A nonhomogeneous elliptic problem involving critical growth in dimension two. ⋆

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Abstract

In this paper we study a class of nonhomogeneous Schrödinger equations

$$-\Delta u + V(x)u = f(u) + h(x)$$

in the whole two-dimension space where $V(x)$ is a continuous positive potential bounded away from zero and which can be "large" at the infinity. The main difficulty in this paper is the lack of compactness due to the unboundedness of the domain besides the fact that the nonlinear term $f(s)$ is allowed to enjoy the critical exponential growth by means of the Trudinger-Moser inequality. By combining variational arguments and a version of the Trudinger-Moser inequality, we establish the existence of two distinct solutions when h is suitably small.

Key words: Schrödinger equation; standing wave solutions; critical growth; Trudinger-Moser inequality, variational methods.

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1 Introduction

The main purpose of this paper is to study the existence and multiplicity of solutions for nonhomogeneous elliptic problems of the form

$$-\Delta u + V(x)u = f(u) + h(x), \quad x \in \mathbb{R}^2 \tag{1.1}$$

when the nonlinear term $f(s)$ is allowed to enjoy the critical exponential growth by means of the Trudinger-Moser inequality.

The above problem appears in many areas of mathematical physics; in particular, solutions of the equation (1.1) provide *standing waves solutions* for the nonlinear Schrödinger equation (see for instance [5], [18], [21], [25] and references therein)

$$i\frac{\partial \psi}{\partial t} = -\Delta\psi + W(x)\psi - g(|\psi|)\psi - e^{i\lambda t}h(x), \quad x \in \mathbb{R}^2,$$

where $\psi = \psi(t,x)$, $\psi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$, λ is a positive constant, $W : \mathbb{R}^2 \to \mathbb{R}$ is a given potential and for suitable functions $g : \mathbb{R}^+ \to \mathbb{R}$, $h : \mathbb{R}^2 \to \mathbb{R}$. Throughout this paper we assume the following hypotheses on V:

(V_1) $V : \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies

$$V(x) \geq V_0 > 0 \quad \text{for all} \quad x \in \mathbb{R}^2;$$

(V_2) The function $[V(x)]^{-1}$ belongs to $L^1(\mathbb{R}^2)$.

The study of existence and multiplicity of solution for nonhomogeneous elliptic equations in euclidian domains involving critical growth have received considerable attention in recent years. Most of these problems are dealt with variational methods, and since the Palais-Smale condition no longer holds for this class of problems this poses an essential difficulty to the existence question. It is well known that in dimensions $N \geq 3$, the maximal possible growth for the nonlinearity is polynomial at infinity, so that the related functional is well defined in a Sobolev space (see [8] and [19]). Limitations on the growth of the nonlinearity vary substantially when we come to dimension two. The nonlinearity may exhibit exponential growth as established by the Trudinger-Moser inequality, which in this case replaces the Sobolev embedding theorem. We are interested in the case where the nonlinear term $f(s)$ has the maximal growth on s which allows us to treat problem (1.1) variationally. Motivated by a Trudinger-Moser type inequality (see Lemma 2.1 below) we say that $f(s)$ has *subcritical growth* at $+\infty$ if for all $\alpha > 0$

$$\lim_{s\to+\infty} \frac{f(s)}{e^{\alpha s^2}} = 0, \tag{1.2}$$

and $f(s)$ has *critical growth* at $+\infty$ if there exists $\alpha_0 > 0$ such that

$$\lim_{s\to+\infty} \frac{f(s)}{e^{\alpha s^2}} = \begin{cases} 0, & \forall\, \alpha > \alpha_0, \\ +\infty, & \forall\, \alpha < \alpha_0. \end{cases} \tag{1.3}$$

We introduce the following assumptions on the nonlinear term:

(f_0) $f \in C(\mathbb{R}, \mathbb{R})$ and $f(0) = 0$;
(f_1) there exist $\theta > 2$ and $s_1 > 0$ such that for all $|s| \geq s_1$,

$$0 < \theta F(s) \doteq \theta \int_0^s f(t)\ \mathrm{d}t \leq s f(s);$$

(f_2) there exist constants $R_0, M_0 > 0$, such that for all $|s| \geq R_0$

$$0 < F(s) \leq M_0 f(s).$$

Next, in order to apply variational methods, we consider the following subspace of $H^1(\mathbb{R}^2)$

$$E = \Big\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} V(x) u^2 \mathrm{d}x < \infty \Big\},$$

which is a Hilbert space endowed with the inner product

$$\langle u, v \rangle = \int_{\mathbb{R}^2} (\nabla u \nabla v + V(x) uv) \mathrm{d}x, \quad u, v \in E \tag{1.4}$$

to which corresponds the norm $\|u\| = \langle u, u \rangle^{1/2}$. Here $H^1(\mathbb{R}^2)$ denotes the usual Sobolev space with the norm

$$\|u\|_{1,2} = \left[\int_{\mathbb{R}^2} (|\nabla u|^2 + |u|^2) \mathrm{d}x \right]^{1/2}.$$

We say that $u \in E$ is a weak solution of the problem (1.1), provided that

$$\int_{\mathbb{R}^2} (\nabla u \nabla v + V(x) uv) \mathrm{d}x - \int_{\mathbb{R}^2} f(u) v\ \mathrm{d}x - \int_{\mathbb{R}^2} hv\ \mathrm{d}x = 0 \tag{1.5}$$

for all $v \in E$. Notice that weak solutions of (1.1) turn out to be critical points of the energy functional

$$I(u) = \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}^2} F(u)\ \mathrm{d}x - \int_{\mathbb{R}^2} hu\ \mathrm{d}x. \tag{1.6}$$

Assumption (V_1) implies that the embedding

$$E \hookrightarrow H^1(\mathbb{R}^2)$$

is continuous whereas condition (V_2), together with the Hölder inequality, implies that

$$\|u\|_{L^1(\mathbb{R}^2)} \leq \left(\int_{\mathbb{R}^2} V(x)^{-1} \mathrm{d}x \right)^{1/2} \|u\|. \tag{1.7}$$

As a consequence,

$$E \hookrightarrow L^q(\mathbb{R}^2) \quad \text{for all} \quad 1 \le q < \infty, \tag{1.8}$$

with continuous embedding. It is also well known that assumption (V_2) implies that these embeddings are compact for all $1 \le q < \infty$ (see [16], [18]). Moreover,

$$\lambda_1 \doteq \inf_{u \in E \setminus \{0\}} \frac{\int_{\mathbb{R}^2} (|\nabla u|^2 + V(x) u^2)\ \mathrm{d}x}{\int_{\mathbb{R}^2} u^2\ \mathrm{d}x} \ge V_0 > 0. \tag{1.9}$$

If $h \ge 0$, it is readily seen that the problem

$$-\Delta u + V(x) u = \lambda_1 u + 2u e^{u^2} + h(x), \quad x \in \mathbb{R}^2$$

does not have positive solutions. Therefore, we assume the following additional condition near the origin:

(f_3) $\lim_{s \to 0} 2F(s) s^{-2} < \lambda_1$.

We want to remark that we have to handle two terms in problem (1.1), the nonlinearity $f(s)$ and the perturbation $h(x)$. Our main interest is to analyze the interplay between them. In this paper we look for conditions that ensure the existence and multiplicity of solutions of (1.1), focusing our attention on the existence and multiplicity of one sign.

We distinguish two cases:

1.1 Subcritical Case

Throughout this paper, we denote by H^{-1} the dual space of $H^1(\mathbb{R}^2)$ with the usual norm $\|\cdot\|_{H^{-1}}$.

Our main results are the following:

Theorem 1.1 *If $f(s)$ has subcritical growth and $(V_1) - (V_2)$, (f_0), (f_1), (f_3) are satisfied, then there exists $\delta_1 > 0$ such that if $0 < \|h\|_{H^{-1}} < \delta_1$, (1.1) has at least two weak solutions. One of them with positive energy, while the other one with negative energy.*

Furthermore, if $h(x)$ has defined sign, the following result holds:

Theorem 1.2 *Under the assumptions of Theorem 1.1, if $h(x) \ge 0$ ($h(x) \le 0$) almost everywhere in $\mathbb{R}^2$, then the solutions obtained in Theorem 1.1 are nonnegative (nonpositive), respectively.*

Example 1.3 *A typical example of functions satisfying assumptions* $(f_1),(f_3)$ *with subcritical growth is* $f(s)=\lambda(2s+s^2)e^s$ *with* $0<\lambda<\lambda_1/2$. *We have that* $F(s)=\lambda s^2e^s$. *In order to prove that* (f_1) *is satisfied, it is enough to notice that*

$$\lim_{|s|\to\infty}\frac{F(s)}{sf(s)}=\lim_{|s|\to\infty}\frac{s^2e^s}{s(2s+s^2)e^s}=\lim_{|s|\to\infty}\frac{1}{2+s}=0.$$

Furthermore, (f_3) *is satisfied,*

$$\lim_{s\to 0}\frac{2F(s)}{s^2}=2\lambda\lim_{s\to 0}e^s=2\lambda<\lambda_1.$$

1.2 Critical Case

When $f(s)$ exhibits critical growth we obtain the following results:

Theorem 1.4 *If* $f(s)$ *has critical growth and* $(V_1)-(V_2)$, (f_0), (f_2), (f_3) *are satisfied, then, there exists* $\delta_1>0$ *such that if* $0<\|h\|_{H^{-1}}<\delta_1$, *problem (1.1) has a weak solution with negative energy.*

Theorem 1.5 *Under the hypotheses of Theorem 1.4, if in addiction we assume that there exists* $\beta_0>0$ *such that*

(f_4^+) $\lim_{s\to+\infty} sf(s)e^{-\alpha_0 s^2}\geq\beta_0>0.$

Then, there exists $\delta_2>0$, *such that if* $0<\|h\|_{H^{-1}}<\delta_2$, *then problem (1.1) has a second weak solution.*

Furthermore, if $h(x)$ has defined sign, the following result holds:

Theorem 1.6 *Under the assumptions of Theorem 1.5, if* $h(x)\geq 0$ *almost everywhere in* $\mathbb{R}^2$, *then the solutions obtained in Theorem 1.5 are nonnegative. Moreover, if* $h(x)\leq 0$ *almost everywhere in* $\mathbb{R}^2$ *and* $f(s)$ *satisfies*

(f_4^-) $\lim_{s\to-\infty} sf(s)e^{-\alpha_0 s^2}\geq\beta_0>0,$

then these solutions are nonpositive.

Example 1.7 *A typical example of functions satisfying the assumptions* $(f_2),(f_3),(f_4^+)$ *with critical growth is* $f(s)=3s^2+2se^{s^2}-2s$. *In order to prove that* (f_2) *is satisfied, it is enough to notice that*

$$\lim_{|s|\to\infty}\frac{F(s)}{f(s)}=\lim_{|s|\to\infty}\frac{s^3+e^{s^2}-1-s^2}{3s^2+2se^{s^2}-2s}=0.$$

Furthermore, it is easy to see that $\lim_{s\to 0} 2F(s)s^{-2} = 0 < \lambda_1$ *and* $\lim_{s\to+\infty} sf(s)e^{-s^2} = +\infty$, *showing that* (f_3) *and* (f_4^+) *hold.*

Remark 1.8 *Condition* (f_2) *is stronger than* (f_1), *in the sense that* (f_2) *implies* (f_1). *One can easily see that integrating condition* (f_1) *there exists positive constants* C_1, C_2 *such that*

$$F(s) \geq C_1|s|^\theta - C_2, \quad s \in \mathbb{R}. \tag{1.10}$$

On the other hand, it follows from (f_2) *that there exist positive constants* C_1, C_2 *such that*

$$F(s) \geq C_1 e^{|s|/M} - C_2, \quad s \in \mathbb{R}. \tag{1.11}$$

In the last years, several papers have been devoted to the study of elliptic problems involving critical growth in terms of the Trudinger-Moser inequality. Problems with critical growth, involving the Laplace operator and in bounded domains of $\mathbb{R}^2$, have been investigated among others by [2], [3], [12]. Quasilinear elliptic problems with critical growth for the $N-$Laplacian in bounded domains of $\mathbb{R}^N$, have been studied in [1], [13], [23]. Cao in [9] treated problem (1.1) in the homogeneous case, that is, $h \equiv 0$, when V and $f(s)$ are asymptotic to a constant function. See also [15] and [4] for related results for homogeneous elliptic problems when the potential V satisfies some geometric condition. In [14], by combining a version of the Trudinger-Moser inequality with the mountain-pass theorem, the author studied the problem $-\Delta_N u + V(x)|u|^{N-2}u = f(x,u)$ imposing a coercivity condition on the potential V, $f(x,u)$ with critical growth and $f(x,0) = 0$. In the present paper, we improve and complement some of the results cited above and ours results can be considered as an extension of the main results in [23] and [14]. Here our approach to obtain multiplicity of solutions is in the spirit of [21] and based on a global variational point of view. The proofs of our results rely on minimization methods in combination with the mountain-pass theorem. In the subcritical case we are able to prove that the associated functional satisfies the Palais-Smale compactness condition which allow us to obtain critical points for the functional. As a consequence we can distinguish the local minimum solution from the mountain-pass solution. However, in the critical case to prove that these solutions are different is more involved, requiring fine energy level estimates. Assumption (f_4^+) in Theorem 1.5 will be used to estimate the mountain-pass level.

The outline of the paper is as follows: Section 2 contains some preliminary results including an extension of Lions' lemma in the whole $\mathbb{R}^2$ (Lemma 2.6). In Section 3, we set up technical results which will allow us to follow a variational approach. Finally, in Section 4 we complete the proofs of our main results.

Notation. In this work we make use of the following notation:

- C, C_0, C_1, C_2, ... denote positive (possibly different) constants;
- B_R denotes the open ball centered at the origin and radius $R > 0$;
- For $1 \leq p < \infty$, $L^p(\mathbb{R}^2)$ denotes the usual Lebesgue spaces with respect the norm

$$\|u\|_p = \left(\int_{\mathbb{R}^2} |u|^p \mathrm{d}x\right)^{1/p};$$

- $C_0^\infty(\Lambda)$ denotes the space of infinitely differentiable functions with compact support in Λ, where Λ is a domain of $\mathbb{R}^2$;
- $H^1(\Lambda)$ denotes the Sobolev spaces modeled in $L^2(\Lambda)$ with the norm

$$\|u\|_{1,2} = \left[\int_\Lambda (|\nabla u|^2 + |u|^2)\mathrm{d}x\right]^{1/2};$$

- By $\langle \cdot, \cdot \rangle$ we denote the duality pairing between X' and X;
- We denote the weak convergence in X by " $\rightharpoonup$ " and the strong convergence by " $\to$ ".

2 Some preliminary results

Let Ω be a bounded domain in $\mathbb{R}^2$; we know by the Trudinger-Moser inequality that for all $\alpha > 0$ and $u \in H_0^1(\Omega)$, $e^{\alpha u^2} \in L^1(\Omega)$ (see [20], [24]). Moreover, there exists a constant $C > 0$ such that

$$\sup_{\|u\|_{H_0^1(\Omega)} \leq 1} \int_\Omega e^{\alpha u^2} \, \mathrm{d}x \leq C|\Omega| \quad \text{if} \quad \alpha \leq 4\pi. \tag{2.12}$$

Here we shall use the following extension of these results for the whole space $\mathbb{R}^2$ obtained by [9] (see also [14,22] for a more complete result):

Lemma 2.1 *If $\alpha > 0$ and $u \in H^1(\mathbb{R}^2)$ then*

$$\int_{\mathbb{R}^2} (e^{\alpha u^2} - 1) \, \mathrm{d}x < \infty.$$

Moreover, if $\|\nabla u\|_2^2 \leq 1$, $\|u\|_2 \leq M < \infty$ and $\alpha < 4\pi$ then there exists a constant $C = C(M, \alpha)$, which depends only on M and α, such that

$$\int_{\mathbb{R}^2} (e^{\alpha u^2} - 1) \, \mathrm{d}x \leq C(M, \alpha). \tag{2.13}$$

The next results are essential to establish the mountain-pass geometry of the associated functional.

Lemma 2.2 *Let $\beta > 0$ and $r > 1$. Then for each $\alpha > r$ there exists a positive constant $C = C(\alpha)$ such that for all $s \in \mathbb{R}$*

$$(e^{\beta s^2} - 1)^r \leq C(e^{\alpha\beta s^2} - 1).$$

In particular, if $u \in H^1(\mathbb{R}^2)$ then $(e^{\beta u^2} - 1)^r$ belongs to $L^1(\mathbb{R}^2)$.

Proof. Since $r > 1$, by L'Hospital's Rule we conclude that

$$\lim_{s\to 0} \frac{(e^{\beta s^2} - 1)^r}{e^{\alpha\beta s^2} - 1} = \lim_{s\to 0} \frac{r(e^{\beta s^2} - 1)^{r-1}e^{\beta s^2}}{\alpha e^{\alpha\beta s^2}} = 0.$$

Moreover, notice that

$$\lim_{|s|\to\infty} \frac{(e^{\beta s^2} - 1)^r}{(e^{\alpha\beta s^2} - 1)} = \lim_{|s|\to\infty} \frac{e^{r\beta s^2}(1 - e^{-\beta s^2})^r}{e^{\alpha\beta s^2}(1 - e^{-\alpha\beta s^2})} = 0.$$

Thus, the result follows. ∎

Remark 2.3 *As a consequence of Lemmas 2.1 and 2.2 and Hölder inequality, we see that if $\beta > 0$ and $q > 0$ then the function $|u|^q(e^{\beta u^2} - 1)$ belongs to $L^1(\mathbb{R}^2)$ for all $u \in H^1(\mathbb{R}^2)$.*

Lemma 2.4 *If $v \in E$, $\beta > 0$, $q > 0$ and $\|v\| \leq M$ with $\beta M^2 < 4\pi$, then there exists $C = C(\beta, M, q) > 0$ such that*

$$\int_{\mathbb{R}^2} (e^{\beta v^2} - 1)|v|^q \,\mathrm{d}x \leq C\|v\|^q.$$

Proof. We consider $r > 1$ close to 1 such that $r\beta M^2 < 4\pi$ and $sq \geq 1$ where $s = r/(r-1)$. Using the Hölder inequality, we have

$$\int_{\mathbb{R}^2} (e^{\beta v^2} - 1)|v|^q \,\mathrm{d}x \leq \left[\int_{\mathbb{R}^2} (e^{\beta v^2} - 1)^r \mathrm{d}x\right]^{1/r} \|v\|_{qs}^q.$$

Now, taking $\alpha > r$ close to r such that $\alpha\beta M^2 < 4\pi$, by Lemmas 2.2 and 2.1 we obtain

$$\begin{aligned}
\int_{\mathbb{R}^2} (e^{\beta v^2} - 1)|v|^q \,\mathrm{d}x &\leq C_1 \left[\int_{\mathbb{R}^2} (e^{\alpha\beta v^2} - 1) \,\mathrm{d}x\right]^{1/r} \|v\|_{qs}^q \\
&\leq C_1 \left\{\int_{\mathbb{R}^2} \left[e^{\alpha\beta M^2\left(\frac{v}{\|\nabla v\|_2}\right)^2} - 1\right]\mathrm{d}x\right\}^{1/r} \|v\|_{qs}^q \\
&\leq C_2\|v\|_{qs}^q.
\end{aligned}$$

Finally, using the continuous embedding $E \hookrightarrow L^{sq}(\mathbb{R}^2)$, we conclude that

$$\int_{\mathbb{R}^2} (e^{\beta v^2} - 1)|v|^q \,\mathrm{d}x \leq C\|v\|^q.$$

∎

The inequality (2.12) was improved by Lions in [17]. More precisely, he proved the following lemma in a bounded domain:

Lemma 2.5 *Let (w_n) be a sequence in $H^1(\Omega)$ such that $\|w_n\|_{1,2} = 1$. Suppose that (w_n) converges weakly to $w_0 \neq 0$ in $H^1(\Omega)$, then for all $0 < p < 4\pi(1 - \|w_0\|_{1,2}^2)^{-1}$ we have*

$$\sup_n \int_\Omega e^{pw_n^2}\, \mathrm{d}x < \infty.$$

Proof. See proof in [17, Theorem I.6] or [3, Lemma 3.5]. ∎

Next, let us establish a version of Lemma 2.5 for the whole $\mathbb{R}^2$.

Lemma 2.6 *Let (w_n) in $H^1(\mathbb{R}^2)$ with $\|w_n\|_{1,2} = 1$ and suppose that $w_n \rightharpoonup w_0$ in $H^1(\mathbb{R}^2)$ with $\|w_0\|_{1,2} < 1$. Then for all $0 < p < 4\pi(1 - \|w_0\|_{1,2}^2)^{-1}$ we have*

$$\sup_n \int_{\mathbb{R}^2} (e^{pw_n^2} - 1)\, \mathrm{d}x < \infty.$$

Proof. Since $w_n \rightharpoonup w_0$ and $\|w_n\|_{1,2} = 1$, we conclude that

$$\|w_n - w_0\|_{1,2}^2 = 1 - 2\langle w_n, w_0\rangle + \|w_0\|_{1,2}^2 \to 1 - \|w_0\|_{1,2}^2 < \frac{4\pi}{p}.$$

Thus, for n large we have $p\|w_n - w_0\|_{1,2}^2 < \alpha < 4\pi$ for some $\alpha > 0$. Now choosing $q > 1$ close to 1 and $\epsilon > 0$ satisfying $q(1+\epsilon^2)p\|w_n - w_0\|_{1,2}^2 < \alpha$, by (2.13) we have

$$\begin{aligned}\int_{\mathbb{R}^2} \left[e^{qp(1+\epsilon^2)(w_n - w_0)^2} - 1\right] \mathrm{d}x &= \int_{\mathbb{R}^2} \left[e^{qp(1+\epsilon^2)\|w_n - w_0\|_{1,2}^2 \left(\frac{w_n - w_0}{\|w_n - w_0\|_{1,2}}\right)^2} - 1\right] \mathrm{d}x \\ &\leq \int_{\mathbb{R}^2} \left[e^{\alpha\left(\frac{|w_n - w_0|}{\|w_n - w_0\|_{1,2}}\right)^2} - 1\right] \mathrm{d}x \leq C.\end{aligned}$$

Moreover, since

$$pw_n^2 \leq p(1+\epsilon^2)(w_n - w_0)^2 + p(1 + \frac{1}{\epsilon^2})w_0^2,$$

it follows that

$$\begin{aligned}e^{pw_n^2} - 1 &\leq (e^{p(1+\epsilon^2)(w_n - w_0)^2} e^{p(1+1/\epsilon^2)w_0^2} - 1) \\ &\leq \frac{1}{q}\left(e^{qp(1+\epsilon^2)(w_n - w_0)^2} - 1\right) + \frac{1}{r}\left(e^{rp(1+1/\epsilon^2)w_0^2} - 1\right),\end{aligned}$$

where in the last inequality we have used that for all $a, b > 0$ and $q^{-1}+r^{-1} = 1$ it holds

$$ab - 1 \leq \frac{1}{q}(a^q - 1) + \frac{1}{r}(b^r - 1).$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}^2}(e^{pw_n^2} - 1)\mathrm{d}x &\leq \frac{1}{q}\int_{\mathbb{R}^2}\left[e^{qp(1+\epsilon^2)(w_n-w_0)^2} - 1\right]\mathrm{d}x + \frac{1}{r}\int_{\mathbb{R}^2}\left[e^{rp(1+1/\epsilon^2)w_0^2} - 1\right]\mathrm{d}x \\ &\leq C,\end{aligned}$$

for n large and the result is proved. ■

We will use the following result which is a converse of the Lebesgue dominated convergence theorem in the space $H^1(\mathbb{R}^2)$.

Proposition 2.7 *Let (u_n) be a sequence in $H^1(\mathbb{R}^2)$ strongly convergent. Then there exist a subsequence (u_{n_k}) of (u_n) and $g \in H^1(\mathbb{R}^2)$ such that $|u_{n_k}(x)| \leq g(x)$ almost everywhere in $\mathbb{R}^2$.*

Proof. Let (u_n) be a sequence in $H^1(\mathbb{R}^2)$ such that $u_n \to u$ in $H^1(\mathbb{R}^2)$. In particular, $u_n \to u$ almost everywhere in $\mathbb{R}^2$. Also we can extract a subsequence (u_{n_k}) de (u_n) which we denote by (u_k) such that for all $k \geq 1$

$$\|u_{k+1} - u_k\|_{1,2} \leq \frac{1}{2^k}.$$

Setting

$$w_n(x) \doteq \sum_{k=1}^{n}\Big|u_{k+1}(x) - u_k(x)\Big|,$$

it follows that $w_n \in H^1(\mathbb{R}^2)$ and $\|w_n\|_{1,2} \leq 1$. Consequently

$$\|w_n\|_2 \leq 1 \;\text{ and }\; \|\nabla w_n\|_2 \leq 1.$$

By monotone convergence theorem, $w_n \to w$ almost everywhere in $\mathbb{R}^2$ for some $w \in L^2(\mathbb{R}^2)$. Furthermore, using Lebesgue dominated convergence theorem we have that $\|w_n - w\|_2 \to 0$. From this convergence in $L^2(\mathbb{R}^2)$ and by the fact that $|\nabla w_n|$ is bounded in $L^2(\mathbb{R}^2)$, we can conclude that $w \in H^1(\mathbb{R}^2)$ (see [6, Remark 4 in Chapter 9]). Now, for $l > k \geq 2$, we have

$$|u_l(x) - u_k(x)| \leq |u_l(x) - u_{l-1}(x)| + \ldots + |u_{k+1}(x) - u_k(x)| \leq w_{l-1}(x) - w_{k-1}(x),$$

and taking $l \to \infty$, we obtain for any $k \geq 2$,

$$|u(x) - u_k(x)| \leq w(x) \;\text{ almost everywhere in }\; \mathbb{R}^2.$$

Therefore

$$|u_k(x)| \leq g(x) \;\text{ almost everywhere in }\; \mathbb{R}^2$$

with $g = |u| + w \in H^1(\mathbb{R}^2)$ and the proof is completed. ■

In order to show that the weak limit of a sequence in E is a weak solution of (1.1) we will use the following convergence result due Figueiredo et al. [12].

Lemma 2.8 *Let $\Omega \subset \mathbb{R}^2$ a bounded domain and $f : \mathbb{R} \to \mathbb{R}$ a continuous function. Then for any sequence (u_n) in $L^1(\Omega)$ such that*

$$u_n \to u \text{ in } L^1(\Omega), \quad f(u_n) \in L^1(\Omega) \quad \text{and} \quad \int_\Omega |f(u_n)u_n| \, \mathrm{d}x \le C,$$

up to a subsequence we have $f(u_n) \to f(u)$ in $L^1(\Omega)$.

3 The variational framework

By the hypothesis (f_3) we have

$$\lim_{s \to 0} \frac{f(s)}{s} < \lambda_1.$$

From this, if $f(s)$ satisfies (1.2), then for each $\alpha > 0$ there exist $b_1, b_2 > 0$ such that for all $s \in \mathbb{R}$

$$|f(s)| \le b_1|s| + b_2(e^{\alpha s^2} - 1). \tag{3.14}$$

Similarly, if $f(s)$ satisfies (1.3), then for each $\alpha > \alpha_0$ there exist $c_1, c_2 > 0$ such that for all $s \in \mathbb{R}$

$$|f(s)| \le c_1|s| + c_2(e^{\alpha s^2} - 1). \tag{3.15}$$

This together with Remark 2.3 and conditions (f_1), (f_3) imply that $F(u) \in L^1(\mathbb{R}^2)$ for all $u \in H^1(\mathbb{R}^2)$. Therefore, the functional energy $I : E \to \mathbb{R}$ given by (1.6) is well defined. Using standard arguments (see [5, Theorem A.VI] and [10]), we can see that $I \in C^1(E, \mathbb{R})$ with

$$\langle I'(u), v \rangle = \int_{\mathbb{R}^2} (\nabla u \nabla v + V(x)uv) \, \mathrm{d}x - \int_{\mathbb{R}^2} f(u)v \, \mathrm{d}x - \int_{\mathbb{R}^2} hv \, \mathrm{d}x,$$

for $u, v \in E$. Consequently, critical points of the functional I are precisely the weak solutions of problem (1.1).

In the next two lemmas we check that the functional I satisfies the geometric conditions of the mountain-pass theorem.

Lemma 3.1 *Assume (f_1), (f_3) and (1.2) (or (1.3)) hold. Then there exists $\delta_1 > 0$ such that for each $h \in H^{-1}$ with $\|h\|_{H^{-1}} < \delta_1$, there exists $\rho_h > 0$ such that*

$$I(u) > 0 \quad if \quad \|u\| = \rho_h.$$

Furthermore, ρ_h can be chosen such that $\rho_h \to 0$ as $\|h\|_{H^{-1}} \to 0$.

Proof. From (f_3), there exist $\epsilon, \delta > 0$ in such a way that $|s| \leq \delta$ implies

$$|F(s)| \leq \frac{(\lambda_1 - \epsilon)}{2}|s|^2. \tag{3.16}$$

By (1.2) (or (1.3)) and (f_1), for each $q > 2$ there exists a constant $C = C(q, \delta)$ such that

$$|F(s)| \leq C|s|^q(e^{\alpha s^2} - 1), \tag{3.17}$$

if $|s| \geq \delta$. From (3.16) and (3.17) we obtain

$$|F(s)| \leq \frac{(\lambda_1 - \epsilon)}{2}|s|^2 + C|s|^q(e^{\alpha s^2} - 1), \tag{3.18}$$

for all $s \in \mathbb{R}$ and $q > 2$. Now, using Lemma 2.4, (1.9) and the continuous embedding (1.8), we obtain

$$\begin{aligned} I(u) \geq\ & \frac{1}{2}\|u\|^2 - \frac{(\lambda_1 - \epsilon)}{2}\|u\|_2^2 - C\|u\|^q - \|h\|_{H^{-1}}\|u\| \\ \geq\ & \frac{1}{2}\left[1 - \frac{(\lambda_1 - \epsilon)}{\lambda_1}\right]\|u\|^2 - C\|u\|^q - \|h\|_{H^{-1}}\|u\|. \end{aligned}$$

Consequently

$$I(u) \geq \|u\|\left[\frac{1}{2}\left(1 - \frac{(\lambda_1 - \epsilon)}{\lambda_1}\right)\|u\| - C\|u\|^{q-1} - \|h\|_{H^{-1}}\right]. \tag{3.19}$$

Since $\epsilon > 0$ and $q > 2$, we may choose $\rho > 0$ such that

$$\frac{1}{2}\left[1 - \frac{(\lambda_1 - \epsilon)}{\lambda_1}\right]\rho - C\rho^{q-1} > 0.$$

Thus, for $\|h\|_{H^{-1}}$ sufficiently small there exists $\rho_h > 0$ such that $I(u) > 0$ if $\|u\| = \rho_h$ and $\rho_h \to 0$ as $\|h\|_{H^{-1}} \to 0$. ■

Lemma 3.2 *Suppose that f satisfies (f_1) or (f_2). Then there exists $e \in E$ with $\|e\| > \rho_h$ such that*

$$I(e) < \inf_{\|u\|=\rho_h} I(u).$$

Proof. Let $u \in H^1(\mathbb{R}^2)$ such that $u \equiv s_1$ in B_1, $u \equiv 0$ in B_2^c and $u \geq 0$. Denoting $K = supp(u)$, by (1.10) we have for $t > 1$ that

$$\begin{aligned} I(tu) &\leq \frac{t^2}{2}\|u\|^2 - Ct^\theta \int_{\{x\,:\,t|u(x)|\geq s_1\}} u^\theta \,\mathrm{d}x + C_1|K| - t\int_{\mathbb{R}^2} hu \,\mathrm{d}x, \\ &\leq \frac{t^2}{2}\|u\|^2 - Ct^\theta \int_{B_1} u^\theta \,\mathrm{d}x + C_1|K| - t\int_{\mathbb{R}^2} hu \,\mathrm{d}x. \end{aligned}$$

Since $\theta > 2$, we get $I(tu) \to -\infty$. Setting $e \doteq tu$ with t large enough, the proof is finished. ■

In order to find an appropriate ball to use minimization argument we need the following result:

Lemma 3.3 *If f satisfies (1.2) (or (1.3)), there exists $\eta > 0$ and $v \in E$ with $\|v\| = 1$ such that $I(tv) < 0$ for all $0 < t < \eta$. In particular,*

$$\inf_{\|u\| \leq \eta} I(u) < 0.$$

Proof. For each $h \in H^{-1}$, by applying the Riesz representation theorem in the space E with the inner product (1.4), the problem

$$-\Delta v + V(x)v = h, \qquad x \in \mathbb{R}^2$$

has a unique weak solution v in E. Thus,

$$\int_{\mathbb{R}^2} hv \, \mathrm{d}x = \|v\|^2 > 0 \quad \text{for each} \quad h \neq 0.$$

Since $f(0) = 0$, by continuity, it follows that there exists $\eta > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} I(tv) = t\|v\|^2 - \int_{\mathbb{R}^2} f(tv)v \, \mathrm{d}x - \int_{\mathbb{R}^2} hv \, \mathrm{d}x < 0,$$

for all $0 < t < \eta$. Using that $I(0) = 0$, it must hold that $I(tv) < 0$ for all $0 < t < \eta$. ■

Lemma 3.4 *Assume (f_1) or (f_2) and (1.2) (or (1.3)). Let (u_n) in E such that $I(u_n) \to c$ and $I'(u_n) \to 0$. Then*

$$\|u_n\| \leq C, \quad \int_{\mathbb{R}^2} f(u_n)u_n \, \mathrm{d}x \leq C \quad \textit{and} \quad \int_{\mathbb{R}^2} F(u_n) \, \mathrm{d}x \leq C.$$

Proof. We have

$$\frac{1}{2}\|u_n\|^2 - \int_{\mathbb{R}^2} F(u_n) \, \mathrm{d}x - \int_{\mathbb{R}^2} hu_n \, \mathrm{d}x = c + o_n(1),$$

and for any $\varphi \in E$

$$\int_{\mathbb{R}^N} (\nabla u_n \nabla \varphi + V(x)u_n\varphi) \, \mathrm{d}x - \int_{\mathbb{R}^N} f(u_n)\varphi \, \mathrm{d}x - \int_{\mathbb{R}^N} h\varphi \, \mathrm{d}x = o_n(\|\varphi\|). \quad (3.20)$$

By (f_1) or (f_2), we obtain

$$\begin{aligned} C + \epsilon_n \|u_n\| &\geq \left(\frac{\theta}{2} - 1\right) \|u_n\|^2 - \int_{\mathbb{R}^2} [\theta F(u_n) - f(u_n)u_n] \,\mathrm{d}x \\ &\geq \left(\frac{\theta}{2} - 1\right) \|u_n\|^2 - \int_{\{x \,:\, |u_n(x)| < s_1\}} [\theta F(u_n) - f(u_n)u_n] \,\mathrm{d}x, \end{aligned}$$

where $\epsilon_n \to 0$ as $n \to \infty$. Using that $|f(s)s - F(s)| \leq C_1|s|$ for all $|s| \leq s_1$ and inequality (1.7) we get

$$C + \epsilon_n \|u_n\| \geq \left(\frac{\theta}{2} - 1\right) \|u_n\|^2 - C_1 \|u_n\|,$$

which implies that $\|u_n\| \leq C$. The other estimates in the statement of the lemma follows easily. ■

For the next result, we will use the Radial Lemma (see [25] or [5, Lemma A.IV]) which asserts that if $u \in L^2(\mathbb{R}^2)$ and u^* is the Schwarz symmetrization of u, then for all $x \neq 0$

$$|u^*(x)| \leq \frac{1}{\sqrt{\pi}|x|} \|u^*\|_2.$$

Lemma 3.5 *Assume that f satisfies (f_2) and (1.3). If $(v_n) \subset E$ is a (P.-S.) sequence for I and u_0 is its weak limit then, up to a subsequence,*

$$F(v_n) \to F(u_0) \quad \textit{in} \quad L^1(\mathbb{R}^2).$$

Proof. As a consequence of Lemmas 2.8 and 3.4, for any $R > 0$ we get

$$f(v_n) \to f(u_0) \quad \text{in} \quad L^1(B_R).$$

Thus, there exists $g \in L^1(B_R)$ such that $|f(v_n)| \leq g$ almost everywhere in B_R. From (f_2) we can conclude that

$$|F(v_n)| \leq \sup_{v_n \in [-R_0, R_0]} |F(v_n)| + M_0 g \;\; \text{almost everywhere in} \;\; B_R.$$

Thus, by Lebesgue dominated convergence theorem

$$F(v_n) \to F(u_0) \;\; \text{in} \;\; L^1(B_R)$$

for all $R > 0$. Using (f_1) together with (3.15), we have

$$\int_{|x|>R} |F(v_n)| \,\mathrm{d}x \leq C_1 \int_{|x|>R} v_n^2 \,\mathrm{d}x + C_2 \int_{|x|>R} |v_n|(e^{\alpha v_n^2} - 1) \,\mathrm{d}x \qquad (3.21)$$

for $\alpha > \alpha_0$. Moreover,

$$\int_{|x|>R} |v_n|(e^{\alpha v_n^2} - 1)\ \mathrm{d}x = \sum_{j=1}^{\infty} \frac{\alpha^j}{j!} \int_{|x|>R} |v_n|^{2j+1}\ \mathrm{d}x$$
$$= \sum_{j=1}^{\infty} \frac{\alpha^j}{j!} \int_{|x|>R} |v_n^*|^{2j+1}\ \mathrm{d}x,$$

where v_n^* is the Schwarz symmetrization of v_n. Notice that using the estimate

$$\int_{|x|>R} \frac{1}{|x|^{2j+1}}\ \mathrm{d}x = 2\pi \int_R^{\infty} \frac{t}{t^{2j+1}}\ \mathrm{d}t = \frac{\pi}{j} R^{-2j} \le \frac{\pi}{R},\ \ j \ge 1$$

and Radial Lemma we achieve

$$\sum_{j=1}^{\infty} \frac{\alpha^j}{j!} \int_{|x|>R} |v_n^*|^{2j+1}\ \mathrm{d}x \le \frac{C}{\sqrt{\pi}} \sum_{j=1}^{\infty} \frac{\alpha^j}{j!} \left(\frac{1}{2\pi}\right)^j C^{2j} \int_{|x|>R} |x|^{-2j-1}\ \mathrm{d}x$$
$$\le \frac{C}{R}.$$

Then given $\delta > 0$ there exists $R > 0$ such that

$$\int_{|x|>R} |u_0|^2\ \mathrm{d}x < \delta \quad \text{and} \quad \int_{|x|>R} |v_n|(e^{\alpha |v_n|^2} - 1)\ \mathrm{d}x < \delta.$$

Thus, from (3.21) we conclude

$$\int_{|x|>R} |F(v_n)|\ \mathrm{d}x \le C\delta \quad \text{and} \quad \int_{|x|>R} |F(u_0)|\ \mathrm{d}x \le C\delta.$$

Since

$$\left| \int_{\mathbb{R}^2} F(v_n)\ \mathrm{d}x - \int_{\mathbb{R}^2} F(u_0)\ \mathrm{d}x \right| \le \left| \int_{B_R} F(v_n)\ \mathrm{d}x - \int_{B_R} F(u_0)\ \mathrm{d}x \right|$$
$$+ \int_{|x|>R} |F(v_n)|\ \mathrm{d}x + \int_{|x|>R} |F(u_0)|\ \mathrm{d}x,$$

we get

$$\lim_{n\to\infty} \left| \int_{\mathbb{R}^2} F(v_n)\ \mathrm{d}x - \int_{\mathbb{R}^2} F(u_0)\ \mathrm{d}x \right| \le C\delta.$$

Since δ is arbitrary, the lemma is proved. ■

4 Proof of the main results

In order to obtain a solution with negative energy, observe by Lemma 3.3 and inequality (3.19) that

$$-\infty < c_0 \equiv \inf_{\|u\| \le \eta} I(u) < 0. \tag{4.22}$$

In this subsection we will give the proof of Theorem 1.1. Thus we are assuming that V satisfies $(V_1)-(V_2)$ and f satisfies (f_0), (f_1) and (f_3). To prove the existence of a local minimum type solution we will use the Ekeland's variational principle.

Lemma 4.1 *The functional I satisfies the Palais-Smale condition.*

Proof. Let (u_n) be a $(P.-S.)$ sequence. By Lemma 3.4, (u_n) is bounded, so, up to subsequence, we may assume that $u_n \rightharpoonup u_0$ in E, $u_n \to u_0$ in $L^q(\mathbb{R}^2)$ for all $q \geq 1$ and $u_n(x) \to u_0(x)$ almost everywhere in $\mathbb{R}^2$. We claim that

$$\int_{\mathbb{R}^2} (f(u_n) - f(u_0))(u_n - u_0)\,\mathrm{d}x \to 0 \quad \text{as} \quad n \to \infty. \tag{4.23}$$

Indeed, using inequality (3.14), for all $\alpha > 0$ we obtain

$$|f(u_n) - f(u_0)||u_n - u_0| \leq C_1\Big[|u_n| + |u_0| + (e^{\alpha u_n^2} - 1) + (e^{\alpha u_0^2} - 1)\Big]|u_n - u_0|.$$

This together with the Hölder inequality and Lemmas 2.1 and 2.2 implies the claim (4.23). Now, observing that

$$\|u_n - u_0\|^2 = \langle I'(u_n) - I'(u_0), u_n - u_0\rangle + \int_{\mathbb{R}^2} (f(u_n) - f(u_0))(u_n - u_0)\,\mathrm{d}x.$$

We conclude that $u_n \to u_0$ and the result follows. ■

In view of Lemmas 3.1 and 3.2 we can apply the mountain-pass theorem to obtain the following result

Proposition 4.2 *There exists $\eta_1 > 0$ such that if $\|h\|_{H^{-1}} \leq \eta_1$ then the functional I has a critical point u_M at the minimax level*

$$c_M = \inf_{g\in\Gamma} \max_{t\in[0,1]} I(g(t)),$$

where

$$\Gamma = \{g \in C([0,1], E) : g(0) = 0,\ I(g(1)) < 0\}.$$

Proposition 4.3 *For each $h \in H^{-1}$ with $h \neq 0$, the equation (1.1) has a minimum type solution u_0 with $I(u_0) = c_0 < 0$, where c_0 is defined in (4.22).*

Proof. Let ρ_h be as in Lemma 3.1. Since $\overline{B}_{\rho_h}$ is a complete metric space with the metric given by the norm of E, convex and the functional I is of class C^1 and bounded below on $\overline{B}_{\rho_h}$, by Ekeland's variational principle there exists a sequence (u_n) in $\overline{B}_{\rho_h}$ such that

$$I(u_n) \to c_0 = \inf_{\|u\|\leq\rho_h} I(u) < 0 \ \text{ and } \ \|I'(u_n)\|_{E'} \to 0,$$

and the proof follows by Lemma 4.1. ∎

Proof of Theorem 1.1: The proof follows from Propositions 4.2 and 4.3.

4.2 Critical case

In order to get a more precise information about the minimax level obtained by the mountain-pass theorem, let us consider the following sequence of scaled and truncated Green's functions also considered by Moser (see [20]):

$$\widetilde{M}_n(x,r) = (2\pi)^{-1/2}\begin{cases} (\log n)^{1/2} & \text{if } |x| \le r/n \\ \dfrac{\log\left(\frac{r}{|x|}\right)}{(\log n)^{1/2}} & \text{if } r/n \le |x| \le r \\ 0 & \text{if } |x| \ge r. \end{cases}$$

Notice that $\widetilde{M}_n(\cdot,r) \in H^1(\mathbb{R}^2)$, $supp(\widetilde{M}_n(x,r)) = \overline{B}_r$,

$$\begin{aligned} &\int_{\mathbb{R}^2} |\nabla \widetilde{M}_n(x,r)|^2 \, dx = 1 \quad \text{and} \\ &\int_{\mathbb{R}^2} |\widetilde{M}_n(x,r)|^2 \, dx = O(1/\log n) \;\; \text{as} \;\; n \to \infty. \end{aligned} \tag{4.24}$$

Moreover, considering $M_n(x,r) = \widetilde{M}_n(x,r)/\|\widetilde{M}_n\|$, we can write

$$M_n^2(x,r) = (2\pi)^{-1}\log n + d_n, \quad \text{for all} \quad |x| \le r/n,$$

where $d_n = (2\pi)^{-1}\log n(\|\widetilde{M}_n\|^{-1} - 1)$. Using (4.24), we conclude that $\|\widetilde{M}_n\| \to 1$ as $n \to \infty$. Consequently,

$$\frac{d_n}{\log n} \to 0 \;\; \text{as} \;\; n \to \infty. \tag{4.25}$$

Lemma 4.4 *Suppose that $(f_2), (f_3), (f_4^+)$ hold. Then there exists $n \in \mathbb{N}$ such that*

$$\max_{t\ge 0}\left[\frac{t^2}{2} - \int_{\mathbb{R}^2} F(tM_n)\, dx\right] < \frac{2\pi}{\alpha_0}.$$

Proof. Let us fix $r > 0$ such that

$$\beta_0 > \frac{2}{r^2\alpha_0}, \tag{4.26}$$

where β_0 has been given in the assumption (f_4^+). Suppose, by contradiction, that for all n we have

$$\max_{t\ge 0}\left[\frac{t^2}{2} - \int_{\mathbb{R}^2} F(tM_n)\, dx\right] \ge \frac{2\pi}{\alpha_0}, \tag{4.27}$$

where $M_n(x) = M_n(x, r)$. In view of (1.11) we get

$$\int_{\mathbb{R}^2} F(tM_n)\ \mathrm{d}x \geq -C_1 + \int_{\{tM_n \geq s_1\}} F(tM_n)\ \mathrm{d}x \geq -C_1 + C_2 \int_{\{tM_n \geq s_1\}} e^{tM_n/M} \mathrm{d}x.$$

If $t > 0$ is sufficiently large and $m > 2$ we have

$$\int_{\{tM_n \geq s_1\}} e^{tM_n/M} \mathrm{d}x \geq C_3 t^m \int_{\{tM_n \geq s_1\}} (M_n)^m \mathrm{d}x \geq C_3 t^m \int_{\{M_n \geq s_1\}} (M_n)^m \mathrm{d}x.$$

Thus, for each n there exists $t_n > 0$ such that

$$\frac{t_n^2}{2} - \int_{\mathbb{R}^2} F(t_n M_n)\ \mathrm{d}x = \max_{t \geq 0} \left[\frac{t^2}{2} - \int_{\mathbb{R}^2} F(tM_n)\ \mathrm{d}x \right]. \tag{4.28}$$

Since at $t = t_n$ holds

$$\frac{d}{\mathrm{d}t} \left(\frac{t^2}{2} - \int_{\mathbb{R}^2} F(tM_n)\ \mathrm{d}x \right) = 0,$$

it follows that

$$t_n^2 = \int_{\mathbb{R}^2} t_n M_n f(t_n M_n)\ \mathrm{d}x = \int_{|x| \leq r} t_n M_n f(t_n M_n)\ \mathrm{d}x. \tag{4.29}$$

Now, using hypothesis (f_4^+), for each $\epsilon > 0$ there exists $R_\epsilon > 0$ such that

$$uf(u) \geq (\beta_0 - \epsilon) e^{\alpha_0 u^2} \quad \text{for all} \quad u \geq R_\epsilon \quad \text{and} \quad |x| \leq r. \tag{4.30}$$

From (4.29) and (4.30), for n large, we obtain

$$\begin{aligned} t_n^2 &\geq (\beta_0 - \epsilon) \int_{|x| \leq r/n} e^{\alpha_0 (t_n M_n)^2}\ \mathrm{d}x \\ &= (\beta_0 - \epsilon) \pi \left(\frac{r}{n} \right)^2 e^{\alpha_0 (2\pi)^{-1} \log n t_n + \alpha_0 t_n^2 d_n}. \end{aligned} \tag{4.31}$$

Thus,

$$1 \geq (\beta_0 - \epsilon) \pi r^2 e^{\alpha_0 (2\pi)^{-1} \log n t_n^2 + \alpha_0 t_n^2 d_n - 2 \log t_n - 2 \log n}.$$

Consequently, the sequence (t_n) is bounded.

We claim that

$$t_n^2 \to \frac{4\pi}{\alpha_0} \quad \text{as} \quad n \to \infty. \tag{4.32}$$

Indeed, condition (f_1) together with (4.27)-(4.28) imply that

$$\frac{t_n^2}{2} \geq \frac{2\pi}{\alpha_0} + \int_{\{t_n M_n \leq s_1\}} F(t_n M_n)\ \mathrm{d}x.$$

Since (t_n) is bounded, using (3.18) we obtain

$$\left| \int_{\{t_n M_n \leq s_1\}} F(t_n M_n)\, \mathrm{d}x \right| \leq C \int_{\mathbb{R}^2} M_n^2 \mathrm{d}x = C \frac{1}{\|\widetilde{M_n}\|^2} \int_{\mathbb{R}^2} \widetilde{M_n}^2 \mathrm{d}x.$$

By using (4.24) and the fact that $\|\widetilde{M_n}\| \to 1$, we obtain

$$\int_{\{t_n M_n \le s_1\}} F\left(t_n M_n\right)\, \mathrm{d}x = o_n(1).$$

Consequently

$$t_n^2 \ge \frac{4\pi}{\alpha_0} + o_n(1).$$

Now suppose by contradiction that $\lim_{n\to+\infty} t_n^2 > 4\pi/\alpha_0$. By (4.31) we get

$$t_n^2 \ge (\beta_0 - \epsilon)\pi r^2 e^{\left(\alpha_0(4\pi)^{-1}t_n^2 - 1\right)2\log n + \alpha_0 t_n^2 d_n}$$

which together with (4.25) contradicts the boundedness of (t_n) and the claim follows.

In order to estimate (4.29) more precisely, we consider the sets (see (4.29) and (4.30))

$$A_n = \{x \in B_r : t_n M_n(x) \ge R_\epsilon\} \quad \text{and} \quad C_n = B_r \setminus A_n.$$

From (4.29) and (4.30) we achieve

$$\begin{aligned} t_n^2 \ge (\beta_0 - \epsilon) \int_{|x|\le r} e^{\alpha_0 (t_n M_n)^2} \mathrm{d}x + \int_{C_n} t_n M_n f\left(t_n M_n\right)\, \mathrm{d}x \\ - (\beta_0 - \epsilon) \int_{C_n} e^{\alpha_0 (t_n M_n)^2} \mathrm{d}x. \end{aligned} \tag{4.33}$$

Notice that $M_n(x) \to 0$ and the characteristic functions $\chi_{C_n} \to 1$ for almost every x such that $|x| \le r$. Therefore, the Lebesgue dominated convergence theorem implies

$$\int_{C_n} t_n M_n f\left(t_n M_n\right)\, \mathrm{d}x \to 0 \quad \text{and} \quad \int_{C_n} e^{\alpha_0 (t_n M_n)^2}\, \mathrm{d}x \to \pi r^2 \ \text{ as } \ n \to \infty.$$

Since $t_n^2 \ge 4\pi/\alpha_0$, we also have

$$\begin{aligned} \int_{|x|\le r} e^{\alpha_0 (t_n M_n)^2}\, \mathrm{d}x &\ge \int_{|x|\le r} e^{4\pi M_n^2}\, \mathrm{d}x \\ &= \int_{|x|\le r/n} e^{4\pi M_n^2}\, \mathrm{d}x + \int_{r/n\le |x|\le r} e^{4\pi M_n^2}\, \mathrm{d}x. \end{aligned} \tag{4.34}$$

For the first integral in (4.34), we notice that

$$\begin{aligned} \int_{|x|\le r/n} e^{4\pi M_n^2}\, \mathrm{d}x &= \int_{|x|\le r/n} e^{2\log n + 4\pi d_n}\, \mathrm{d}x \\ &= \pi \frac{r^2}{n^2} n^{2+4\pi(\log n)^{-1}} d_n \to \pi r^2 \quad \text{as} \quad n \to \infty, \end{aligned}$$

where we have used (4.25).

For the second integral, using the change of variable $\tau = \log(r/s)/(\zeta_n \log n)$ with $\zeta_n = \|\widetilde{M}_n\| > 1$, we obtain

$$\int_{r/n \le |x| \le r} e^{4\pi M_n^2} \, \mathrm{d}x = 2\pi r^2 \zeta_n \log n \int_0^{\zeta_n^{-1}} e^{2\log n (t^2 - \zeta_n t)} \mathrm{d}t.$$

Since

$$t^2 - \zeta_n t \ge \begin{cases} -\zeta_n t & \text{if} \quad 0 \le t \le \dfrac{\zeta_n^{-1}}{2} \\ (2\zeta_n^{-1} - \zeta_n)(t - \zeta_n^{-1}) + (\zeta_n^{-2} - 1) & \text{if} \quad \dfrac{\zeta_n^{-1}}{2} \le t \le \zeta_n^{-1}, \end{cases}$$

By straightforward calculation we can see that

$$\lim_{n \to \infty} \int_{r/n \le |x| \le r} e^{4\pi M_n^2} \, \mathrm{d}x \ge 2\pi r^2.$$

Finally, taking $n \to \infty$ in (4.33) and using (4.32) we obtain

$$\frac{4\pi}{\alpha_0} \ge (\beta_0 - \epsilon) 2\pi r^2$$

which yields $\beta_0 \le 2/(\alpha_0 r^2)$, contradicting (4.26), and the proof is finished. ■

Corollary 4.5 *Under the hypotheses* (V_1) *and* $(f_2) - (f_4^+)$*, if* $\|h\|_{H^{-1}}$ *is sufficiently small then*

$$\max_{t \ge 0} I(tM_n) = \max_{t \ge 0} \left\{ \frac{t^2}{2} - \int_{\mathbb{R}^2} F(tM_n) \, \mathrm{d}x - t \int_{\mathbb{R}^2} hM_n \, \mathrm{d}x \right\} < \frac{2\pi}{\alpha_0}.$$

Proof. Notice that $\|hM_n\|_1 \le \|h\|_{H^{-1}}$. Thus, taking $\|h\|_{H^{-1}}$ sufficiently small and using Lemma 4.4 the result follows. ■

In order to obtain convergence results, we need to improve the estimate of Lemma 4.4.

Corollary 4.6 *Under the hypotheses* $(f_2) - (f_4^+)$*, there exists* $\delta_2 > 0$ *such that for all* $h \in H^{-1}$ *with* $0 < \|h\|_{H^{-1}} < \delta_2$ *there exists* $u \in H^1(\mathbb{R}^2)$ *with compact support verifying*

$$I(tu) < c_0 + \frac{2\pi}{\alpha_0}, \quad \textit{for all} \quad t \ge 0.$$

Proof. It is possible to raise the infimum c_0 by reducing $\|h\|_{H^{-1}}$. By Lemma 3.1, $\rho_h \to 0$ as $\|h\|_{H^{-1}} \to 0$. Consequently, c_0 increases as $\|h\|_{H^{-1}}$ decreases and $c_0 \to 0$ as $\|h\|_{H^{-1}} \to 0$. Thus, there exists $\delta_2 > 0$ such that if $0 < \|h\|_{H^{-1}} < \delta_2$

then, by Corollary 4.5, we have

$$\max_{t\geq 0} I(tM_n) < c_0 + \frac{2\pi}{\alpha_0}.$$

Taking $u = M_n \in H^1(\mathbb{R}^2)$, the result is proved. ■

Lemma 4.7 *If (u_n) is a (P.-S.) sequence for I at any level with*

$$\liminf_{n\to\infty} \|u_n\|^2 < \frac{4\pi}{\alpha_0},$$

then (u_n) possesses a subsequence which converges strongly to a solution u_0 of (1.1).

Proof. Since $\|u_n\|$ is bounded, up to a subsequence if necessary, we may assume that

$$\liminf_{n\to\infty} \|u_n\| = \lim_{n\to\infty} \|u_n\|.$$

By Lemma 3.4, we may assume that $u_n \rightharpoonup u_0$ weakly in E, $u_n \to u_0$ in $L^q(\mathbb{R}^2)$ for all $q \geq 1$ and $u_n(x) \to u_0(x)$ almost everywhere in $\mathbb{R}^2$. Moreover, by Lemma 2.8,

$$f(u_n) \to f(u_0) \quad \text{in} \quad L^1_{loc}(\mathbb{R}^2).$$

Passing to the limit in (3.20), we have

$$\int_{\mathbb{R}^2} (\nabla u_0 \nabla \varphi + V(x) u_0 \varphi)\, \mathrm{d}x - \int_{\mathbb{R}^2} f(u_0)\varphi \,\mathrm{d}x - \int_{\mathbb{R}^2} h\varphi \,\mathrm{d}x = 0,$$

for all $\varphi \in C_0^\infty(\mathbb{R}^2)$. Since $C_0^\infty(\mathbb{R}^2)$ is dense in E, we conclude that u_0 is a weak solution of (1.1).

We claim that $u_n \to u_0$. Indeed, writing $u_n = u_0 + w_n$, it follows that $w_n \rightharpoonup 0$ in E. Thus $w_n \to 0$ in $L^q(\mathbb{R}^2)$ for all $1 \leq q < \infty$. By the Brezis-Lieb Lemma (see [7]), we get

$$\|u_n\|^2 = \|u_0\|^2 + \|w_n\|^2 + o_n(1). \tag{4.35}$$

We first claim that

$$\int_{\mathbb{R}^2} f(u_n)u_0 \,\mathrm{d}x \to \int_{\mathbb{R}^2} f(u_0)u_0 \,\mathrm{d}x \quad \text{as} \quad n \to \infty. \tag{4.36}$$

In fact, since $C_0^\infty(\mathbb{R}^2)$ is dense in E, for all $\tau > 0$ there exists $\varphi \in C_0^\infty(\mathbb{R}^2)$ such that $\|\varphi - u_0\| < \tau$. Observe that

$$\begin{aligned}\left|\int_{\mathbb{R}^2} f(u_n)u_0 \,\mathrm{d}x - \int_{\mathbb{R}^2} f(u_0)u_0 \,\mathrm{d}x\right| \leq & \left|\int_{\mathbb{R}^2} f(u_n)(u_0 - \varphi) \,\mathrm{d}x\right| \\ & + \left|\int_{\mathbb{R}^2} f(u_0)(u_0 - \varphi) \,\mathrm{d}x\right| \\ & + \|\varphi\|_\infty \int_{supp\varphi} |f(u_n) - f(u_0)| \,\mathrm{d}x.\end{aligned}$$

To estimate the first integral we use that $|\langle I'(u_n), u_0-\varphi\rangle| \le \tau_n\|u_0-\varphi\|$ with $\tau_n \to 0$ and we conclude that

$$\begin{aligned}\left|\int_{\mathbb{R}^2} f(u_n)(u_0-\varphi)\ \mathrm{d}x\right| &\le \tau_n\|u_0-\varphi\| + \left(\int_{\mathbb{R}^2}|\nabla u_n|^2\ \mathrm{d}x\right)^{1/2}\|u_0-\varphi\| \\ &\quad + \left(\int_{\mathbb{R}^2} V(x)|u_n|^2\ \mathrm{d}x\right)^{1/2}\|u_0-\varphi\| \\ &\quad + \|h\|_{H^{-1}}\|u_0-\varphi\| \\ &\le C\|u_0-\varphi\| < C\tau,\end{aligned}$$

where C is independent of n and τ. Similarly, using that $\langle I'(u_0), u_0-\varphi\rangle = 0$, we can estimate the second integral obtaining

$$\left|\int_{\mathbb{R}^2} f(u_0)(u_0-\varphi)\ \mathrm{d}x\right| < C\tau.$$

To estimate the last integral we use that $f(u_n) \to f(u_0)$ in $L^1_{loc}(\mathbb{R}^2)$ and conclude by the previous inequalities that

$$\lim_{n\to\infty}\left|\int_{\mathbb{R}^2} f(u_n)u_0\ \mathrm{d}x - \int_{\mathbb{R}^2} f(u_0)u_0\ \mathrm{d}x\right| < 2C\tau;$$

this implies (4.36) because τ is arbitrary.

From (4.35) and (4.36), we can write

$$\langle I'(u_n), u_n\rangle = \langle I'(u_0), u_0\rangle + \|w_n\|^2 - \int_{\mathbb{R}^2} f(u_n)w_n\ \mathrm{d}x + o_n(1),$$

that is,

$$\|w_n\|^2 = \int_{\mathbb{R}^2} f(u_n)w_n\ \mathrm{d}x + o_n(1). \tag{4.37}$$

From (3.15), Hölder inequality and Lemma 2.2, for any $\alpha > \alpha_0$, we get

$$\begin{aligned}\left|\int_{\mathbb{R}^2} f(u_n)w_n\ \mathrm{d}x\right| &\le\ b_1\int_{\mathbb{R}^2}|u_n||w_n|\ \mathrm{d}x + b_2\int_{\mathbb{R}^2}(e^{\alpha u_n^2}-1)|w_n|\ \mathrm{d}x \\ &\le C_1\|w_n\|_2 + b_2\left[\int_{\mathbb{R}^2}(e^{\alpha\|u_n\|^2(u_n/\|u_n\|)^2}-1)^r\ \mathrm{d}x\right]^{1/r}\|w_n\|_p \\ &\le C_1\|w_n\|_2 + C_2\left[\int_{\mathbb{R}^2}(e^{\alpha q\|u_n\|^2(u_n/\|u_n\|)^2}-1)\ \mathrm{d}x\right]^{1/r}\|w_n\|_p.\end{aligned}$$

where $r > 1$, $p = r/(r-1)$ and $q > r$. By hypothesis, $\alpha_0\|u_n\|^2 < 4\pi$ for n sufficiently large. Now, we consider $\alpha > \alpha_0$ and $q > r$, with $r > 1$ close to 1, such that we still have $\alpha q\|u_n\|^2 < 4\pi$. Using Lemma 2.1 and the compact embedding (1.8), we conclude that

$$\int_{\mathbb{R}^2} f(u_n)w_n\ \mathrm{d}x \to 0.$$

This together with (4.37) implies that $\|w_n\| \to 0$ and the result follows. ∎

Next, we will prove the existence of a local minimum type solution.

Lemma 4.8 *For each $h \in H^{-1}$ with $0 < \|h\|_{H^{-1}} < \delta_1$, the equation (1.1) has a minimum type solution u_0 with $I(u_0) = c_0 < 0$, where c_0 is defined in (4.22).*

Proof. Let ρ_h be as in Lemma 3.1. We can choose $\|h\|_{H^{-1}}$ sufficiently small such that $\rho_h < (4\pi/\alpha_0)^{1/2}$. Since $\overline{B}_{\rho_h}$ is a complete metric space with the metric given by the norm of E, convex and the functional I is of class C^1 and bounded below on $\overline{B}_{\rho_h}$, by Ekeland's variational principle there exists a sequence (u_n) in $\overline{B}_{\rho_h}$ such that

$$I(u_n) \to c_0 = \inf_{\|u\| \leq \rho_h} I(u) \quad \text{and} \quad \|I'(u_n)\|_{E'} \to 0.$$

Observing that $\|u_n\|^2 \leq \rho_h^2 < 4\pi/\alpha_0$, by Lemma 4.7, there exists a subsequence of (u_n) which converges strongly to a solution u_0 of (1.1). Therefore, $I(u_0) = c_0 < 0$. ■

Lemma 4.9 *Under the assumptions $(V_1) - (V_2)$ and $(f_2) - (f_4^+)$, if $\|h\|_{H^{-1}} < \delta_1$ the problem (1.1) has a mountain-pass type solution u_M.*

Proof. By Lemmas 3.1 and 3.2 we have that I has a mountain-pass geometry. Thus, using the mountain-pass theorem without the (PS) condition (see [26]), there exists a sequence (u_n) in E satisfying

$$I(u_n) \to c_M > 0 \quad \text{and} \quad \|I'(u_n)\|_{E'} \to 0,$$

where c_M is the mountain-pass level. Now, by Lemma 3.4, the sequence (u_n) is bounded and using the density of $C_0^\infty(\mathbb{R}^2)$ in E, it follows that (u_n) converges weakly to a solution u_M of (1.1). ■

Remark 4.10 *By Corollary 4.6, we can conclude that*

$$0 < c_M < c_0 + \frac{2\pi}{\alpha_0}.$$

Proposition 4.11 *If $\delta_2 > 0$ is small enough, then the solutions of (1.1) obtained in Lemma 4.8 and Lemma 4.9 are distinct.*

Proof. By Lemmas 4.8 and 4.9, there exist sequences (u_n) and (v_n) in E such that

$$u_n \to u_0, \quad I(u_n) \to c_0 < 0, \quad \langle I'(u_n), u_n \rangle \to 0, \tag{4.38}$$

and

$$v_n \rightharpoonup u_M, \quad I(v_n) \to c_M > 0, \quad \langle I'(v_n), v_n \rangle \to 0. \tag{4.39}$$

Now, suppose by contradiction that $u_0 = u_M$. Since we also have $v_n \rightharpoonup u_0$ in $H^1(\mathbb{R}^2)$, up to subsequence, $\lim_{n\to\infty} \|v_n\|_{1,2} \geq \|u_0\|_{1,2} > 0$. Setting

$$w_n \doteq \frac{v_n}{\|v_n\|_{1,2}} \quad \text{and} \quad w_0 \doteq \frac{u_0}{\lim_{n\to\infty} \|v_n\|_{1,2}},$$

we get $\|w_n\|_{1,2} = 1$ and $w_n \rightharpoonup w_0$ in $H^1(\mathbb{R}^2)$.

Now, we consider two possibilities:
(i) $\|w_0\|_{1,2} = 1$ and (ii) $\|w_0\|_{1,2} < 1$.

If (i) happens, we have $\lim_{n\to\infty} \|v_n\|_{1,2} = \|u_0\|_{1,2}$, so that $v_n \to u_0$ in $H^1(\mathbb{R}^2)$. By Proposition 2.7, there exists $g \in H^1(\mathbb{R}^2)$ such that

$$|v_n| \leq g \ \text{ almost everywhere in } \ \mathbb{R}^2.$$

This together with (3.15) implies that

$$|f(v_n)v_n| \leq c_1|g|^2 + c_2|g|(e^{\alpha g^2} - 1) \ \text{ almost everywhere in } \ \mathbb{R}^2,$$

for each $\alpha > \alpha_0$. By Remark 2.3, the function $c_1|g|^2 + c_2|g|(e^{\alpha g^2} - 1) \in L^1(\mathbb{R}^2)$ and using Lebesgue dominated convergence theorem we conclude that

$$\int_{\mathbb{R}^2} f(v_n)v_n \, \mathrm{d}x \to \int_{\mathbb{R}^2} f(u_0)u_0 \, \mathrm{d}x.$$

Similarly,

$$\int_{\mathbb{R}^2} f(u_n)u_n \, \mathrm{d}x \to \int_{\mathbb{R}^2} f(u_0)u_0 \, \mathrm{d}x,$$

because $u_n \to u_0$ in E. Since

$$\langle I'(u_n), u_n\rangle = \|u_n\|^2 - \int_{\mathbb{R}^2} f(u_n)u_n \, \mathrm{d}x - \int_{\mathbb{R}^2} hu_n \, \mathrm{d}x \to 0$$

and

$$\langle I'(v_n), v_n\rangle = \|v_n\|^2 - \int_{\mathbb{R}^2} f(v_n)v_n \, \mathrm{d}x - \int_{\mathbb{R}^2} hv_n \, \mathrm{d}x \to 0,$$

we conclude that

$$\lim_{n\to\infty} \|v_n\|^2 = \lim_{n\to\infty} \|u_n\|^2 = \|u_0\|^2.$$

Therefore, $v_n \to u_0$ in E and consequently $I(v_n) \to I(u_0) = c_0$. This is a contradiction with (4.38) - (4.39).

Now, suppose that (ii) happens. We claim that there exists $\delta > 0$ such that

$$q\alpha_0\|v_n\|_{1,2}^2 \leq 4\pi \frac{1}{1 - \|w_0\|_{1,2}^2} - \delta \tag{4.40}$$

for n large. Indeed, by Remark 4.10, we have

$$\alpha_0 < \frac{2\pi}{c_M - I(u_0)}.$$

Thus, we can choose $q > 1$ sufficiently close to 1 and $\delta > 0$ such that

$$q\alpha_0\|v_n\|_{1,2}^2 \leq \frac{2\pi}{c_M - I(u_0)}\|v_n\|_{1,2}^2 - \delta.$$

Since $v_n \rightharpoonup u_0$, by Lemma 3.5 and the compactness embedding (1.8), up to a subsequence, we conclude that

$$\begin{aligned}&\frac{1}{2}\|v_n\|_{1,2}^2 = \\ c_M - \frac{1}{2}\lim_{n\to\infty}&\int_{\mathbb{R}^2} V(x)v_n^2 \mathrm{d}x + \int_{\mathbb{R}^2}\left[F(u_0) + hu_0 + \frac{1}{2}u_0^2\right]\mathrm{d}x + o_n(1).\end{aligned} \tag{4.41}$$

Thus, for n sufficiently large we get

$$\begin{aligned}&q\alpha_0\|v_n\|_{1,2}^2 \\ &\quad\leq 4\pi\frac{c_M - \frac{1}{2}\lim_{n\to\infty}\int_{\mathbb{R}^2} V(x)v_n^2 \mathrm{d}x + \int_{\mathbb{R}^2}\left[F(u_0) + hu_0 + \frac{1}{2}u_0^2\right]\mathrm{d}x + o_n(1)}{c_M - I(u_0)} - \delta.\end{aligned} \tag{4.42}$$

Notice that

$$\begin{aligned}&\left\{c_M - \frac{1}{2}\lim_{n\to\infty}\int_{\mathbb{R}^2} V(x)v_n^2 \mathrm{d}x + \int_{\mathbb{R}^2}\left[F(u_0) + hu_0 + \frac{1}{2}u_0^2\right]\mathrm{d}x\right\}(1 - \|w_0\|_{1,2}^2) \\ &= c_M - c_M\|w_0\|_{1,2}^2 - I(u_0) + \frac{1}{2}\|u_0\|_{1,2}^2 + \frac{1}{2}\int_{\mathbb{R}^2} V(x)u_0^2 \mathrm{d}x - \frac{1}{2}\lim_{n\to\infty}\int_{\mathbb{R}^2} V(x)v_n^2 \mathrm{d}x \\ &\quad - \left\{-\frac{1}{2}\lim_{n\to\infty}\int_{\mathbb{R}^2} V(x)v_n^2 \mathrm{d}x + \int_{\mathbb{R}^2}\left[F(u_0) + hu_0 + \frac{1}{2}u_0^2\right]\mathrm{d}x\right\}\|w_0\|_{1,2}^2 \\ &\leq c_M - I(u_0),\end{aligned}$$

where we have used that

$$\begin{gathered}\int_{\mathbb{R}^2}\left[F(u_0) + hu_0 + \frac{1}{2}u_0^2\right]\mathrm{d}x = -I(u_0) + \frac{1}{2}\|u_0\|_{1,2}^2 + \frac{1}{2}\int_{\mathbb{R}^2} V(x)u_0^2 \,\mathrm{d}x, \\ \int_{\mathbb{R}^2} V(x)u_0^2 \mathrm{d}x \leq \lim_{n\to\infty}\int_{\mathbb{R}^2} V(x)v_n^2 \mathrm{d}x,\end{gathered}$$

the equality (4.41) and the definition of w_0. This, together with (4.42) implies (4.40) for n large.

Now taking $p = (q+\epsilon)\alpha_0\|v_n\|_{1,2}^2$, it follows from (4.40) and Lemma 2.5 that

$$\int_{\mathbb{R}^2}(e^{(q+\epsilon)\alpha_0\|v_n\|_{1,2}^2|w_n|^2} - 1)\mathrm{d}x \leq C, \tag{4.43}$$

for $\epsilon > 0$ sufficiently small. Using (3.15), the Hölder inequality and the Sobolev embedding we get

$$\begin{aligned}\left|\int_{\mathbb{R}^2} f(v_n)(v_n - u_0)\,\mathrm{d}x\right| \leq& b_1\|v_n\|_2\|v_n - u_0\|_2 \\ &+ b_2\|v_n - u_0\|_{q'}\left[\int_{\mathbb{R}^2}(e^{\alpha_0\|v_n\|_{1,2}^2 w_n^2} - 1)^q \mathrm{d}x\right]^{1/q},\end{aligned}$$

where $q' = q/(q-1)$. Now using Lemma 2.2, estimate (4.43) and the compactness of the embedding (1.8), we obtain

$$\left| \int_{\mathbb{R}^2} f(v_n)(v_n - u_0)\, \mathrm{d}x \right| \leq C_1 \|v_n - u_0\|_2 + C_2 \|v_n - u_0\|_{q'} \to 0$$

as $n \to \infty$. This convergence together with the fact that $I'(v_n)(v_n - u_0) \to 0$ show that

$$\int_{\mathbb{R}^2} \nabla v_n (\nabla v_n - \nabla v_0)\, \mathrm{d}x + \int_{\mathbb{R}^2} V(x) v_n (v_n - v_0)\, \mathrm{d}x \to 0.$$

Since $v_n \rightharpoonup u_0$ we have

$$\int_{\mathbb{R}^2} \nabla u_0 (\nabla v_n - \nabla v_0)\, \mathrm{d}x + \int_{\mathbb{R}^2} V(x) u_0 (v_n - v_0)\, \mathrm{d}x \to 0.$$

Consequently, $v_n \to u_0$ in E. Thus $I(v_n) \to I(u_0) = c_0$, which contradicts (4.38) - (4.39). Therefore $u_0 \neq u_M$. ■

Now, the proof of Theorems 1.4 and 1.5 follows directly from Lemmas 4.8, 4.9 and Proposition 4.11.

4.3 Proof of Theorems 1.2 and 1.6:

In order to prove Theorems 1.2 and 1.6 in the case $h(x) \geq 0$, we redefine $f(s) = 0$ for $s < 0$. Thus, in the subcritical case (f_1) holds for $s \geq s_1$ and in the critical case (f_2) holds for $s \geq R_0$. Notice that hypotheses (f_1) and (f_2) was required to help verify the Palais-Smale condition and Lemmas 3.2, 3.4 and 3.5, which are valid also for this modified nonlinearity.

The proof is a consequence of the following result.

Corollary 4.12 *If $h(x) \geq 0$ almost everywhere in $\mathbb{R}^2$, then the weak solutions of (1.1) are nonnegative.*

Proof. Let $u \in E$ be a weak solution of (1.1). Setting $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$ and taking $v = u^-$ in (1.5), we obtain

$$\|u^-\|^2 = -\int_{\mathbb{R}^2} h u^- \mathrm{d}x \leq 0,$$

because $f(u(x))u^-(x) = 0$ in $\mathbb{R}^2$. Consequently, $u = u^+ \geq 0$. ■

Now, in the case $h(x) \leq 0$, in order to prove Theorems 1.2 and 1.6 let us define the following function

$$\widetilde{f}(s) = \begin{cases} -f(-s), & \text{if} \quad s < 0 \\ f(s), & \text{if} \quad s \geq 0. \end{cases}$$

In this case, the proof of Theorems 1.2 and 1.6 is given in the following corollary:

Corollary 4.13 *Suppose that* (f_4^-) *holds and* $h(x) \leq 0$ *almost everywhere in* $\mathbb{R}^2$*. Then there exist at least two nonpositive weak solutions of (1.1).*

Proof. Consider the functional defined by

$$\widetilde{I}(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^2} \widetilde{F}(u)\, \mathrm{d}x - \int_{\mathbb{R}^2} (-h)u\, \mathrm{d}x,$$

where $\widetilde{F}$ is the primitive of $\widetilde{f}$. Notice that $\widetilde{f}$ satisfies the same hypotheses of f. Since $-h(x) \geq 0$ almost everywhere in $\mathbb{R}^2$, by Corollary 4.12, $\widetilde{I}(u)$ has two nonnegative nontrivial critical points. Let $\tilde{u}$ be one such critical point, that is

$$\int_{\mathbb{R}^2} (\nabla\tilde{u}\nabla v + V(x)\tilde{u}v)\mathrm{d}x - \int_{\mathbb{R}^2} \widetilde{f}(\tilde{u})v\, \mathrm{d}x + \int_{\mathbb{R}^2} hv\, \mathrm{d}x = 0, \quad \forall\, v \in E.$$

Recalling the construction of $\widetilde{f}$, we have that $\widetilde{f}(\tilde{u}) = -f(-\tilde{u})$ and replace v by $-v$ in the last equality, we obtain

$$\int_{\mathbb{R}^2} [\nabla(-\tilde{u})\nabla v + V(x)(-\tilde{u})v]\mathrm{d}x - \int_{\mathbb{R}^2} f(-\tilde{u})v\, \mathrm{d}x - \int_{\mathbb{R}^2} hv\, \mathrm{d}x = 0, \quad \forall\, v \in E.$$

which implies that $-\tilde{u}$ is a nonpositive solution of (1.1). ∎

Remark 4.14 *Finally, we observe that the same procedures used in this paper, along with obvious modification, can be used to obtain analogous results for the problem of the form*

$$-\Delta u + V(x)u = f(x,u) + h(x), \quad x \in \mathbb{R}^2.$$

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