On a Fourth Order Elliptic Problem with a Singular Nonlinearity

Daniele* Cassani

Dipartimento di Matematica "F. Enriques",
Università degli Studi di Milano, Via C. Saldini 50, 20133 Milano (Italy)
e-mail: `cassani@mat.unimi.it`

João Marcos do Ó

Departamento de Matemática,
Universidade Federal de Paraíba, 58059.900, João Pessoa, Pb, (Brazil)
e-mail: `jmbo@math.ufpb.br`

Nassif Ghoussoub

Department of Mathematics,
University of British Columbia, 58059.900, Vancouver, BC, V6T 1Z2 (Canada)
e-mail: `nassif@math.ubc.ca`

April 24, 2008

Abstract

We study the following semilinear biharmonic equation: $\Delta^2 u = \lambda \frac{f(x)}{(1-u)^2}$, $x \in B_R$, where $0 \leq f \leq 1$ and $B_R \subset \mathbb{R}^N$, $N \geq 1$, is the ball centered in the origin of radius R. We prove, under Dirichlet boundary conditions $u = \partial u/\partial \eta = 0$ on ∂B_R, the existence of $\lambda^* = \lambda^*(R, f) > 0$ such that for $\lambda \in (0, \lambda^*)$ there exists a minimal (classical) solution $\underline{u}_\lambda$, which satisfies $0 < \underline{u}_\lambda < 1$. In the extremal case $\lambda = \lambda^*$, we prove the existence of a weak solution which has finite energy and which is the unique solution even in a very weak sense. For $\lambda > \lambda^*$ there are no solutions of any kind. Estimates on λ^*, stability properties of solutions and nonexistence results in the whole space are also established.

2000 Mathematics Subject Classification. 35G30, 35J40.

Key words. Biharmonic operator, semilinear elliptic equations, singular nonlinearity, minimal solutions, extremal solutions, nonlinear eigenvalue problems, MEMS devices.

*Corresponding author.

1 Introduction

The main purpose of this paper is to investigate a class of fourth order problems which arise in applications in the study of *Micro Electro Mechanical Systems* (MEMS), micro-devices developed since late 1960s and which nowadays enter as a key ingredient in many industrial products beyond having made inroads into many different areas of applied sciences; we refer to [23] and reference therein for a technical insight on the subject and an overview on applications. As a naive reference model, let us think of a dielectric membrane coated with a conducting film, building up a plate which is clamped at the boundary of a region $\Omega \subset \mathbb{R}^N$. Once that a drop voltage is suitably applied, the plate deflects from the steady state $u = 0$ towards a conducting plate (*ground plate*) positioned at hight $u = 1$. The deformation u of the membrane is then governed, in the stationary case, by the following model:

$$(M_\lambda) \begin{cases} \alpha\Delta^2 u = \left(\beta \displaystyle\int_\Omega |\nabla u|^2\, dx + \gamma\right) \Delta u + \dfrac{\lambda f(x)}{(1-u)^2 \left(1 + \chi \displaystyle\int_\Omega \dfrac{dx}{(1-u)^2}\right)}, & \text{in } \Omega \\ 0 < u < 1, & \text{in } \Omega \\ u = \dfrac{\partial u}{\partial \eta} = 0, & \text{on } \partial\Omega \end{cases}$$

where $\Delta^2(\,\cdot\,) := -\Delta(-\Delta\,\cdot\,)$ denotes the biharmonic operator, $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, η denotes the outward pointing unit normal to $\partial\Omega$ and $\alpha, \beta, \gamma, \chi \geq 0$ are physically relevant constants: α is proportional to the thickness of the plate, for $\beta > 0$ the presence of a non-local term represents the self-stretching contributions to the potential energy of deformation, γ is the tension related to the stretching energy whereas in the electrostatic actuation, the dependence of the capacitance on the deformation variable u does not allow one to keep the drop voltage at a given supply voltage and this yields an extra non-local term for $\chi > 0$. The parameter $\lambda \geq 0$ is proportional to the applied voltage and, as we are going to see, it plays an important role: if λ crosses an extremal value λ_*, the so-called *pull-in voltage*, then the problem has no solutions which physically corresponds to the fact that the membrane hits the ground plate and a snap-through occurs. This "touch-down" phenomenon, which is the main feature of a MEMS device, is allowed by the presence of the singular nonlinearity. The function $f : \overline{\Omega} \longrightarrow \mathbb{R}$ is the *permittivity profile* and is related to a varying dielectric property of the material; in what follows we assume that f is a continuous function satisfying: $0 \leq f(x) \leq 1$ and $f(x) > 0$ on a set of positive Lebesgue measure.

We mention that in the applications framework, from one side one is interested to prevent or push back the *pull-in instability*, on the other side, as in the case of security systems, one needs to facilitate the occurrence of the snap-through in order to improve the performance of the device; this two-fold employment of MEMS devices turns into a rich and challenging mathematical problem. However, if from the point of view of applications, many progresses have been made as well as numerical results obtained, just recently problem (M_λ) has been investigated with a rigorous mathematical approach. In the limit case of zero plate thickness, hence for a thin membrane with zero rigidity and neglecting inertial effects as well as non-local effects, that is in dimensionless constants we set $\alpha = \beta = \chi = 0$, (M_λ) reduces to the following second order semilinear elliptic problem:

$$(\widetilde{M}_\lambda) \begin{cases} -\Delta u = \lambda \dfrac{f(x)}{(1-u)^2}, & \text{in } \Omega \\ 0 < u < 1, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

where we have set for simplicity $\gamma = 1$ and where $u \in \mathcal{C}^2(\overline{\Omega})$. When $f \equiv 1$ this problem was studied in a very general context in [19] whereas the situation of a non constant permittivity profile has been investigated in a series of papers: the effects on the pull-in instability of a special tailoring of the permittivity profile has been studied in [22]; in [15] non-existence results and upper bounds for the pull-in voltage λ_* were established in terms of material and geometric properties of the membrane, results further complemented in [13], where the existence of minimal solutions for $0 < \lambda < \lambda_*$ was proved as well as the existence and uniqueness of the extremal (classical) solution for $\lambda = \lambda_*$ provided that the dimension $1 \leq N \leq 7$. In this dimensional range, the existence of non minimal solutions was obtained in [11], where the authors study the branch of semi-stable solutions and where existence results in higher dimensions, for a suitable class of functions f, were also established.

Very recently, some existence results for the non-local equation (M_λ), avoiding self-stretching effects ($\beta = 0$) and for a constant permittivity profile, have been obtained in [16], where a contraction iteration scheme is used to build up a unique small-amplitude solution for small voltage λ and which tends to zero as $\lambda \to 0$; however in [16] the authors deal mainly with *Navier boundary conditions:* $u = \Delta u = 0$ on $\partial\Omega$. Sometimes called *pinned* boundary conditions, physically this situation corresponds to a device which is ideally hinged along the boundary without experiencing any torque or bending moment. Even though too loose from the point of view of applications, the mathematical advantage of Navier boundary conditions consists of the availability of two important tools: the Maximum Principle and the Gidas-Ni-Nirenberg [14] symmetry type results, by which radial domains induce spherically symmetric and radially decreasing solutions. These key-ingredients are missing in general when *Dirichlet boundary conditions* are considered (see [25, 3]) and just recently some symmetry results are obtained in [4].

In the present paper we consider the corresponding fourth order problem of $(\widetilde{M}_\lambda)$, namely with the biharmonic operator Δ^2 in place of Δ and subject to Dirichlet boundary conditions, thus we set $\beta = \gamma = \chi = 0$ in (M_λ): in the physical model, neglecting non-local contributions, we consider the plate situation in which flexural rigidity is now allowed whose effects however dominates over the stretching tension (see [23, Sec. 7.6]). In order to exploit the positivity of the Green function for the biharmonic operator, we restrict ourself to the ball which will enable us to use a positivity preserving property due to T. Boggio [5, 1905]. We set

$B_R := \{x \in \mathbb{R}^N \,:\, |x| < R\}$ and we study the following semilinear problem:

$$(P_\lambda) \begin{cases} \Delta^2 u = \lambda \dfrac{f(x)}{(1-u)^2}, & \text{in } B_R \\ 0 < u < 1, & \text{in } B_R \\ u = \dfrac{\partial u}{\partial \eta} = 0, & \text{on } \partial B_R \end{cases}$$

where $\lambda \geq 0$ and notice that the nonlinearity is given by a singular function. We mention that closely related problems for smooth nonlinearities and in particular the fourth order Gelfand problem with exponential nonlinearty, have been investigated in [2]. It is worth to point out that problem (M_λ) in its whole generality as stated remains essentially open, however some identification problem for the dynamic version of (M_λ) has been studied in [8]. Before stating our main results, we give some basic definitions and precise the class of solutions we are going to consider.

1.1 Preliminaries

Besides classical solutions i.e. $u_\lambda \in \mathcal{C}^4(\overline{B_R})$ which satisfy (P_λ), let us introduce the class of weak solutions we will be dealing with. We denote by $H_0^2(B_R)$ the usual Sobolev space which can be defined by completion as follows:

$$H_0^2(B_R) := cl\,\{u \in \mathcal{C}_c^\infty(B_R) \,:\, \|\Delta u\|_2 < \infty\}$$

and which is an Hilbert space endowed with the scalar product

$$(u, v)_{H_0^2(B_R)} := \int_{B_R} \Delta u \Delta v \, dx$$

Definition 1.1 *We say that* $u_\lambda \in L^1(B_R)$ *is a weak solution of* (P_λ) *provided* $0 \leq u_\lambda \leq 1$ *almost everywhere,* $1/(1-u_\lambda)^2 \in L^1(B_R)$ *and*

$$\int_{B_R} u_\lambda \Delta^2 \varphi \, dx = \lambda \int_{B_R} \frac{f(x)}{(1-u_\lambda)^2} \varphi \, dx, \quad \forall\, \varphi \in \mathcal{C}^4(\overline{B_R}) \cap H_0^2(B_R) \tag{1.1}$$

When in (1.1) *the equality is replaced by the inequality* $\geq$ *and* $\varphi \geq 0$*, we say that* u_λ *is a weak super-solution of* (P_λ) *provided the following boundary conditions are satisfied:* $u_\lambda = 0$ *and* $\partial u_\lambda / \partial \eta \leq 0$ *on* ∂B_R*. (Indeed, the second boundary condition turns out to be automatically satisfied, integrating by parts, in the case of a smooth super-solution.)*

Definition 1.2 *If* u_λ *is a solution of* (P_λ) *such that for any other solution* v_λ *of* (P_λ) *one has*

$$u_\lambda(x) \leq v_\lambda(x), \quad a.e. \quad x \in B_R$$

we say that u_λ *is a minimal solution of* (P_λ)*, which we denote in the sequel by* $\underline{u}_\lambda$*.*

If u_λ is a classical solution of (P_λ), then it turns out to be well defined the linearized operator at u_λ

$$L_{u_\lambda} := \Delta^2 - \frac{2\lambda f(x)}{(1-u_\lambda)^3}$$

which yields the following notion of stability

Definition 1.3 *A classical solution u_λ of (P_λ) is semi-stable provided*

$$\mu_1(u_\lambda) := \inf\left\{\langle L_{u_\lambda}\varphi, \varphi\rangle \,:\, \varphi \in \mathcal{C}^4(\overline{B_R}) \cap H_0^2(B_R),\, \|\varphi\|_2 = 1\right\} \geq 0$$

If $\mu_1(u_\lambda) > 0$ we say that u_λ is stable.

(Analogously one defines (semi-) unstable solutions by reversing the above inequalities).

As far as we are concerned with weak solutions, the linearized operator is no longer well defined, however we introduce the following weaker notion of stability

Definition 1.4 *A weak solution u_λ to (P_λ) is said to be weakly stable if $1/(1-u_\lambda)^3 \in L^1(B_R)$ and the following holds:*

$$\int_{B_R} |\Delta\varphi|^2\, dx \geq \int_{B_R} \frac{2\lambda f(x)}{(1-u_\lambda)^3}\varphi^2, \quad \varphi \in \mathcal{C}^4(\overline{B_R}) \cap H_0^2(B_R), \quad \varphi \geq 0$$

Accordingly to the class of solutions which we consider, let us introduce the following values:

$$\begin{aligned} \lambda^* &:= \sup\{\lambda \geq 0 \,:\, (P_\lambda) \quad \text{posses a weak solution}\} \\ \lambda_* &:= \sup\{\lambda \geq 0 \,:\, (P_\lambda) \quad \text{posses a classical solution}\} \end{aligned} \tag{1.2}$$

Remark 1.1 *Clearly, a classical solution is also a weak solution, so that one has $\lambda_* \leq \lambda^*$.*

Let us denote by ν_R the first eigenvalue of the biharmonic operator on B_R with Dirichlet boundary conditions, which is characterized variationally as follows:

$$\nu_R := \min\left\{\int_{B_R} |\Delta u|^2\, dx \,:\, u \in H_0^2(B_R),\, \|u\|_2 = 1\right\}$$

It is well known that $\nu_R > 0$, that it is simple, isolated and that the corresponding eigenfunction $\psi_R > 0$, spherically symmetric and radially decreasing.

1.2 Main results

Theorem 1.1 *There exists $\lambda_* = \lambda_*(R, f) > 0$ such that for $0 < \lambda < \lambda_*$, problem (P_λ) posses a minimal classical solution $\underline{u}_\lambda$ which is positive and stable. Moreover, λ_* satisfies the following bounds:*

$$\frac{\max\left\{\frac{32(10N-N^2-12)}{27}, \frac{128-240N+72N^2}{81}\right\}}{R^4 \sup\limits_{x\in B_R} f(x)} \leq \lambda_* \leq \min\left\{\frac{4\nu_R}{27 \inf\limits_{x\in B_R} f(x)}, \frac{\nu_R\|\psi_R\|_1}{\int_{B_R}\psi_R f(x)\, dx}\right\}$$

If in addition, $f(x)$ is radially symmetric, then the minimal solution is radially symmetric and radially decreasing. In particular, for $f(x) = |x|^\alpha$, $x \in B_1$ and $N \geq 3$, we have also the following lower bound

$$\lambda_* \geq \max_{\alpha > 0} \frac{(4+\alpha)(2-\alpha)[(1+\alpha)(\alpha-5)+6(N-1)\alpha+3(N-1)(3N-7)]}{81}$$

Theorem 1.2 *The following holds:*

$$\boxed{\lambda_* = \lambda^*}$$

In particular, for $\lambda > \lambda^*$ there are no solutions, even in the weak sense. Furthermore, for almost every $x \in B_R$, there exists

$$u^*(x) := \lim_{\lambda \nearrow \lambda^*} \underline{u}_\lambda(x)$$

and u^* is a weak solution of (P_{λ^*}) such that $u^* \in H_0^2(B_R)$.

Theorem 1.3 *Let v be a weak super-solution of (P_{λ^*}). Then $v = u^*$; in particular (P_{λ^*}) has a unique weak solution (which is called the extremal solution).*

As a consequence of Theorem 1.2 and Theorem 1.3, we obtain in the spirit of [7, Brezis-Vásquez] the following characterization of possibly *"singular"* solutions:

Corollary 1.1 *Let $u_\lambda \in H_0^2(B_R)$ be a weak solution of (P_λ) such that $\|u_\lambda\|_\infty = 1$. Then u_λ is weakly stable if and only if $\lambda = \lambda^*$ and $u_\lambda = u^*$.*

As complementary to the existence results in the ball, we conclude this paper by proving a general non-existence result in the whole space, precisely we have the following

Theorem 1.4 *Let $\lambda > 0$ and $g(x) \geq \lambda |x|^\gamma$, $\gamma \geq 0$. Then the following problem*

$$\begin{cases} \Delta^2 u \geq \dfrac{g(x)}{(1-u)^2}, & \\ & \text{in} \quad \mathbb{R}^N \\ 0 < u < 1, & \end{cases} \tag{1.3}$$

has no solutions (even in the weak sense) for any dimension $N \geq 1$.

1.3 Key-ingredients

We are going to make an extensive use of the following weak version of a positivity preserving property, due to Boggio [5], for the biharmonic operator subject to Dirichlet boundary conditions, see [2, Lemma 16]:

Lemma 1.1 (Boggio's Principle) *Let $u \in L^1(B_R)$ such that*

$$\int_{B_R} u \Delta^2 \varphi \, dx \geq 0, \quad \varphi \in \mathcal{C}^4(\overline{B_R}) \cap H_0^2(B_R),\ \varphi \geq 0$$

Then $u \geq 0$ a.e. in B_R. Moreover, $u \equiv 0$ or $u > 0$ a.e. in B_R.

As a consequence, any solution of (P_λ) is necessarily positive a.e. inside the ball. It is well known (see [3]) that the positivity preserving property fails for general domains and this is the main reason for us to setting up the problem in the ball. The following existence result has been proved in [2, Lemma 17]:

Lemma 1.2 *Let $g \in L^1(B_R)$, $g \geq 0$ almost everywhere. Then there exists a unique $u \in L^1(B_R)$ such that $u \geq 0$ and*

$$\int_{B_R} u\Delta^2\varphi\,dx = \int_{B_R} g\varphi\,dx, \quad \varphi \in \mathcal{C}^4(\overline{B_R}) \cap H_0^2(B_R)$$

Moreover, there exists $C > 0$ which does not depend on g such that $\|u\|_1 \leq C\|g\|_1$.

Proposition 1.1 *Let $\lambda > 0$ and assume that there exists a weak super-solution U_λ of (P_λ). Then there exists a weak solution u_λ of (P_λ) such that $0 \leq u_\lambda \leq U_\lambda$.*

Proof. By means of a standard monotone iteration argument, set $u_0 := U_\lambda$ and define recursively $u_{n+1} \in L^1(B_R)$ as the unique solution of

$$\int_{B_R} u_{n+1}\Delta^2\varphi\,dx = \lambda\int_{B_R} \frac{f(x)\varphi}{(1-u_n)^2}\,dx, \quad \varphi \in \mathcal{C}^4(\overline{B_R}) \cap H_0^2(B_R)$$

then we have

$$\int_{B_R} (u_n - u_{n+1})\Delta^2\varphi\,dx \geq 0, \quad \varphi \in \mathcal{C}^4(\overline{B_R}) \cap H_0^2(B_R) \tag{1.4}$$

and Lemma 1.1 yields $0 \leq u_{n+1}(x) \leq u_n(x)$ a.e. for all $n \in \mathbb{N}$ and the claim follows from the Beppo-Levi monotone convergence theorem.

Remark 1.2 *Notice that by a standard approximation procedure one easily shows that $\mathcal{C}^4(\overline{B_R}) \cap H_0^2(B_R)$ is dense into $H^4(B_R) \cap H_0^2(B_R)$; see e.g. [2, Lemma 16]. Moreover, by standard elliptic regularity theory for the biharmonic operator [1] (see also [17]), any weak solution of (P_λ) which satisfies $\|u_\lambda\|_\infty < 1$ turns out to be smooth.*

We complete these preliminary results by proving a key lemma which provides a comparison principle:

Lemma 1.3 *Let $u, U \in H_0^2(B_R)$ such that in the weak sense: u is a (weakly) stable sub-solution of (P_λ) and U is super-solution of (P_λ). Then, $u(x) \leq U(x)$ almost everywhere in B_R. Moreover, if $0 < u < 1$ is a solution of (P_λ) such that $\mu_1(u) = 0$ and U is any classical super-solution of (P_λ), then $u \equiv U$.*

Proof. Set $v := u - U$. By means of the Moreau decomposition [21] for the biharmonic operator [12], there exist $v_1, v_2 \in H_0^2(B_R)$ such that $v = v_1 + v_2$ with $v_1 \geq 0$ and $v_1 \perp v_2$, $\Delta^2 v_2 \leq 0$ in the $H_0^2(B_R)$ sense; in particular Boggio's principle yields $v_2 \leq 0$ from which $v_1 \geq v$. For $\varphi \in \mathcal{C}_c^\infty(B_R)$, $\varphi \geq 0$ we have

$$\int_{B_R} \Delta(u-U)\Delta\varphi\,dx \leq \lambda\int_{B_R} f(x)\left[\frac{1}{(1-u)^2} - \frac{1}{(1-U)^2}\right]\varphi\,dx \tag{1.5}$$

By a standard density argument we may test (1.5) with v_1 and then, the stability assumption together with the orthogonality condition $(v_1, v_2)_{H_0^2(B_R)} = 0$ yield

$$2\lambda \int_{B_R} \frac{f(x)}{(1-u)^3} v_1^2 \, dx \leq \int_{B_R} |\Delta v_1|^2 \, dx = \int_{B_R} \Delta(u-U)\Delta v_1 \, dx$$
$$\leq \lambda \int_{B_R} f(x) \left[\frac{1}{(1-u)^2} - \frac{1}{(1-U)^2} \right] v_1 \, dx$$

Since $v_1 \geq v$, we obtain

$$0 \leq \int_{B_R} f(x) \left[\frac{1}{(1-u)^2} - \frac{1}{(1-U)^2} - 2\frac{(u-U)}{(1-u)^3} \right] v_1 \, dx \tag{1.6}$$

Now observe that, by convexity of the map $s \mapsto 1/(1-s)^2$, the factor between brackets is non positive, thus necessarily

$$0 = v_1 \geq v = u - U, \quad \text{a.e.} \quad x \in B_R$$

and the first part of the lemma follows. Next consider the map $Q : [0,1] \to \mathbb{R}$ given by

$$Q(\tau) := \Delta^2[u + \tau(U-u)] - \frac{\lambda f(x)}{\{1 - [u + \tau(U-u)]\}^2}$$

Notice that $Q(0) = 0$ and if $\mu_1(u) = 0$ also $Q'(0) = 0$ moreover, by convexity of the map $s \mapsto 1/(1-s)^2$ we have $Q(\tau) \geq 0$ for all $\tau \in [0,1]$. Thus necessarily $Q''(0) \geq 0$, that is

$$-6\lambda f(x) \frac{(U-u)^2}{(1-u)^4} \geq 0$$

and hence $U \equiv u$.

Remark 1.3 *From the first part of the previous lemma we deduce in particular that weakly stable solutions, belonging to a suitable energy class are necessarily minimal.*

2 Existence results: proof of Theorems 1.1 and 1.2

2.1 The pull-in voltage

Let us define

$$\Lambda := \{\lambda \geq 0 \, : \, (P_\lambda) \text{ has a classical solution}\}$$

Lemma 2.1 Λ *is a bounded interval. Moreover,*

$$\sup \Lambda =: \lambda_* \leq \min \left\{ \frac{4\nu_R}{27 \inf\limits_{x \in B_R} f(x)}, \frac{\nu_R \|\psi_R\|_1}{\int_{B_R} \psi_R f(x) \, dx} \right\}$$

Proof. First we show that Λ does not consist of just $\lambda = 0$. To this end, let Ψ_{R_2} be the first eigenfunction of the biharmonic operator subject to Dirichlet boundary conditions on $B_{R_2} \supset B_R$ which we normalize by $\sup_{B_{R_2}} \psi_{R_2} = 1$ and let $\nu_{R_2} > 0$ be the corresponding eigenvalue. Next, we are going to prove that for $\theta \in (0,1)$ the function $\psi := \theta\psi_{R_2}$ is a super-solution of (P_λ) as long as λ is sufficiently small. We have

$$0 < 1 - \theta\psi_{R_2} < 1, \quad \text{in} \quad B_R$$

and moreover,

$$\Delta^2\psi = \nu_{R_2}\theta\psi_{R_2} \geq \frac{\lambda f(x)}{(1-\theta\psi_{R_2})^2} = \frac{\lambda f(x)}{(1-\psi)^2}, \quad \text{in } B_R$$

provided that

$$\nu_{R_2}\theta\psi_{R_2}(1-\theta\psi_{R_2})^2 \geq \lambda f(x)$$

Notice that

$$0 < s_1 := \inf_{x\in B_R} \psi < s_2 := \sup_{x\in B_R} \psi < 1$$

and that $\partial\psi/\partial\eta < 0$ on ∂B_R. Thus, looking at the function $g(s) := s(1-s)^2$, for $s \in [s_1, s_2]$, it is readily seen that we can choose $\lambda > 0$ sufficiently small such that

$$\nu_{R_2} \inf_{x\in B_R} g(\theta\psi_{R_2}(x)) > \lambda \sup_{x\in B_R} f(x) \tag{2.7}$$

Since $\underline{u} \equiv 0$ is a sub-solution of (P_λ), the classical sub-super solution theorem provides a (classical) solution to (P_λ). We conclude with a straightforward upper bound for λ_*. Indeed, let u_λ be a solution of (P_λ) then,

$$\begin{aligned}
\nu_R \geq \nu_R \int_{B_R} u_\lambda \frac{\psi_R}{\|\psi_R\|_1}\, dx &= \int_{B_R} u_\lambda \Delta^2 \left(\frac{\psi_R}{\|\psi_R\|_1}\right) dx \\
&= \int_{B_R} \left(\Delta^2 u_\lambda\right) \frac{\psi_R}{\|\psi_R\|_1}\, dx \\
&= \lambda \int_{B_R} \frac{f(x)\psi_{1,R}}{\|\psi_R\|_1(1-u_\lambda)^2}\, dx \\
&\geq \lambda \int_{B_R} f(x)\frac{\psi_R}{\|\psi_R\|_1}\, dx
\end{aligned} \tag{2.8}$$

and this implies

$$\sup\Lambda = \lambda_* \leq \frac{\nu_R\|\psi_R\|_1}{\int_{B_R} f(x)\psi_R\, dx} < \infty \tag{2.9}$$

The upper bound (2.9) can be improved in the case: $\inf_{B_R} f > 0$. Indeed, resuming resuming calculations in (2.8), notice that

$$\int_{B_R} \left(-\nu_R u_\lambda + \frac{\lambda f(x)}{(1-u_\lambda)^2}\right) \psi_R\, dx = 0$$

can not occur if the following holds

$$\lambda > \frac{\nu_R}{\inf_{B_R} f(x)} u_\lambda (1 - u_\lambda)^2$$

and clearly this is the case when

$$\lambda > \frac{\nu_R}{\inf_{B_R} f(x)} \max_{s \in [0,1]} s(1-s)^2 = \frac{4}{27} \frac{\nu_R}{\inf_{B_R} f(x)}$$

Remark 2.1 *Notice that as a consequence of the argument in the first part of the above proof, we have from* (2.7) *the following "max-min" estimate from below:*

$$\lambda_* \geq \frac{1}{\sup\limits_{x \in B_R} f(x)} \sup_{\substack{0<\theta<1 \\ R_2 > R}} \left[\nu_{R_2} \inf_{x \in B_R} g(\theta \psi_{R_2}(x)) \right]$$

and one may proceed exactly as in [13] to which we refer, to get a slight more explicit estimate of this lower bound. Finally, by means of Lemma 1.1, if $\mu \leq \lambda$ one has that solutions of (P_λ) provide super-solution to (P_μ) and thus Λ is an interval.

Remark 2.2 *Clearly, by the very definition of λ_*, there are no classical solutions of (P_λ) for $\lambda > \lambda_*$. However one may wonder about the existence of weak solutions; see Section 2.2.*

In applications, upper bounds for the pull-in voltage are important in order to shorten the lapse of time to get the snap-through, improving the efficiency of the MEMS device as in the case of security systems. However, in many situations one is interested in preventing the so called *pull-in instability* which in turn can be achieved by rising the pull-in voltage λ_*; this conveys in obtaining (preferably computationally accessible) lower bounds.

Lemma 2.2 *The following lower bound holds:*

$$\lambda_* \geq \frac{32}{27} \frac{(10N - N^2 - 12)}{R^4 \|f\|_\infty}$$

Proof. Consider the function

$$w_\alpha(x) := \alpha \left(1 - \frac{|x|^4}{R^4}\right), \quad \alpha \in (0,1)$$

wich satisfies $0 \leq w_\alpha(x) < 1$ for $x \in B_R$ and $w_\alpha(x) = 0$, $\partial w_\alpha / \partial \eta \leq 0$, for $x \in \partial B_R$; for all $\alpha \in (0,1)$. Now the idea is to obtain from $w_\alpha(x)$ a super-solution of (P_λ), for a suitable choice of α and for λ in a suitable range of the form $0 < \lambda \leq \widetilde{\lambda}(N, R, \|f\|_\infty)$. We have

$$\begin{aligned} \Delta^2 w_\alpha(r) &= \frac{d^4 w_\alpha}{dr^4} + \frac{2(N-1)}{r} \frac{d^3 w_\alpha}{dr^3} + \frac{(N-1)(N-3)}{r^2} \frac{d^2 w_\alpha}{dr^2} - \frac{(N-1)(N-3)}{r^3} \frac{d w_\alpha}{dr} \\ &= [-24 + 48(N-1) - 8(N-1)(N-3)] \frac{\alpha}{R^4} =: C(N) \frac{\alpha}{R^4} \end{aligned}$$

and thus

$$\Delta^2 w_\alpha(x) = \frac{C(N)\alpha}{R^4}(1-\alpha)^2 \frac{1}{(1-\alpha)^2} \geq \frac{C(N)\alpha(1-\alpha)^2}{R^4\|f\|_\infty} \frac{f(x)}{\left[1-\alpha\left(1-\frac{|x|^4}{R^4}\right)\right]^2}$$
$$= \frac{C(N)\alpha(1-\alpha)^2}{R^4\|f\|_\infty}\frac{f(x)}{(1-w_\alpha)^2}$$

from which we deduce that

$$\lambda_* \geq \sup_{\alpha\in(0,1)} \frac{C(N)\alpha(1-\alpha)^2}{R^4\|f\|_\infty} = \frac{4}{27}\frac{C(N)}{R^4\|f\|_\infty}$$

and the result follows by evaluating $C(N) = 80N - 8N^2 - 96$.

2.2 The branch of minimal solutions

Proposition 2.1 *For all $0 \leq \lambda < \lambda_*$, there exists a minimal classical solution $\underline{u}_\lambda$ of (P_λ) which is smooth and stable. Moreover,*

(i) The map $\lambda \mapsto \underline{u}_\lambda$, for $\lambda \in (0, \lambda_*)$ is differentiable and strictly increasing.

(ii) The map $\lambda \mapsto \mu_1(\underline{u}_\lambda)$ is decreasing on $(0, \lambda_*)$.

Proof. As a consequence of Lemma 2.1 there exists a classical solution u_λ of (P_λ) which we are going to exploit as a super solution while $u_0 = 0$ is a sub solution. Let us define recursively a sequence $\{u_{n,\lambda}\}_{n=0}^\infty$ as follows:

$$\begin{cases} \Delta^2 u_{n,\lambda} = \lambda \dfrac{f(x)}{(1-u_{n-1,\lambda})^2}, & \text{in } B_R \\ u_{n,\lambda} = \dfrac{\partial u_{n,\lambda}}{\partial \eta} = 0, & \text{on } \partial B_R \end{cases}$$

which is well defined since $u_0 \leq u_\lambda < 1$ and if we assume $u_{n-1,\lambda} \leq u_\lambda$ then

$$\begin{cases} \Delta^2(u_\lambda - u_{n,\lambda}) = \lambda f(x)\left[\dfrac{1}{(1-u_\lambda)^2} - \dfrac{1}{(1-u_{n-1,\lambda})^2}\right] \geq 0, & \text{in } B_R \\ (u_\lambda - u_{n,\lambda}) = \dfrac{\partial(u_\lambda - u_{n,\lambda})}{\partial\eta} = 0, & \text{on } \partial B_R \end{cases}$$

and the Boggio principle yields $0 \leq u_{n,\lambda} \leq u_\lambda < 1$ for all $n \in \mathbb{N}$; similarly one proves that $u_{n-1,\lambda} \leq u_{n,\lambda}$ for all $n \in \mathbb{N}$. Therefore, the sequence $\{u_{n,\lambda}\}_{n=0}^\infty$ is monotone increasing and the minimal solution $\underline{u}_\lambda$ is obtained as the increasing limit

$$\underline{u}_\lambda(x) := \lim_{n\to\infty} u_{n,\lambda}(x)$$

Again from the Boggio positivity preserving property (Lemma 1.1) we obtain $0 \leq \underline{u}_\lambda < 1$; in particular, from standard elliptic regularity theory for the biharmonic operator [1] (see also [17]) follows that $\underline{u}_\lambda$ is smooth. In order to prove stability, let us argue as follows: set

$$\lambda_{**} := \sup\left\{\lambda \in (0, \lambda_*) \,:\, \mu_1\left(\underline{u}_\lambda\right) > 0\right\}$$

clearly $\lambda_{**} \leq \lambda_*$. Now suppose by contradiction that $\lambda_{**} < \lambda_*$ and let $\varepsilon > 0$ sufficiently small such that $\lambda_{**} + \varepsilon < \lambda_*$ and v_ε be the corresponding minimal solution. By definition and left continuity of the map $\lambda \mapsto \mu_1(\underline{u}_\lambda)$ we have necessarily $\mu_1(\underline{u}_{\lambda_{**}}) = 0$. Since v_ε is a super-solution of $(P_{\lambda_{**}})$ by Lemma 1.3 we get $v_\varepsilon = \underline{u}_{\lambda_{**}}$ and thus $\varepsilon = 0$; a contradiction. Next consider the map $F : (0, \lambda_*) \times C^4(\overline{B}_R) \to \mathbb{R}$ given by

$$F(\lambda, v) = \Delta^2 v - \frac{\lambda f(x)}{(1-v)^2}$$

Since $\underline{u}_\lambda$ is stable, we have

$$\frac{\partial F}{\partial v}(\lambda, \underline{u}_\lambda) = \Delta^2 \underline{u}_\lambda - \frac{2\lambda f(x)\underline{u}_\lambda}{(1-\underline{u}_\lambda)^3} > 0$$

and statement (i) follows by means of the Implicit Function Theorem. The last claim of the lemma easily follows from the variational characterization

$$\mu_1(\underline{u}_\lambda) = \inf_{w \in H_0^2(B_R)\setminus\{0\}} \frac{\int_{B_R}\left[(\Delta w)^2 - 2\lambda f(x)(1-\underline{u}_\lambda)^{-3} w^2\right]\, dx}{\int_{B_R} w^2\, dx}. \tag{2.10}$$

together with the monotonicity of the map $s \mapsto 1/(1-s)^3$, $s \in (0,1)$ and part (i).

2.3 Weak solutions versus classical solutions

Lemma 2.3 *Let u_μ be a weak solution of (P_μ) with $\mu < \lambda^*$. Then, for $\varepsilon > 0$ sufficiently small, the problem $(P_{(1-\varepsilon)\mu})$ posses a classical solution.*

Proof. Let $\widetilde{u} \in L^1(B_R)$ be the unique solution of

$$\int_{B_R} \widetilde{u}\Delta^2\varphi\, dx = (1-\varepsilon)\mu \int_B \frac{f(x)}{(1-u_\mu)^2}\varphi\, dx, \quad \varphi \in \mathcal{C}^4(\overline{B_R}) \cap H_0^2(B_R) \tag{2.11}$$

provided by Lemma 1.2. By hypothesis we have

$$\int_{B_R} u_\mu \Delta^2\varphi\, dx = \mu \int_{B_R} \frac{f(x)}{(1-u_\mu)^2}\varphi\, dx, \quad \varphi \in \mathcal{C}^4(\overline{B_R}) \cap H_0^2(B_R) \tag{2.12}$$

By uniqueness we get

$$(1-\varepsilon)u_\mu = \widetilde{u}$$

whereas Lemma 1.1 yields $\widetilde{u} > 0$ a.e. in B_R and hence we may assume

$$u_\mu > \widetilde{u}, \quad x \in B_R \setminus \{x \in B_R \,:\, \widetilde{u} = 0\}$$

Therefore,

$$\int_{B_R} \widetilde{u}\Delta^2\varphi\,dx = (1-\varepsilon)\mu\int_{B_R}\frac{f(x)}{\left(1-\frac{1}{1-\varepsilon}\widetilde{u}\right)^2}\varphi\,dx$$
$$\geq (1-\varepsilon)\mu\int_{B_R}\frac{f(x)}{(1-\widetilde{u})^2}\,dx,\quad \varphi\in\mathcal{C}^4(\overline{B_R})\cap H_0^2(B_R)$$

thus $\widetilde{u}$ is a weak super-solution of $(P_{(1-\varepsilon)\mu})$ and Proposition 1.1 yields a weak solution v of $(P_{(1-\varepsilon)\mu})$ which satisfies

$$0\leq v\leq \widetilde{u}$$

and then classical by Remark 1.2, since $\widetilde{u} < u_\mu \leq 1$.

Proposition 2.2 *Up to a subsequence, the convergence*

$$u^*(x) := \lim_{\lambda\nearrow\lambda_*} u_\lambda(x)$$

holds in $H_0^2(B_R)$ and the extremal solution u^* satisfies

$$\int_{B_R}\Delta u^*\Delta\varphi\,dx = \lambda_*\int_{B_R}\frac{f(x)\varphi}{(1-u^*)^2}\,dx,\quad \varphi\in\mathcal{C}_c^\infty(B_R) \tag{2.13}$$

In particular, u^* is a weak solution of (P_{λ_*}). Furthermore, the extremal solution is weakly stable and if $\|u^*\|_\infty < 1$ then $\mu_1(u^*) = 0$.

Proof. Since $\underline{u}_\lambda$ is stable, we have

$$2\lambda\int_B\frac{f(x)\underline{u}_\lambda^2}{(1-\underline{u}_\lambda)^3}\,dx \leq \int_B|\Delta\underline{u}_\lambda|^2\,dx = \int_B\underline{u}_\lambda\Delta^2\underline{u}_\lambda\,dx = \lambda\int_B\frac{f(x)\underline{u}_\lambda}{(1-\underline{u}_\lambda)^2}\,dx \tag{2.14}$$

Next, it is easy to check that the following elementary inequality holds: there exist $C>0$ such that

$$(1+C)\frac{s}{(1-s)^2}\leq\frac{s^2}{(1-s)^3}+(1+C),\quad s\in(0,1) \tag{2.15}$$

which used in (2.14) yields

$$\lambda\int_B\frac{f(x)\underline{u}_\lambda}{(1-\underline{u}_\lambda)^2}\,dx \geq 2\lambda\int_B\frac{f(x)\underline{u}_\lambda^2}{(1-\underline{u}_\lambda)^3}\,dx$$
$$\geq 2\lambda(1+C)\int_B\frac{f(x)\underline{u}_\lambda}{(1-\underline{u}_\lambda)^2}\,dx - 2\lambda(1+C)\int_B f(x)\,dx$$

from which we get

$$\|\Delta\underline{u}_\lambda\|_2^2 = \lambda\int_B\frac{f(x)\underline{u}_\lambda}{(1-\underline{u}_\lambda)^2}\,dx \leq 2\lambda_*^2\frac{(1+C)}{C}\|f\|_1$$

Therefore, we may assume $\underline{u}_\lambda \rightharpoonup u^*$ in $H_0^2(B_R)$ and by monotone convergence (2.13) holds after integration by parts. Clearly, (2.13) implies that u^* is a weak solution in terms of Definition 1.1. Since $\mu_1(\underline{u}_\lambda) > 0$ for all $\lambda \in (0, \lambda^*)$, in particular we have

$$\int_{B_R} |\Delta\varphi|^2 \, dx \geq \int_{B_R} \frac{2\lambda f(x)}{(1-\underline{u}_\lambda)^3}\varphi^2, \quad \varphi \in \mathcal{C}_c^\infty(B_R)$$

and passing to the limit as $\lambda \nearrow \lambda^*$ we obtain that u^* is weakly stable. Finally, if $\|u^*\|_\infty < 1$ and hence u^* is a classical solution of (P_{λ_*}), the linearized operator at u^*

$$L(\lambda^*, u^*) := \Delta^2 - \frac{2\lambda^* f(x)}{(1-u^*)^3}$$

is well defined on the space $\mathcal{C}^{4,\alpha}(B_R) \times \mathbb{R}^+$. If $\mu_1(u^*) > 0$ then the implicit function theorem applied to the function

$$F(\lambda, u) := \Delta^2 u - \lambda \frac{f(x)}{(1-u)^2}$$

would yield a solution for $\lambda > \lambda_*$ contradicting the definition of λ_*; thus $\mu_1(u^*) = 0$.

Remark 2.3 *Notice that from* (2.13) *the extremal solution solves problem* (P_{λ_*}) *in a stronger sense with respect to Definition 1.1 and one may think to use* (2.13) *as a suitable definition of weak-solution. Actually this benefit comes from the fact that we are approaching* λ_* *following the branch of smooth stable solutions. However, in general one can not expect this extra regularity to hold for possibly singular solutions, that is* $\|u_\lambda\|_\infty = 1$ *and* $1/(1-u_\lambda)^2 \in L^1(B_R)$*, which do not belong to a suitable energy class; compare with [18, 6]. In this case one has to resort to the weaker notion of solution as in Definition 1.1 in order to provide an existence tool as in Proposition 1.1 (see also [7]). (We mention that this delicate key point, which may prevent to use standard arguments, affects somehow [10] where the stronger notion of weak solution is adopted).*

Proof. [Proof of Theorem 1.2] We already have $\lambda_* \leq \lambda^*$. Suppose $\lambda_* < \lambda^*$, then Lemma 2.3 yields a classical solution of $(P_{\lambda_*+\varepsilon})$, $\varepsilon > 0$, contradicting the definition of λ_*. The second part of Theorem 1.2 follows from Proposition 2.2. *Proof.* [Proof of Theorem 1.1] Notice that for $N \geq 3$ the function

$$\widetilde{u}(x) = 1 - \left(\frac{|x|}{R}\right)^{\frac{4}{3}}$$

enjoys

$$\frac{1}{(1-\widetilde{u})^2} \in L^1(B_R)$$

and satisfies in the weak sense

$$\Delta^2 \widetilde{u} = \frac{c(N)}{R^4} \frac{1}{(1-\widetilde{u})^2}, \qquad c(N) = \frac{72N^2 - 240N + 128}{81}$$

Therefore, $\widetilde{u}$ turns out to be a weak super-solution of (P_λ) provided

$$\lambda f(x) \le \frac{c(N)}{R^4}$$

thus necessarily

$$\lambda_* = \lambda^* \ge \frac{c(N)}{R^4 \|f\|_\infty} \tag{2.16}$$

If f is radially symmetric then the iteration scheme used in the proof of Proposition 2.1 yields a radially symmetric minimal solution which is also strictly radially decreasing in view of [24]. In the case of a power-like permittivity profile $f(x) = |x|^\alpha$, $\alpha > 0$, we can improve the lower bound for λ_* at least in dimension $N \ge 3$ by considering functions

$$u_\alpha(x) = 1 - |x|^{\frac{4+\alpha}{3}}, \quad x \in B_1$$

which satisfy in the weak sense the following equation

$$\Delta^2 u_\alpha = \lambda(N, \alpha) \frac{|x|^\alpha}{(1 - u_\alpha)^2}, \quad x \in B_1 \tag{2.17}$$

where

$$\lambda(N, \alpha) = \frac{1}{81}(4 + \alpha)(2 - \alpha)[(1 + \alpha)(\alpha - 5) + 6(N - 1)\alpha + 3(N - 1)(3N - 7)]$$

Hence, we necessarily have

$$\lambda_* \ge \max_{\alpha > 0} \lambda(N, \alpha)$$

The proof of Theorem 1.1 then follows from Lemma 2.1, Proposition 2.1 and Lemma 2.2.

3 Uniqueness in the extremal case $\lambda = \lambda^*$: proof of Theorem 1.3

Let us consider a perturbation of problem (P_{λ^*}) as follows:

$$(\widetilde{P}_{\lambda^*}) \begin{cases} \Delta^2 u = \lambda^* \dfrac{f(x)}{(1-u)^2} + \mu_0 \dfrac{\xi(x) f(x)}{(1-u)^2}, & \text{in } B_R \\ 0 \le u \le 1, & \text{in } B_R \\ u = \dfrac{\partial u}{\partial \eta} = 0, & \text{on } \partial B_R \end{cases}$$

for a standard cut-off function $\xi \in \mathcal{C}_c^\infty(B_R)$ and $\mu_0 > 0$ to be suitably chosen; clearly a solution is understood in the weak sense. Let v be as in Theorem 1.3 a super-solution of problem (P_{λ^*}); assuming by contradiction that $v \not\equiv u^*$ we are going to prove the existence of a super-solution for the perturbed problem $(\widetilde{P}_{\lambda^*})$ and this will enable us to build up a weak solution of (P_λ) for $\lambda > \lambda^*$ and thus necessarily

$v \equiv u^*$. We mention that a similar idea was introduced in [18] in the second order case for regular nonlinearities, afterwards extended in [10] to the fourth order Gelfand problem; however here we propose a different technique. *Proof.* [Proof of Theorem 1.3] Notice that the construction of minimal solutions in Proposition 2.1, for $\lambda \in (0, \lambda^*)$ carries over to $\lambda = \lambda^*$ but just in the weak sense; precisely, we may assume that for $\lambda = \lambda^*$ there exists a minimal weak solution. Next we are going to show that any weak super-solution of (P_{λ^*}) coincides with the minimal solution. In other words, it is legitimate to assume:

$$v(x) \geq u^*(x), \quad \text{a.e.} \quad x \in B_R$$

Therefore, for $\varphi \in \mathcal{C}_c^\infty(B_R)$, $\varphi \geq 0$, we have

$$\int_{B_R} (v - u^*)\Delta^2 \varphi \, dx \geq \int_{B_R} \lambda^* f(x) \left[\frac{1}{(1-v)^2} - \frac{1}{(1-u^*)^2} \right] \varphi \, dx \geq 0$$

and the Boggio principle (Lemma 1.1) yields:

$$v \equiv u^* \quad \text{a.e.} \quad x \in B_R \quad \text{or} \quad v - u^* \geq c_0 > 0, \quad \text{a.e.} \quad |x| \leq \rho < R$$

In the first case we are done, otherwise let us set

$$u_0 := \frac{u^* + v}{2}$$

so that $0 \leq u_0 \leq 1$ and as one can easily check, the following elementary inequality holds

$$\frac{1}{(1-v)^2} + \frac{1}{(1-u^*)^2} \geq \frac{1}{(1-u_0)^2} \left[\left(1 + \frac{v - u_0}{1 - u_0}\right)^2 + \left(1 + \frac{u^* - u_0}{1 - u_0}\right)^2 \right]$$
$$= \frac{1}{(1-u_0)^2} \left[2 + \frac{(v-u^*)^2}{2(1-u_0)^2} \right]$$

and in turn we obtain,

$$\int_{B_R} u_0 \Delta^2 \varphi \, dx \geq \int_{B_R} \left[\frac{\lambda^* f(x)}{(1-u_0)^2} + \frac{\lambda^* f(x)(v-u^*)^2}{4(1-u_0)^2} \right] \varphi \, dx$$
$$\geq \int_{B_R} \left[\frac{\lambda^* f(x)}{(1-u_0)^2} + \frac{\lambda^* c_0^2 \xi(x) f(x)}{4(1-u_0)^2} \right] \varphi \, dx$$

Thus, u_0 is a weak super-solution of $(\widetilde{P}_{\lambda^*})$ with $\mu_0 = (\lambda^* c_0^2)/4$ and the cut-off ξ with support in B_ρ. Now reasoning as in Lemma 2.3, we may assume for $\varepsilon > 0$ sufficiently small, that $(\widetilde{P}_{\lambda^* - \varepsilon})$ posses a classical solution $0 \leq u_\varepsilon < 1$. Set $\mu_\varepsilon := [(\lambda^* - \varepsilon)c_0^2]/4$ and let $\psi \in \mathcal{C}^4(\overline{B}_R)$ be the unique classical solution (by Boggio [5]) of the following

$$\begin{cases} \Delta^2 \psi = \mu_\varepsilon \dfrac{\xi(x) f(x)}{(1-u_\varepsilon)^2}, & \text{in } B_R \\ \psi = \dfrac{\partial \psi}{\partial \eta} = 0, & \text{on } \partial B_R \end{cases} \tag{3.18}$$

We also have that there exists $M > 0$ sufficiently large such that $u_\varepsilon \le M\psi$. Next let $\delta > 0$ and set

$$w := \frac{(\lambda^* - \varepsilon) + \delta}{\lambda^* - \varepsilon} u_\varepsilon - \psi$$

and choosing δ sufficiently small, we obtain $w \le u_\varepsilon < 1$; moreover, from

$$\begin{cases} \Delta^2 (u_\varepsilon - \psi) = (\lambda^* - \varepsilon) \dfrac{f(x)}{(1-u_\varepsilon)^2} \ge 0, & \text{in } B_R \\ u_\varepsilon - \psi = \dfrac{\partial(u_\varepsilon - \psi)}{\partial \eta} = 0, & \text{on } \partial B_R \end{cases} \tag{3.19}$$

we have again by the Boggio principle that $\psi \le u_\varepsilon$ and eventually that $w \ge 0$. Finally we have

$$\begin{aligned} \Delta^2 w = (\lambda^* - \varepsilon + \delta) \frac{f(x)}{(1-u_\varepsilon)^2} + \frac{(\lambda^* - \varepsilon + \delta) c_0^2}{4} \frac{\xi(x) f(x)}{(1-u_\varepsilon)^2} - \mu_\varepsilon \frac{\xi(x) f(x)}{(1-u_\varepsilon)^2} \\ \ge (\lambda^* - \varepsilon + \delta) \frac{f(x)}{(1-w)^2} \end{aligned} \tag{3.20}$$

since $w \le u_\varepsilon$. Thus it is enough to choose $0 < \varepsilon < \delta$ to provide a classical solution to (P_λ) for $\lambda > \lambda^*$ which is a contradiction; this completes the proof of Theorem 1.3.

3.1 Characterization of singular solutions

Proof. [Proof of Corollary 1.1] From Proposition 2.2 we have that u^* is weakly stable, thus we have to prove just the necessary part. If $\lambda = \lambda^*$, the result easily follows from Theorem 1.3. On the other hand, if $\lambda < \lambda^*$, by Theorem 1.1 there exists a minimal solution $\underline{u}_\lambda$ and the comparison Lemma 1.3 yields $u_\lambda = \underline{u}_\lambda < 1$, thus a contradiction.

4 Final remarks and open problems

4.1 Nonexistence results in $\mathbb{R}^N$: proof of Theorem 1.4

The result essentially follows from [20, Section I.5] and we just sketch the proof; the main tool is the nonlinear capacity method. Let us first make precise what we mean by weak solution of (1.3), namely: $u \in L^1_{loc}(\mathbb{R}^N)$, $u \ge 0$, such that $1/(1-u)^2 \in L^1_{loc}(\mathbb{R}^N)$ and the following holds

$$\int_{\mathbb{R}^N} u \Delta^2 \varphi \, dx \ge \int_{\mathbb{R}^N} \frac{g(x)}{(1-u)^2} \varphi \, dx, \qquad \varphi \in \mathcal{C}^4_c(\mathbb{R}^N; \mathbb{R}^+)$$

Now notice that

$$\int_{\mathbb{R}^N} \frac{g(x)}{(1-u)^2} \varphi \, dx \ge \int_{\mathbb{R}^N} \lambda |x|^\gamma u^p \varphi \, dx, \quad \forall p \ge 1$$

Set

$$\varphi(x) := \left[\varphi_0\left(\frac{|x|}{R}\right)\right]^{\alpha}$$

with $\alpha > 0$ to be suitably chosen and where $\varphi_0 : \mathbb{R} \longrightarrow \mathbb{R}^+$ is a smooth cut-off function such that $\varphi_0(s) = 1$, for $0 \le s \le 1$ and $\varphi_0(s) = 0$ for $s \ge 2$. Let $\varepsilon > 0$, then by Hölder's inequality with exponents $p = 1 + \varepsilon$, $p' = (1 + \varepsilon)/\varepsilon$, we have

$$\int_{\mathbb{R}^N} \lambda |x|^{\gamma} u^p \varphi \, dx \le \left(\int_{\mathbb{R}^N} \lambda |x|^{\gamma} u^p \varphi \, dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^N} \frac{|\Delta^2 \varphi|^{p'}}{(|x|^{\gamma}\varphi)^{p'-1}} \, dx\right)^{\frac{1}{p'}}$$

from which

$$\int_{\mathbb{R}^N} \lambda |x|^{\gamma} u^p \varphi \, dx \le \int_{\mathbb{R}^N} \frac{|\Delta^2 \varphi|^{p'}}{(|x|^{\gamma}\varphi)^{p'-1}} \, dx \tag{4.21}$$

where we may choose $\alpha > 0$ such that the right hand side in (4.21) (the so-called nonlinear capacity induced by the operator: $\Delta^2 u - |x|^{\gamma}|u|^p$) is finite. Moreover, by straightforward calculations one has, performing the change of variable $x = R\xi$, the following estimate:

$$\int_{\mathbb{R}^N} \frac{|\Delta^2 \varphi|^{p'}}{(|x|^{\gamma}\varphi)^{p'-1}} \, dx \le \frac{C_0}{R^{4p' + \gamma(p'-1) - N}}$$

for a positive constant $C_0 = C_0(p, \alpha, N)$, depending on the choice of φ_0 but not on R. Thus, if the following condition is satisfied

$$N \le \frac{4(1+\varepsilon) + \gamma}{\varepsilon} \tag{4.22}$$

the claim follows from (4.21) by letting $R \to +\infty$; however, since $\varepsilon > 0$ is arbitrary, condition (4.22) turns out to be always satisfied provided $\varepsilon > 0$ is sufficiently small. This completes the proof of Theorem 1.4.

4.2 Compactness issues

The main problem we leave open in this paper conveys into the following question:

For which class of solutions u_λ of problem (P_λ) and under which conditions does the following property:

$$(\mathcal{C}) \qquad \sup_{\lambda \in [0, \lambda^*]} \|u_\lambda\|_\infty < 1$$

hold true?

This compactness property has many important consequences which are relevant both from the point of view of applications and mathematics. Clearly, in MEMS devices condition $(\mathcal{C})$ would prevent the occurrence of snap-through phenomena, representing a stability properties of the device. On the other side, if $(\mathcal{C})$ holds along the branch of minimal solutions $\underline{u}_\lambda$, we end up in $\lambda = \lambda^*$ with a smooth extremal solution and by means of the implicit function theorem, this would enable us to continue, at least locally, the minimal branch with a new branch of unstable solutions.

It is worth to point out that similar issues have been intensively studied in literature (see e.g. [19, 7, 9]) for the second order problem

$$\begin{cases} -\Delta u = \lambda g(u), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

where $g : \mathbb{R} \longrightarrow \mathbb{R}^+$ is increasing, convex and super-linear at infinity. Here one expects that the extremal solution can be regular or singular, that is $u^* \notin L^\infty(\Omega)$. In this respect, a crucial role is played by the underlying space dimension: in particular for the Gelfand problem, when $g(u) = e^u$, it is well known that in the ball, the extremal solution is bounded provided $N \leq 9$ and unbounded in dimension $N \geq 10$; the threshold $N^* = 10$ between regular and singular solutions is called the *critical dimension*.

In the case of singular nonlinearity, it was shown in [13] by means of energy estimates that property (C) holds following the branch of minimal solutions of problem $(\widetilde{M}_\lambda)$ as long as $1 \leq N \leq 7$, result further extended in [11] where finer blow-up arguments are used to prove compactness along the branch of unstable solutions. Thus, in this case the critical dimension $N^* = 8$.

In our situation the plot thickens and the picture is far from being clear: from one side the techniques used in the second order case do not seem working in the fourth order problem, on the other side even basic evidences, which in the second order problem are based on Hardy's inequality and which may be used to guess the *critical* candidate (see [11]), here fail because of the too stringent Dirichlet boundary conditions.

Acknowledgments

This research started when the first two named authors were visiting the Pacific Institute for the Mathematical Sciences and the Department of Mathematics at the University of British Columbia: they widely thank the UBC environment for the kind hospitality. The first named author greatly thanks also the Departamento de Matematica of the Universidade Federal de Paraiba, where during a visiting period this research advanced.

References

[1] S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I*, Comm. Pure Appl. Math. **12** (1959), 623–727.

[2] G. Arioli, F. Gazzola, H.-C. Grunau, and E. Mitidieri, *A Semilinear Fourth Order Elliptic Problem with Exponential Nonlinearity*, SIAM J. Math. Anal. **36** (2005), 1226–1258.

[3] E. Berchio, F. Gazzola, and E. Mitidieri, *Positivity preserving property for a class of biharmonic elliptic problems*, J. Diff. Eq. **229** (2006), 1–23.

[4] E. Berchio, F. Gazzola, and T. Weth, *Radial symmetry of positive solutions to nonlinear polyharmonic Dirichlet problems*, Preprint (2007).

[5] T. Boggio, *Sulle funzioni di Green d'ordine m*, Rend. Circ. Mat. Palermo (1905), 97–135.

[6] H. Brezis, T. Cazenave, and Y. Martel, *Blow-up for $u_t - \Delta u = g(u)$ revisited*, Adv. Diff. Eq. **1** (1996), 73–90.

[7] H. Brezis and J.L. Vásquez, *Blow-up solutions of some nonlinear elliptic problem*, Rev. Mat. Univ. Complut. Madrid **10** (1997), 443–468.

[8] D. Cassani, B. Kaltenbacher, and A. Lorenzi, *Direct and inverse problems related to MEMS devices*, Preprint (2008).

[9] M.C. Crandall and P.H. Rabinowitz, *Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems*, Arch. Ration. Mech. Anal. **58** (1975), 207–218.

[10] J. Dávila, L. Dupaigne, I. Guerra, and M. Montenegro, *Stable solutions for the bilaplacian with exponential nonlinearity*, SIAM J. Math. Anal. **39** (2007), 565–592.

[11] P. Esposito, N. Ghoussoub, and Y. Guo, *Compactness along the branch of simi-stable and unstable solutions for an elliptic problem with a singular nonlinearity*, Comm. Pure Appl. Math. **60** (2007), 1731–1768.

[12] F. Gazzola and H.-Ch. Grunau, *Critical dimensions and and higher order Sobolev inequalities with reminder terms*, NoDEA **8** (2001), 35–44.

[13] N. Ghoussoub and Y. Guo, *On the partial differential equations of electrotatic MEMS devices: stationary case*, SIAM J. Math. Anal. **38** (2007), 1423–1449.

[14] B. Gidas, W.M. Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209–243.

[15] Y. Guo, Z. Pan, and M.J. Ward, *Touchdown and pull-in voltage behavior of a mems device with varying dielectric properties*, SIAM J. Appl. Math. **66** (2005), 309–338.

[16] F.H. Lin and Y.S. Yang, *Nonlinear non-local elliptic equation modelling electrostatic actuation*, Proc. R. Soc. London, Ser. A **463** (2007), 1323–1337.

[17] S. Luckhaus, *Existence and regularity of weak solutions to the Dirichlet problem for semilinear elliptic systems of higher order*, J. Reine Angew. Math. **306** (1979), 192–207.

[18] Y. Martel, *Uniqueness of weak extremal solutions of nonlinear elliptic problems*, Houston Journal of Mathematics **23** (1997), 161–168.

[19] F. Mignot and J-P. Puel, *Sur une classe de problemes non lineaires avec non linearite positive, croissante, convexe*, Comm. Partial Differential Equations **5** (1980), 791–836.

[20] E. Mitidieri and S.I. Pohožaev, *A priori estimates and blow-up of solutions to nonlinear partial differential equations and inequalities*, Proc. Steklov Institute of Math. **234** (2001), 1–362.

[21] J.-J. Moreau, *Decomposition orthogonale d'un espace hilbertien selon deux cônes mutuellement polaires.*, C. R. Acad. Sci. Paris **255** (1962), 238–240.

[22] J.A. Pelesko, *Mathematical modeling of electrostatic mems with tailored dielectric properties*, SIAM J. Appl. Math. **62** (2002), 888–908.

[23] J.A. Pelesko and A.A. Bernstein, *Modeling MEMS and NEMS*, Chapman Hall and CRC Press, 2002.

[24] R. Soranzo, *A priori estimates and existence of positive solutions of a superlinear polyharmonic equation*, Dynamic Systems and Applications **3** (1994), 465–488.

[25] G. Sweers, *No Gidas-Ni-Nirenberg type result for biharmonic problems*, Math. Nachr. **246/247** (2002), 202–206.