frac{1}{4}\left(1 - \frac{\eta}{\lambda_1}\right)\rho^2$$

which proves the lemma.

3.2 On the mountain-pass level

As a consequence of Lemmas 3.1 and 3.2, and Theorem 3.1 we have

$$C_0 = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I(\gamma(t)) \geq \alpha > 0,$$

where

$$\Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^2));\ \gamma(0) = 0, \gamma(1) \neq 0,\ I(\gamma(1)) < 0\}.$$

Lemma 3.3 *Assume that h has critical growth and satisfies (H_0), (H_2) and (H_3). Then there exists λ_∞ such that for $\lambda \geq \lambda_\infty$, the critical value C_0 is bounded from above by $(\mu-4)/4\mu$.*

Proof.
Fix a positive function $\bar\phi \in H^1_V$ such that $\bar\phi^2 \in H^1$. Take λ_∞ large enough in such a way that

$$0 < \frac{\|\bar\phi\|_V^2}{\lambda_\infty \int_{\mathbb{R}^2} |\bar\phi|^4\,dx - 2\int_{\mathbb{R}^2} \bar\phi^2|\nabla\bar\phi|^2\,dx} < \frac{\mu-4}{\mu} \tag{3.1}$$

Consider a large number n such that $J(n\bar\phi) < 0$. Set $\phi(x) = n\bar\phi(x)$ and $\gamma_1(t) := f(t\phi)$. It follows from the characterization of the mountain-pass level that

$$C_0 = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I(\gamma(t)) \leq \sup_{t\in[0,1]} I(\gamma_1(t)) = \sup_{t\in[0,1]} I(f(t\phi)) = \sup_{t\in[0,1]} J(t\phi).$$

Therefore, it suffices to show that $\sup_{t\in[0,1]} J(t\phi) \leq (\mu-4)/4\mu$. Indeed,

$$\begin{aligned} C_0 &\leq \sup_{t\in[0,1]} J(t\phi) \\ &\leq \sup_{t\in[0,1]} \left\{ \frac{t^2}{2}\|\phi\|_V^2 + \frac{t^4}{2}\int_{\mathbb{R}^2} \phi^2|\nabla\phi|^2\,dx - \int_{\mathbb{R}^2} H(x,t\phi)dx \right\} \end{aligned}$$

It follows from (H_3) with $\lambda \geq \lambda_\infty$ that

$$\begin{aligned} C_0 &\leq \sup_{t\in[0,1]} \left\{ \frac{t^2}{2}\|\phi\|_V^2 + \frac{t^4}{2}\int_{\mathbb{R}^2} \phi^2|\nabla\phi|^2\,dx - \frac{\lambda t^4}{4}\int_{\mathbb{R}^2} \phi^4 dx \right\} \\ &= \sup_{t\in[0,1]} \left\{ \frac{t^2}{2}\|\phi\|_V^2 - \frac{t^4}{4}(\lambda\int_{\mathbb{R}^2} \phi^4 dx - 2\int_{\mathbb{R}^2} \phi^2|\nabla\phi|^2\,dx) \right\} \\ &= \frac{\|\phi\|_V^2}{4(\lambda\int_{\mathbb{R}^2} |\phi|^4\,dx - 2\int_{\mathbb{R}^2} \phi^2|\nabla\phi|^2\,dx)} \\ &\leq \frac{\|\bar\phi\|_V^2}{4(\lambda_\infty\int_{\mathbb{R}^2} |\bar\phi|^4\,dx - 2\int_{\mathbb{R}^2} \bar\phi^2|\nabla\bar\phi|^2\,dx)} \end{aligned}$$

from which together with (3.1) we obtain

$$C_0 < \frac{\mu-4}{4\mu},$$

and the proof is complete.

3.3 Palais-Smale sequences

Proposition 3.1 *Suppose (v_n) is a Palais-Smale sequence for I at level C_0, that is, $(v_n) \subset E$ such that, as $n \to \infty$,*

$$I(v_n) \to C_0 \quad and \quad I'(v_n) \to 0 \quad in \quad E^*$$

Then (v_n) is bounded in E.

Proof. Since (v_n) is a $(\text{P.-S.})_{C_0}$ sequence, we have

$$\begin{aligned} I(v_n) &= \frac{1}{2}\int |\nabla v_n|^2 dx + \frac{1}{2}\int V(x) g(v_n)^2 dx - \int H(x, g(v_n)) dx \\ &= C_0 + \delta_n, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} |\langle I'(v_n), \phi\rangle| = \bigg| \int \nabla v_n . \nabla \phi dx + \int V(x) g(v_n) g'(v_n) \phi dx \\ - \int h(x, g(v_n)) g'(v_n) \phi dx \bigg| \leq \epsilon_n \|\phi\| \end{aligned} \tag{3.3}$$

where $\delta_n,\ \epsilon_n \to 0$ as $n \to \infty$. Next, we pick

$$\phi = \frac{g(v_n)}{g'(v_n)} = \sqrt{1 + g(v_n)^2}\, g(v_n)$$

as a test function in (3.3). One can easily deduce that

$$\|\phi\|_G \leq C \|v_n\|_G \quad \text{and} \quad |\nabla \phi| = \left[1 + \frac{g(v_n)^2}{1 + g(v_n)^2}\right] |\nabla v_n| \leq 2 |\nabla v_n|,$$

which implies $\|\phi\| \leq C\|v_n\|$. Substituting ϕ in (3.3), gives

$$\begin{aligned} |\,\langle I'(v_n), \phi\rangle\,| = \bigg| \int \left[1 + \frac{g(v_n)^2}{1 + g(v_n)^2}\right] |\nabla v_n|^2 dx + \int V(x) g(v_n)^2 dx \\ - \int h(x, g(v_n)) g(v_n) dx \bigg| \leq \epsilon_n \|v_n\|. \end{aligned} \tag{3.4}$$

Taking into account assumption (H_2) and (3.2)–(3.4) we have

$$\begin{aligned} C_0 + \delta_n + \epsilon_n \|v_n\| \geq & \frac{1}{2}\int_{\mathbb{R}^2} |\nabla v_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^2} V(x) g(v_n)^2 dx \\ & - \frac{1}{\mu}\int_{\mathbb{R}^2} \left[1 + \frac{g(v_n)^2}{1 + g(v_n)^2}\right] |\nabla v_n|^2 dx - \frac{1}{\mu}\int_{\mathbb{R}^2} V(x) g(v_n)^2 dx \\ & + \int_{\mathbb{R}^2} \left[\frac{1}{\mu} h(x, g(v_n)) g(v_n) - H(x, g(v_n))\right] dx \\ & \geq \int_{\mathbb{R}^2} \left[\frac{1}{2} - \frac{1}{\mu}\left(1 + \frac{g(v_n)^2}{1 + g(v_n)^2}\right)\right] |\nabla v_n|^2 dx \\ & + \left(\frac{1}{2} - \frac{1}{\mu}\right) \int_{\mathbb{R}^2} V(x) g(v_n)^2 dx \end{aligned}$$

Now by considering (2.4) with $k=1$ we have $\int_{\mathbb{R}^2} V(x)g(v_n)^2dx \geq \|v_n\|_G - 1$ and therefore we obtain

$$\begin{aligned} C_0+\delta_n+\epsilon_n\|v_n\| \geq & \left(\frac{1}{2}-\frac{2}{\mu}\right)\int_{\mathbb{R}^2}|\nabla v_n|^2dx+\left(\frac{1}{2}-\frac{1}{\mu}\right)\int_{\mathbb{R}^2}V(x)g(v_n)^2dx \\ \geq & \frac{(\mu-4)}{2\mu}\int_{\mathbb{R}^2}\left[|\nabla v_n|^2+V(x)g(v_n)^2\right]\,dx \\ \geq & \frac{(\mu-4)}{2\mu}\left(\int_{\mathbb{R}^2}|\nabla v_n|^2\,dx+\|v_n\|_G-1\right), \end{aligned}$$

from which together with the elementary inequality $\|\nabla v_n\|_2 \leq \|\nabla v_n\|_2^2+1$ we have

$$\begin{aligned} C_0+\delta_n+\epsilon_n\|v_n\| \geq & \frac{(\mu-4)}{2\mu}\left(\|\nabla v_n\|_2+\|v_n\|_G-2\right) \\ = & \frac{(\mu-4)}{2\mu}\left(\|v_n\|_E-2\right). \end{aligned} \tag{3.5}$$

Since $\mu > 4$ it follows from the above estimate that

$$C_0+\delta_n+\epsilon_n\|v_n\| \geq C_1\|v_n\|,$$

which implies that (v_n) is bounded in E.

Remark 3.1 *Even though the space E is not reflexive, we may assume by Proposition 2.1 that, up to a subsequence, $\nabla v_n \rightharpoonup \nabla v$ in $L^2(\mathbb{R}^N)$ and $g(v_n) \to g(v)$ in $L^p(\mathbb{R}^2)$ and then $v_n(x) \to v(x)$ almost everywhere. By a result due to Berestycki, Capuzzo-Dolcetta and Nirenberg, (see [3, Proposition 8]) we may assume that $v_n \geq 0$.*

Proposition 3.2 *Let (v_n) be a (P.-S.)$_{C_0}$ sequence, then the following statements hold.*

(a) *Assuming that h has critical growth, (H_0) and (H_3), and taking a subsequence if necessary we have $\|\nabla v_n\|_2 \leq K < 1$.*

(b) *If $v_n \geq 0$ converges weakly to v in E, then for every nonnegative test function $\phi \in E$ we have*

$$\lim_{n\to+\infty}\langle I'(v_n),\phi\rangle = \langle I'(v),\phi\rangle.$$

Proof. In order to prove part (a) we notice that from estimates in (3.5) and Lemma 3.3 we have

$$\limsup_{n\to\infty}\int_{\mathbb{R}^2}|\nabla v_n|^2dx\;dx := K \leq \frac{2\mu C_0}{\mu-4} < 1.$$

To prove (b) note first that

$$\begin{aligned}\langle I'(v_n),\phi\rangle &= \int \nabla v_n\cdot\nabla\phi\,dx \\ &\quad + \int V(x)g(v_n)g'(v_n)\phi\,dx - \int h(x,g(v_n))g'(v_n)\phi\,dx.\end{aligned} \tag{3.6}$$

Now by the result of part (a) and Trudinger-Moser inequality, there exists $q>1$ sufficiently close to one that $T_n(x) := e^{\alpha_0 g(v_n)^4}-1$ is bounded in L^q. Since, $v_n \to v$ a.e. in $\mathbb{R}^2$ so $T_n(x) \rightharpoonup T(x) = e^{\alpha_0 g(v)^2}-1$ weakly in L^q. Now for each $\phi\in E$ since $E\subseteq L^t$ for $t\geq 2$ we have

$$\int T_n(x)\phi\,dx \to \int T(x)\phi\,dx.$$

Since, g is increasing and $g(0)=0$, hence $g(v_n)\geq 0$ and $g(v)\geq 0$. Now it follows from H_0 that

$$h(x,g(v_n))g'(v_n)\phi \leq b_1 g(v_n)\phi + b_2 T_n(x)\phi.$$

Hence, the dominated convergence theorem implies

$$\int h(x,g(v_n))g'(v_n)\phi\,dx \to \int h(x,g(v))g'(v)\phi\,dx. \tag{3.7}$$

For the second term on the right hand side of (3.6), we have

$$V(x)g(v_n)g'(v_n)\phi \leq V(x)g(v_n)\phi,$$

and since $v_n \rightharpoonup v$ weakly in E, for the right hand side of the above inequality we have

$$\lim_{n\to\infty}\int V(x)g(v_n)\phi\,dx = \int V(x)g(v)\phi\,dx.$$

Hence by the dominated convergence theorem and the fact that $v_n \to v$ a.e. we obtain

$$\lim_{n\to\infty}\int V(x)g(v_n)g'(v_n)\phi\,dx = \int V(x)g(v)g'(v)\phi\,dx. \tag{3.8}$$

It follows from (3.6), (3.7) and (3.8) that

$$\lim_{n\to+\infty}\langle I'(v_n),\phi\rangle = \langle I'(v),\phi\rangle,$$

and the proof is complete.

Lemma 3.4 *Assume conditions* (H_0) *and* (H_2)*. If* (v_n) *is a (P.-S.)*$_{C_0}$ *sequence for* I *and* $v_n \rightharpoonup 0$ *then taking subsequence if necessary we have*

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbb{R}^2} h(x,g(v_n))g(v_n)\,dx &= 0 \quad \textit{and} \\ \lim_{n\to\infty}\int_{\mathbb{R}^2} H(x,g(v_n))\,dx &= 0.\end{aligned}$$

Proof. It follows from (H_0) that

$$\begin{aligned}
&\int h(x,g(v_n))g(v_n)\ dx \\
\leq & b_1 \int g(v_n)^2\ dx + b_2 \int (e^{\alpha_0 g(v_n)^4} - 1) g(v_n)\ dx \qquad (3.9)\\
\leq & b_1 \int g(v_n)^2\ dx + b_2 \Big(\int (e^{2\alpha_0 g(v_n)^4} - 1\Big)\ dx \Big)^{\frac{1}{2}} \|g(v_n)\|_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

It follows from $v_n \rightharpoonup 0$ that $\|\nabla v_n\|_{L^2(\mathbb{R}^2)}$ is bounded and $\|g(v_n)\|_{L^2(\mathbb{R}^2)} \to 0$ due to part (3) of Proposition 2.1. Notice that

$$\|\nabla g^2(v_n)\|_2 \leq 2\|g(v_n)\nabla v_n\|_2 \leq 2\|g(v_n)\|_2 \|\nabla v_n\|_2 \to 0$$

from which together with Trudinger-Moser inequality (1.3) we obtain for large n, $(e^{2\alpha_0 |g(v_n)|^4} - 1)$ is bounded in $L^1(\mathbb{R}^2)$. Therefore it follows from (3.9) that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} h(x,g(v_n))g(v_n)\ dx \quad = \quad 0,$$

and consequently from (H_2),

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} H(x,g(v_n))\ dx \quad = \quad 0$$

which proves the lemma.

3.4 Proof of Theorems 1.1 and 1.2

It follows from Lemmas 3.1 and 3.2 that the functional I has the geometry of the mountain-pass theorem. Therefore applying Theorem 3.1 we obtain a bounded Palais-Smale sequence (cf. Proposition 3.1) (v_n) in E, that is,

$$I(v_n) \to C_0 \text{ and } I'(v_n) \to 0.$$

Using Proposition 3.2 and taking a subsequence if necessary we can conclude that (v_n) converges weakly to a critical point $v \in E$ of I. Thus, it remains only to prove that v is nontrivial. Assume by contradiction that $v \equiv 0$ and take

$$\phi = \frac{g(v_n)}{g'(v_n)} = \sqrt{1 + g(v_n)^2}\, g(v_n)$$

as a test function in (3.3) and using Lemma 3.4 we conclude that

$$\lim_{n\to\infty} \int |\nabla v_n|^2 dx + \int V(x) g(v_n)^2 dx = 0.$$

On the other hand, from (3.2) and using once more Lemma 3.4 we conclude that $C_0 = 0$, which is a contradiction.

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