Available online at www.sciencedirect.com

ScienceDirect

J. Math. Anal. Appl. 342 (2008) 432–445

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

On the existence of signed and sign-changing solutions for a class of superlinear Schrödinger equations ☆

João Marcos do Ó *, Everaldo Medeiros, Uberlandio Severo

Departamento de Matemática, Universidade Federal da Paraíba, 58051-900 João Pessoa, PB, Brazil

Received 16 June 2007

Available online 14 December 2007

Submitted by P.J. McKenna

Abstract

This paper deals with a semilinear Schrödinger equation whose nonlinear term involves a positive parameter λ and a real function $f(u)$ which satisfies a superlinear growth condition just in a neighborhood of zero. By proving an a priori estimate (for a suitable class of solutions) we are able to avoid further restrictions on the behavior of $f(u)$ at infinity in order to prove, for λ sufficiently large, the existence of one-sign and sign-changing solutions. Minimax methods are employed to establish this result.

Keywords: Nonlinear eigenvalue problems; Schrödinger equations; Klein–Gordon equations; Standing waves; A priori estimate; Variational approach; Minimax methods

1. Introduction

In this paper we are concerned with *nonlinear Schrödinger equations* of the form

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi + W(x)\psi - \lambda|\psi|^{-1}g(|\psi|)\psi \quad \text{in } \mathbb{R}^N,$$

or *nonlinear equations of the Klein–Gordon* type

$$\frac{\partial^2\psi}{\partial t^2} = \Delta\psi + W(x)\psi - \lambda|\psi|^{-1}g(|\psi|)\psi \quad \text{in } \mathbb{R}^N,$$

where $\psi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, λ is a positive parameter, $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential and $g : \mathbb{R} \to \mathbb{R}$ is a nonlinear term. Here our special interest is in the existence of standing wave solutions, namely, solutions of type

$$\psi(t,x) = \exp(-iEt)u(x),$$

☆ Research partially supported by the Millennium Institute for the Global Advancement of Brazilian Mathematics IM-AGIMB, CNPq/Brazil Grants # 200648/2006-3, 473929/2006-6 and 471111/2006-6, and CAPES/MEC/Brazil.

* Corresponding author.

E-mail addresses: jmbo@mat.ufpb.br (J.M. do Ó), everaldo@mat.ufpb.br (E. Medeiros), uberlandio@mat.ufpb.br (U. Severo).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.11.058

where $E \in \mathbb{R}$ and $u > 0$ is a real function. It is known that if we seek for standing wave solutions we are led to look for solutions of nonlinear elliptic equations of the form

$$-\Delta u + V(x)u = \lambda f(u) \quad \text{in } \mathbb{R}^N, \tag{P_λ}$$

where $V : \mathbb{R}^N \to \mathbb{R}$ is the new potential for which is assumed to be uniformly positive, λ is a positive parameter and $f : \mathbb{R} \to \mathbb{R}$ is the new nonlinearity. Such equations arise in various branches of mathematical physics and mathematical biology and they have been subject of extensive study in the past years, among others we refer to [8,9,18,25,28–30] and references therein.

The main purpose of the present paper is to establish the existence of signed and sign-changing solutions for problem (P_λ) with the nonlinearity $f(u)$ satisfying a superlinear growth condition just in a neighborhood of zero. This result can be considered as an extension of the main result in [13] concerning the nonlinear eigenvalue problem $-\Delta u = \lambda f(u)$ with homogeneous Dirichlet boundary condition on a bounded domain $\Omega \subset \mathbb{R}^N$. See also [11] where the authors proved a multiplicity result for this class of problems. For related results under global superlinear conditions we refer to [1,14,22–24] and references therein. The approach proposed here is in the spirit of [25] and based on a global variational point of view. Throughout the paper, we assume the following basic hypotheses on the potential:

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{x \in \mathbb{R}^N} V(x) > 0$.

We consider the situation in which the potential $V(x)$ is possibly unbounded from above and also the case when actually the potential is "large" at infinity. Indeed, we prove the existence under either of the following assumptions on the potential:

(V_2) $V(x) \to \infty$ as $|x| \to \infty$; or more generally, for every $M > 0$, the set
$$\{x \in \mathbb{R}^N:\ V(x) \leqslant M\}$$
has finite Lebesgue measure;

(V_3) The function $[V(x)]^{-1}$ belongs to $L^1(\mathbb{R}^N)$.

We assume that $f : \mathbb{R} \to \mathbb{R}$ is a function of class C^1 with primitive $F(s) = \int_0^s f(t)\,dt$ satisfying the following conditions:

(f_1) there exists $p \in (2, 2^*)$ such that
$$\limsup_{|s| \to 0} \frac{f(s)s}{|s|^p} < +\infty;$$

(f_2) there exists $q \in (2, 2^*)$ such that
$$\liminf_{|s| \to 0} \frac{F(s)}{|s|^q} > 0;$$

(f_3) there exists $\theta \in (2, 2^*)$ such that
$$0 < \theta F(s) \leqslant s f(s) \quad \text{for } |s| \neq 0 \text{ small}.$$

Here $N \geqslant 3$, $2^* = 2N/(N-2)$ is the critical Sobolev exponent. Of course, it follows from (f_1) and (f_2) that $p \leqslant q$. Assumptions like (f_1)–(f_3) were already used in [11] and [13] in order to prove multiplicity results for a class of nonlinear eigenvalue problems on a bounded domain. Condition (f_3) is a local version of the classical Ambrosetti–Rabinowitz condition.

A main difficulty in treating this class of semilinear Schrödinger equations (P_λ) is the possible lack of compactness due to the unboundedness of the domain besides that ours assumptions on the nonlinear term $f(u)$ refer solely to its behavior in neighborhood of zero. By using minimax methods and proving an a priori estimate (for a suitable class of solutions) we are able to avoid further restrictions on the behavior of $f(u)$ at infinity in order to prove, for λ sufficiently large, the existence of three solutions of problem (P_λ). Next, we state our main result in a more precise way.

Theorem 1.1. *Assume* (V_1)–((V_2) *or* (V_3)) *and* (f_1)–(f_3). *Then problem* (P_λ) *has at least one positive solution, one negative solution and a sign-changing solution for all* λ *sufficiently large.*

Example 1.2. Note that the hypotheses of our main result are satisfied by nonlinear functions of the form

(a) $f(s) = s|s|^{\alpha-1}\ln(1+|s|)$ with $0<\alpha<1$ and $\alpha+2<2^*$. Since $\lim_{s\to 0} sf(s)/F(s) = \alpha+2$, $\lim_{|s|\to\infty} sf(s)/F(s) = \alpha+1$ and $\lim_{s\to+\infty} f(s)/s = 0$ we see that this nonlinear function satisfies the Ambrosetti–Rabinowitz condition (f_3) near the origin but does not satisfy the usual global superlinear condition.
(b) $f \in C^1(\mathbb{R})$ such that $f(s) = |s|^{p-2}s$ for $|s| \leqslant 1$ and $f(s) = e^{s^2}$ for $|s| \geqslant 2$ with $p \in (2, 2^*)$.
(c) $f(s) = a|s|^{p-2}s + b|s|^{r-2}s$ where $2<p<2^*<r$ and a, b are positive constants.
(d) $f(s) = a|s|^{p-2}s + b|s|^{q-2}se^{s^2}$ where $2<p<q<2^*$ and a, b are positive constants.

Here, the nonlinear functions given in (b)–(d) do not satisfy the usual global subcritical conditions.

Remark 1.3. It is readily seen that using classical regularity arguments for elliptic equations one can see that weak solutions of (P_λ) are indeed classical (see [16]).

The existence of a positive and a negative solution for the semilinear elliptic partial differential equation $-\Delta u + V(x)u = f(x,u)$ in $\mathbb{R}^N$ can be found in [25] provided that $V(x)\to\infty$ as $|x|\to\infty$, and $f(x,u)$ is subcritical and superlinear. In [6], among other things, the authors weakened the conditions on the potential V and still obtained a positive and a negative solution. These results have been generalized in [5] and [7] where the existence of a sign changing solution has been obtained. Note that sign-changing solutions for elliptic semilinear problems have attracted much attention in the last decade and numerous papers were published; see, for instance [4,15,20,22,26,27,33] and references therein. To study sign-changing solutions, several authors have established an abstract critical point theory in partially ordered Hilbert space. The methods and the abstract critical point theory of [2,18] involve the density of the Banach space $C(\Omega)$ of continuous functions in the Hilbert space $H_0^1(\Omega)$, where the cone of the positive functions has nonempty interior and this framework imposes stronger hypotheses on the nonlinearity and the domain. Indeed, it is required the boundedness of the domain and the stronger smoothness on the nonlinearity. The existence of sign changing solutions using properties of invariants set of descending flow defined by a pseudogradient field has been investigated by several authors (see [4,19,26]). Our results can also be considered as an extension of the above mention papers in the sense that we are considering only superlinear conditions in a neighborhood of the origin. Finally we mention that existence of sign-changing solutions for problems involving the p-Laplacian was studied recently in [3], and see also [32] for Kirchhoff type problems.

The outline of the paper is as follows: In the forthcoming section we have the modified problem and some preliminary results. In the third section we shall deal with the existence of signed solutions, while the fourth section is devoted to prove the existence of a sign-changing solution by using a linking type theorem together with an appropriated energy estimate.

Notation. In this paper we make use of the following notation:

- $C, C_0, C_1, C_2, \ldots$ denote positive (possibly different) constants.
- B_R denotes the open ball centered at origin and radius $R>0$.
- $C_0^\infty(\mathbb{R}^N)$ denotes the functions infinitely differentiable with compact support in $\mathbb{R}^N$.
- For $1 \leqslant p \leqslant \infty$, $L^p(\mathbb{R}^N)$ denotes the usual Lebesgue space with norms

$$|u|_p := \left(\int_{\mathbb{R}^N} |u|^p\,dx\right)^{1/p}, \quad 1 \leqslant p < \infty;$$

$$|u|_\infty := \inf\bigl\{C>0:\ \bigl|u(x)\bigr| \leqslant C \text{ almost everywhere on } \mathbb{R}^N\bigr\}.$$

- $H^1(\mathbb{R}^N)$ denotes the Sobolev space modeled in $L^2(\mathbb{R})$ with its usual norm

$$\|u\|_{1,2} := \bigl(|\nabla u|_2^2 + |u|_2^2\bigr)^{1/2}.$$

- By $\langle\cdot,\cdot\rangle$ we denote the duality pairing between X and its dual X'.
- We denote the weak convergence in X and X' by "$\rightharpoonup$" and the strong convergence by "$\to$."

2. Preliminaries and reformulation of the problem

At this stage, in order to apply variational methods, we consider the subspace of $H^1(\mathbb{R}^N)$,

$$E=\left\{u\in H^1(\mathbb{R}^N):\ \int_{\mathbb{R}^N} V(x)u^2\,dx<\infty\right\},$$

which is a Hilbert space when endowed with the inner product

$$\langle u,v\rangle=\int_{\mathbb{R}^N}\big(\nabla u\nabla v+V(x)uv\big)\,dx,\quad u,v\in E,$$

and its correspondent norm $\|u\|=\langle u,u\rangle^{1/2}$.

Notice that, under assumption (V_1), for all $2\leqslant r\leqslant 2^*$ we have

$$E\hookrightarrow L^r(\mathbb{R}^N)$$

with continuous embedding and with compact embedding if $2\leqslant r<2^*$ and V satisfies condition (V_2) or (V_3) (these facts can be found in [6,12,17,21]).

Remark 2.1. It is readily seen that our method applies to other potentials, although we focus our attention on the case where the potential satisfies condition (V_2) or (V_3); this does not require significant changes in our argument.

We observe that formally (P_λ) is the Euler–Lagrange equation associated to the following functional:

$$\Psi_\lambda(u)=\frac{1}{2}\|u\|^2-\lambda\int_{\mathbb{R}^N}F(u)\,dx.$$

From the variational point of view, the first difficulty we have to deal with this problem, is the fact that since (f_1)–(f_3) give the behavior of $f(s)$ just in a neighborhood of zero, the functional Ψ_λ is not well defined in E. To overcome this difficulty we use here a penalization technique in the spirit of the argument developed by Costa and Wang in [13] to obtain a new functional well defined in E. To this end, we first observe that (f_1) and (f_2) imply the existence of positive constants C_0, C_1 such that for $|s|$ small,

$$F(s)\leqslant C_0|s|^p \tag{2.1}$$

and

$$F(s)\geqslant C_1|s|^q. \tag{2.2}$$

Let $\rho(s)$ be an even cut-off function verifying $s\rho'(s)\leqslant 0$, $|s\rho'(s)|\leqslant 2/\delta$ and

$$\rho(s)=\begin{cases}1 & \text{if } |s|\leqslant\delta,\\ 0 & \text{if } |s|\geqslant 2\delta,\end{cases}$$

where δ is chosen such that (2.1), (2.2) and (f_3) hold for $|s|\leqslant 2\delta$.

Now, setting

$$\begin{cases}F_\infty(s)=C_0|s|^p,\\ G(s)=\rho(s)F(s)+\big(1-\rho(s)\big)F_\infty(s),\\ g(s)=G'(s),\end{cases}$$

we introduce the auxiliary problem

$$-\Delta u+V(x)u=\lambda g(u)\quad\text{in }\mathbb{R}^N, \tag{2.3}$$

with variational structure. More precisely, weak solutions of (2.3) are critical points of the C^2 functional $I_\lambda : E \to \mathbb{R}$,

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 - \lambda \int_{\mathbb{R}^N} G(u)\,dx.$$

This fact is a consequence of the following result (see also [10,24,25] and [31] for regularity properties of the associated functional).

Lemma 2.2.

(1) *There exists $C > 0$ such that*

$$\left|g(s)\right| \leqslant C|s|^{p-1} \quad \textit{for all } s \in \mathbb{R}. \tag{2.4}$$

(2) *The Ambrosetti–Rabinowitz condition*:

$$0 < \alpha G(s) \leqslant s g(s) \quad \textit{for all } s \in \mathbb{R} \setminus \{0\},$$

where $\alpha = \min\{p, \theta\}$.

Proof. If $|s| \leqslant \delta$, we have $G(s) = F(s)$. It follows from (f_1) and (f_3) that

$$\left|g(s)\right| = \left|F'(s)\right| = \left|f(s)\right| \leqslant C_1|s|^{p-1}.$$

For $|s| \geqslant 2\delta$, we have $G(s) = F_\infty(s) = C_0|s|^p$, consequently $|g(s)| \leqslant C_2|s|^{p-1}$. By definition

$$g(s) = \rho(s) f(s) + \rho'(s)\bigl(F(s) - F_\infty(s)\bigr) + \bigl(1 - \rho(s)\bigr)F'_\infty(s). \tag{2.5}$$

Since $|\rho'(s)s| \leqslant \frac{2}{\delta}$, by (2.1) we get $|\rho'(s)F(s)| \leqslant C_3|s|^{p-1}$ for all $\delta \leqslant |s| \leqslant 2\delta$. Choosing $C = \max\{C_1, C_2, C_3\}$, we obtain (2.4).

In order to prove (2) we observe that for $\alpha = \min\{p, \theta\}$ and $|s| \leqslant 2\delta$ we have

$$\begin{aligned}
\alpha G(s) &= \alpha\rho(s)F(s) + \alpha\bigl(1 - \rho(s)\bigr)F_\infty(s)\\
&\leqslant \frac{\alpha}{\theta}\rho(s)sf(s) + \frac{\alpha}{p}\bigl(1 - \rho(s)\bigr)sF'_\infty(s)\\
&\leqslant \rho(s)sf(s) + \bigl(1 - \rho(s)\bigr)sF'_\infty(s)
\end{aligned}$$

which together with (2.5) implies that

$$\alpha G(s) - sg(s) \leqslant \rho'(s)s\bigl(F_\infty(s) - F(s)\bigr) \leqslant 0,$$

since $\rho'(s)s \leqslant 0$ and $F_\infty(s) \geqslant F(s)$. This shows (2) for $|s| \leqslant 2\delta$. If $|s| \geqslant 2\delta$ the inequality (2) is immediate and the proof is finished. □

Lemma 2.3. *The functional I_λ exhibits the mountain-pass geometry*:

(1) *There exist $\rho > 0$ and $C = C(\rho, \lambda) > 0$ such that*

$$I_\lambda(u) \geqslant C \quad \textit{for } \|u\| = \rho.$$

(2) *There exists $e \in E$ with $\|e\| > \rho$ and $I_\lambda(e) \leqslant 0$.*

Proof. Using Lemma 2.2 together with the Sobolev embedding we obtain

$$I_\lambda(u) \geqslant \frac{1}{2}\|u\|^2 - \lambda C_2|u|_p^p \geqslant \|u\|^2\left(\frac{1}{2} - \lambda C_4\|u\|^{p-2}\right),$$

for $\|u\| = \rho$ sufficiently small. Hence (1) holds, since $p > 2$.

Next we prove (2). Using property (2) in Lemma 2.2 we obtain

$$G(u) \geqslant C_1|u|^\alpha - C_2 \quad \text{for all } s \in \mathbb{R}.$$

Thus, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$ and $t > 0$, we have

$$I_\lambda(t\varphi) \leqslant \frac{t^2}{2}\|\varphi\|^2 - \lambda C_1 t^\alpha |\varphi|_\alpha^\alpha - C_2,$$

which implies that $I_\lambda(t\varphi) \to -\infty$ as $t \to +\infty$, since $\alpha > 2$. Hence $I_\lambda(e) \leqslant 0$ for $e = t\varphi$ and t large enough. □

In order to show that solutions of *penalized problem* (2.3) are solutions of the original problem (P_λ), we will use the following L^∞ estimate.

Lemma 2.4. *If $u \in E$ is a weak solution of problem* (2.3)*, then $u \in L^\infty(\mathbb{R}^N)$. Moreover, there exists $C = C(p, N) > 0$ such that*

$$|u|_\infty \leqslant C\big(\lambda\|u\|^{p-2}\big)^{1/(2^*-p)}\|u\|. \tag{2.6}$$

Proof. Let $u \in E$ be a weak solution of problem (2.3), that is,

$$\int_{\mathbb{R}^N} \nabla u \nabla\varphi\, dx + \int_{\mathbb{R}^N} V(x)u\varphi\, dx = \lambda\int_{\mathbb{R}^N} g(u)\varphi\, dx, \quad \varphi \in E. \tag{2.7}$$

We can assume, without lost of generality, that u is nonnegative. Otherwise, we argue with the positive and negative parts of u separately. For each $k > 0$, we define $v_k = u_k^{2(\beta-1)}u$ and $w_k = uu_k^{\beta-1}$ with $\beta > 1$ to be determined later, where

$$u_k = \begin{cases} u & \text{if } u \leqslant k, \\ k & \text{if } u \geqslant k. \end{cases}$$

Notice that $0 \leqslant u_k \leqslant u$, $\nabla u_k \nabla u \geqslant 0$ and $|\nabla u_k| \leqslant |\nabla u|$. Taking v_k as a test function in (2.7) and using (2.4), we obtain

$$\int_{\mathbb{R}^N} u_k^{2(\beta-1)}|\nabla u|^2\, dx \leqslant -\int_{\mathbb{R}^N} V(x)u_k^{2(\beta-1)}u^2\, dx - 2(\beta-1)\int_{\mathbb{R}^N} u_k^{2(\beta-1)-1}u\nabla u_k\nabla u\, dx + C\lambda\int_{\mathbb{R}^N} u^p u_k^{2(\beta-1)}\, dx.$$

Now, observing that the first and the second terms in the right-hand side of the inequality above are nonpositive, we have

$$\int_{\mathbb{R}^N} u_k^{2(\beta-1)}|\nabla u|^2\, dx \leqslant \lambda C\int_{\mathbb{R}^N} u^p u_k^{2(\beta-1)}\, dx = \lambda C\int_{\mathbb{R}^N} u^{p-2}w_k^2\, dx.$$

This together with the Gagliardo–Nirenberg–Sobolev inequality implies that

$$\begin{aligned}
\bigg(\int_{\mathbb{R}^N} w_k^{2^*}\, dx\bigg)^{2/2^*} &\leqslant C_1\int_{\mathbb{R}^N} |\nabla w_k|^2\, dx \\
&\leqslant C_2\int_{\mathbb{R}^N} \big[u_k^{2(\beta-1)}|\nabla u|^2\, dx + (\beta-1)^2u^2u_k^{2(\beta-2)}|\nabla u_k|^2\big]\, dx \\
&\leqslant C_4\beta^2\int_{\mathbb{R}^N} u_k^{2(\beta-1)}|\nabla u|^2\, dx \\
&\leqslant \lambda C_5\beta^2\int_{\mathbb{R}^N} u^{p-2}w_k^2\, dx,
\end{aligned}$$

where we have used that $1+(\beta-1)^2 \leqslant \beta^2$ for $\beta \geqslant 1$. Using the Hölder inequality, we get

$$\Big(\int_{\mathbb{R}^N} w_k^{2^*}\,dx\Big)^{2/2^*} \leqslant \lambda\beta^2 C_5\Big(\int_{\mathbb{R}^N} u^{2^*}\,dx\Big)^{(p-2)/2^*}\Big(\int_{\mathbb{R}^N} w_k^{22^*/(2^*-p+2)}\,dx\Big)^{(2^*-p+2)/2^*}.$$

Observing that $|w_k| \leqslant |u|^\beta$ and since the embedding $E \hookrightarrow L^{2^*}(\mathbb{R}^N)$ is continuous, we conclude that

$$\Big(\int_{\mathbb{R}^N} \big|uu_k^{\beta-1}\big|^{2^*}\,dx\Big)^{2/2^*} \leqslant \lambda\beta^2 C_6\|u\|^{p-2}\Big(\int_{\mathbb{R}^N} u^{\beta 22^*/(2^*-p+2)}\,dx\Big)^{(2^*-p+2)/2^*}.$$

Choosing $\beta = 1+(2^*-p)2^{-1}$, we have $2\beta(2^*-p+2)^{-1}=1$. Thus,

$$\Big(\int_{\mathbb{R}^N} \big|uu_k^{\beta-1}\big|^{2^*}\,dx\Big)^{2/2^*} \leqslant \lambda\beta^2 C_6\|u\|^{p-2}|u|^{2\beta}_{\beta\alpha^*},$$

where $\alpha^* = 2(2^*)(2^*-p+2)^{-1}$. By the Fatou's lemma in k, we obtain

$$|u|_{\beta 2^*} \leqslant \big(\lambda\beta^2 C_6\|u\|^{p-2}\big)^{1/2\beta}|u|_{\beta\alpha^*}. \tag{2.8}$$

Taking $\beta_0=\beta$ and inductively $\beta_{m+1}\alpha_* = 2^*\beta_m$ for $m=1,2,\dots,$ and applying the previous processes for β_1, by (2.8) we have

$$\begin{aligned}|u|_{\beta_1 2^*} &\leqslant \big(\lambda\beta_1^2 C_6\|u\|^{p-2}\big)^{1/2\beta_1}|u|_{\beta_1\alpha^*}\\ &\leqslant \big(\lambda\beta_1^2 C_6\|u\|^{p-2}\big)^{1/2\beta_1}\big(\lambda\beta^2 C_6\|u\|^{p-2}\big)^{1/2\beta}|u|_{\beta\alpha^*}\\ &\leqslant \big(\lambda C_6\|u\|^{p-2}\big)^{1/2\beta_1+1/2\beta}(\beta)^{1/\beta}(\beta_1)^{1/\beta_1}|u|_{2^*}.\end{aligned}$$

Observing that $\beta_m = \chi^m\beta$ where $\chi = 2^*/\alpha^*$, by iteration we obtain

$$|u|_{\beta_m 2^*} \leqslant \big(\lambda C_6\|u\|^{p-2}\big)^{1/2\beta\sum_{i=0}^m \chi^{-i}}\beta^{1/\beta\sum_{i=0}^m \chi^{-i}}\chi^{1/\beta\sum_{i=0}^m i\chi^{-i}}|u|_{2^*}.$$

Since $\chi > 1$ and

$$\lim_{m\to\infty}\frac{1}{2\beta}\sum_{i=0}^m \chi^{-i} = \frac{1}{2^*-p},$$

we can take the limit as $m\to\infty$ to get

$$|u|_\infty \leqslant C_7\big(\lambda\|u\|^{p-2}\big)^{1/(2^*-p)}\|u\|.$$

Thus, the proof is completed. $\square$

In order to get estimates on the critical level of the functional I_λ, important role will be played by the following energy functional:

$$J_\lambda(u) = \frac{1}{2}\|u\|^2 - \lambda\int_{\mathbb{R}^N} C_1|u|^q\,dx,$$

where $q\in(2,2^*)$ and C_1 is the positive constant given in (2.2). It is clear that J_λ is a C^1-functional and enjoys the mountain-pass geometry. Moreover, in view of the compact embedding $E \hookrightarrow L^q(\mathbb{R}^N)$ we can apply the mountain-pass theorem (see [1,10,24,25,31]) to conclude that the minimax level

$$b_\lambda = \inf_{u\neq 0}\max_{t\geqslant 0} J_\lambda(tu)$$

is a critical value of J_λ. Besides, we have the following estimate.

Lemma 2.5. *There exists $C = C(N, q, V) > 0$ independent of λ such that*

$$b_\lambda \leqslant C\lambda^{-2/(q-2)}. \tag{2.9}$$

Proof. Using one more time the compact embedding $E \hookrightarrow L^q(\mathbb{R}^N)$ it is easy to see that the infimum

$$S_V(\mathbb{R}^N) := \inf\left\{\frac{\|u\|^2}{|u|_q^2}\colon u \in E \setminus \{0\}\right\}$$

is attained. Let $u_0 \in E$ be such that

$$S_V(\mathbb{R}^N) = \frac{\|u_0\|^2}{|u_0|_q^2}.$$

Then we can obtain through straightforward calculations that

$$\max_{t\geqslant 0} J_\lambda(tu_0) = \left(\frac{q-2}{2q}\right)(C_1\lambda)^{-2/(q-2)}\left(\frac{\|u_0\|^2}{|u_0|_q^2}\right)^{q/(q-2)}.$$

Consequently, $b_\lambda \leqslant C_2\lambda^{-2/(q-2)} S_V(\mathbb{R}^N)^{q/(q-2)}$, which completes the proof. $\square$

The next result will be used in several arguments through this paper.

Lemma 2.6. *If $u \in E$ is a critical point of I_λ, then*

$$\|u\|^2 \leqslant \frac{2\alpha}{\alpha-2} I_\lambda(u). \tag{2.10}$$

Proof. Let $u \in E$ be a critical point of I_λ. By Lemma 2.2 we have

$$\|u\|^2 = \lambda \int_{\mathbb{R}^N} g(u)u\,dx \geqslant \lambda\alpha \int_{\mathbb{R}^N} G(u)\,dx$$

which implies that

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 - \lambda \int_{\mathbb{R}^N} G(u)\,dx \geqslant \left(\frac{1}{2} - \frac{1}{\alpha}\right)\|u\|^2,$$

and (2.10) is proved. $\square$

3. Signed solution via mountain-pass

In order to obtain positive and negative solutions of (P_λ) we consider the following auxiliary functions:

$$G_1(s) = \begin{cases} G(s) & \text{if } s > 0, \\ 0 & \text{if } s \leqslant 0, \end{cases} \quad \text{and} \quad G_2(s) = \begin{cases} G(s) & \text{if } s < 0, \\ 0 & \text{if } s \geqslant 0, \end{cases}$$

and the associated functionals

$$I_{i,\lambda}(u) = \frac{1}{2}\|u\|^2 - \lambda \int_{\mathbb{R}^N} G_i(u)\,dx \quad \text{for } i = 1, 2.$$

From Lemma 2.3, applying the mountain-pass theorem (see [1,10,24,25,31]), there exists a sequence $(u_{i,\lambda}^n) \subset E$ such that

$$I_{i,\lambda}(u_{i,\lambda}^n) \to c_{i,\lambda} \quad \text{and} \quad \|I'_{i,\lambda}(u_{i,\lambda}^n)\|_{E'} \to 0,$$

where $c_{i,\lambda}$ is the mountain-pass level

$$c_{i,\lambda} = \inf_{h\in\Gamma_i} \max_{t\in[0,1]} I_{i,\lambda}(h(t))$$

and

$$\Gamma_i = \{h \in C([0,1], E)\colon\ h(0) = 0,\ I_{i,\lambda}(h(1)) < 0\}$$

for $i = 1, 2$. It is straightforward to verify as consequence of Lemma 2.2 that the sequence above is bounded in E and since the embedding $E \hookrightarrow L^r(\mathbb{R}^N)$ is compact for $2 \leqslant r < 2^*$, so that passing to the subsequence if necessary, we can assume that $u^n_{i,\lambda} \rightharpoonup u_{i,\lambda}$ weakly in E and $u^n_{i,\lambda} \to u_{i,\lambda}$ in $L^r(\mathbb{R}^N)$. Observe that

$$\|u^n_{i,\lambda} - u_{i,\lambda}\|^2 = \langle I'_{i,\lambda}(u^n_{i,\lambda}) - I'_{i,\lambda}(u_{i,\lambda}), u^n_{i,\lambda} - u_{i,\lambda}\rangle + \int\limits_{\mathbb{R}^N} \big(g_i(u^n_{i,\lambda}) - g_i(u_{i,\lambda})\big)\big(u^n_{i,\lambda} - u_{i,\lambda}\big)\,dx,$$

where $g_i(s) = G'_i(s)$. Since $|g_i(s)| \leqslant C|s|^{p-1}$, by Hölder's inequality we find

$$\int\limits_{\mathbb{R}^N} \big|\big(g_i(u^n_{i,\lambda}) - g_i(u_{i,\lambda})\big)\big(u^n_{i,\lambda} - u_{i,\lambda}\big)\big|\,dx \leqslant C\big(|u^n_{i,\lambda}|_p^{p-1} + |u_{i,\lambda}|_p^{p-1}\big)\big|u^n_{i,\lambda} - u_{i,\lambda}\big|_p$$
$$\to 0 \quad \text{as } n \to \infty.$$

Hence

$$\langle I'_{i,\lambda}(u^n_{i,\lambda}) - I'_{i,\lambda}(u_{i,\lambda}), u^n_{i,\lambda} - u_{i,\lambda}\rangle \to 0,$$

which implies that $\|u^n_{i,\lambda} - u_{i,\lambda}\|^2 \to 0$. Therefore, $I_{\lambda,i}$ satisfies the Palais–Smale condition and $c_{i,\lambda}$ is a critical value of $I_{i,\lambda}$ for $i = 1, 2$.

Lemma 3.1. *Let $u_{i,\lambda}$ be a critical point of $I_{i,\lambda}$ in the level $c_{i,\lambda}$ for $i = 1, 2$. Then there exists $C > 0$ such that*

$$\|u_{i,\lambda}\| \leqslant C\lambda^{-1/(q-2)}. \tag{3.11}$$

Proof. For $i = 1, 2$ we have

$$c_{i,\lambda} \leqslant d_{i,\lambda} = \inf_{u>0} \max_{t\geqslant 0} I_{i,\lambda}(tu). \tag{3.12}$$

We are going to show (3.11) for $u_{1,\lambda}$. One can use the same argument to $u_{2,\lambda}$. Since $I_{1,\lambda}(u) = I_\lambda(u)$ for $u > 0$, we deduce from Lemma 2.6 and (3.12) that

$$\|u_{1,\lambda}\|^2 \leqslant C_1 I_\lambda(u_{1,\lambda}) = C_1 c_{1,\lambda} \leqslant C_1 d_{1,\lambda}. \tag{3.13}$$

We claim that there exists $C > 0$ such that $d_{1,\lambda} \leqslant C\lambda^{-2/(q-2)}$. Indeed, by Lemma 2.5, J_λ has a critical point u^*_λ in the level b_λ satisfying

$$\left(\frac{1}{2} - \frac{1}{q}\right)\|u^*_\lambda\|^2 = J_\lambda(u^*_\lambda) = b_\lambda \leqslant C_2\lambda^{-2/(q-2)}. \tag{3.14}$$

This implies that $\lambda\|u^*_\lambda\|^{q-2} \leqslant C_3$. The same argument employed in the proof of Lemma 2.4 shows that

$$\begin{aligned} |u^*_\lambda|_\infty &\leqslant C\big(\lambda\|u^*_\lambda\|^{q-2}\big)^{1/(2^*-p)}\|u^*_\lambda\| \\ &\leqslant CC_3^{1/(2^*-p)}\|u^*_\lambda\| \leqslant C_4\lambda^{-1/(q-2)} \leqslant 2\delta, \end{aligned}$$

for $\lambda > 0$ sufficiently large. Also notice that by the choice of δ, we have

$$G_1(u) = G(u) \geqslant C_1|u|^q \quad \text{for } 0 < u < 2\delta,$$

and hence

$$d_{1,\lambda} \leqslant \inf_{0<u\leqslant 2\delta} \max_{t\geqslant 0} I_{1,\lambda}(tu) \leqslant \inf_{0<u\leqslant 2\delta} \max_{t\geqslant 0} J_\lambda(tu) \leqslant \max_{t\geqslant 0} J_\lambda(tu^*_\lambda). \tag{3.15}$$

We may easily check that $\max_{t\geqslant 0} J_\lambda(tu^*_\lambda) = J_\lambda(u^*_\lambda)$. From (3.14) and (3.15) we get $d_{1,\lambda} \leqslant C\lambda^{-\frac{2}{q-2}}$. This inequality and (3.13) complete the proof of (3.11). □

Now, we immediately deduce:

Proposition 1. *The problem* (2.3) *has one positive solution* $u_{1,\lambda}$ *and one negative solution* $u_{2,\lambda}$. *Furthermore, we can choose* λ *sufficiently large such that*

$$|u_{i,\lambda}|_\infty \leqslant \delta \quad \text{for } i = 1, 2.$$

Proof. It is a consequence of Lemmas 2.4 and 3.1. □

4. Sign-changing solution

Using results obtained in Liu and Sun [19], the existence of sign-changing solution for the problem (2.3) was shown by Bartsch, Liu and Weth [4]. We will use recent results of Schechter and Zou [27] to obtain a solution with variational characterization. To do this, let us recall some basic results which are useful. We start by one of spectral theory.

Proposition 2. *If* (V_1) *and* $((\mathrm{V}_2)$ *or* $(\mathrm{V}_3))$ *hold, then the eigenvalue problem*

$$-\Delta w + V(x)w = \mu w \quad \text{in } \mathbb{R}^N,$$

possesses a sequence of eigenvalues $0 < \mu_1 < \mu_2 \leqslant \cdots \leqslant \mu_k \to \infty$, *where each* μ_k *has finite multiplicity, the first eigenvalue* μ_1 *is simple with positive eigenfunction* φ_1 *and the eigenfunctions* φ_k *correspondent to* μ_k $(k \geqslant 2)$ *are sign-changing.*

Proof. We sketch the argument here. For every $f \in L^2(\mathbb{R}^N)$, there exists a unique $w \in E$ such that

$$-\Delta w + V(x)w = f \quad \text{in } \mathbb{R}^N.$$

Denote $L = -\Delta + V$. Then the operator L has an inverse L^{-1}. Moreover, using the fact that the embedding $E \hookrightarrow L^2(\mathbb{R}^N)$ is compact, we conclude that the operator $L : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is compact. Then, from spectral theory of symmetric compact operators on Hilbert space, we obtain the result. □

Remark 4.1. For each integer positive k, we denote by N_k the eigenspace associated to μ_k and $E_k \doteq N_1 \oplus N_2 \oplus \cdots \oplus N_k$. Since the norms $\|u\|$ and $|u|_2$ are equivalent in E_k, there exists a positive constant ν_k such that

$$\|u\|^2 \leqslant \nu_k |u|_2^2 \tag{4.16}$$

for all $u \in E_k$.

The following result holds.

Lemma 4.2. *Let* $b_{k,\lambda} = \sup_{u \in E_k} I_\lambda(u)$ *and* $2 < p \leqslant q < 2^*$. *For each* $\lambda \in [1, \infty)$ *we have*

$$b_{k,\lambda} \leqslant C_* C_k \lambda^{-2/(q-2)}, \tag{4.17}$$

where $C_k = \nu_k^{q/(q-2)} + \nu_k^{p/(p-2)}$ *and* C_* *depends only on* p, q *and* k.

Proof. For each $u \in E_k$, we define $\Omega_1 = \{x \in \mathbb{R}^N \colon |u(x)| \geqslant 2\delta\}$, $\Omega_2 = \{x \in \mathbb{R}^N \colon |u(x)| < 2\delta\}$, $u_1 = u_{|\Omega_1}$ and $u_2 = u_{|\Omega_2}$. By choice of $\delta > 0$, we have

$$G(u) \geqslant F(u) \geqslant C_1|u|^q \quad \text{if } |u| < 2\delta, \tag{4.18}$$

$$G(u) = F_\infty(u) = C_0|u|^p \quad \text{if } |u| \geqslant 2\delta. \tag{4.19}$$

Since E_k has finite dimension, we deduce from (4.18) and (4.19) that

$$\int_{\mathbb{R}^N} G(u_2)\,dx \geqslant C_1|u_2|_q^q \geqslant C_*|u_2|_2^q, \qquad \int_{\mathbb{R}^N} G(u_1)\,dx \geqslant C_0|u_1|_p^p \geqslant C_*|u_1|_2^p.$$

From these estimates and (4.16), we obtain

$$\begin{aligned} I_\lambda(u) &\leqslant \frac{\nu_k}{2}|u|_2^2 - \lambda C_*|u_1|_2^p - \lambda C_*|u_2|_2^q \\ &= \frac{\nu_k}{2}|u_1|_2^2 - \lambda C_*|u_1|_2^p + \frac{\nu_k}{2}|u_2|_2^2 - \lambda C_*|u_2|_2^q \\ &\leqslant C_*\nu_k^{p/(p-2)}\lambda^{-2/(p-2)} + C_*\nu_k^{q/(q-2)}\lambda^{-2/(q-2)}. \end{aligned}$$

Since $p \leqslant q$, we deduce

$$b_{k,\lambda} \leqslant C_* C_k \lambda^{-2/(q-2)}, \tag{4.20}$$

where $C_k = \nu_k^{p/(p-2)} + \nu_k^{q/(q-2)}$ and $C_* = C_*(p,q,k)$. Hence (4.17) is proved. □

In the following we consider the convex cones $P^+ = \{u \in E\colon u(x) \geqslant 0\}$ and $P^- = \{u \in E\colon u(x) \leqslant 0\}$. For $\epsilon > 0$, we define

$$\mathcal{D}_\epsilon^+ = \{u \in E\colon \operatorname{dist}(u, P^+) < \epsilon\}, \qquad \mathcal{D}_\epsilon^- = \{u \in E\colon \operatorname{dist}(u, P^-) < \epsilon\},$$

and

$$\mathcal{D}_\epsilon = \mathcal{D}_\epsilon^+ \cup \mathcal{D}_\epsilon^-, \qquad \mathcal{S}_\epsilon = E \setminus \mathcal{D}_\epsilon.$$

We set $\mathcal{K}[a,b] = \{u \in E\colon I'_\lambda(u) = 0,\ a \leqslant I_\lambda(u) \leqslant b\}$. Notice that the gradient of I_λ at the point u, $I'_\lambda(u) : E \to E$ has the form

$$I'_\lambda = \mathrm{Id}_E - K_{I_\lambda},$$

where $K_{I_\lambda} : E \to E$ is given by $K_{I_\lambda}(u) = (-\Delta + V)^{-1}[\lambda g(u(.))]$. In other words $K_{I_\lambda}(u)$ is uniquely determined by the relation

$$\big(K_{I_\lambda}(u), \varphi\big) = \lambda \int_{\mathbb{R}^N} g\big(u(x)\big)\varphi\, dx \quad \text{for all } \varphi \in E.$$

Here, we assume the following assumption:

(a_1) $K_{I_\lambda}(\mathcal{D}_\epsilon^\pm) \subset \mathcal{D}_{\epsilon/2}^\pm$.

Next, we consider the following class (see [27]):

$$\varPhi := \left\{ \begin{array}{l} h \in C([0,1] \times E, E) \text{ such that } h(0,.) = \mathrm{Id}_E;\ h(t,.) : E \to E \text{ is a homeomorphism of } E \\ \text{onto itself for all } t \in [0,1) \text{ and } h^{-1} \text{ is continuous on } [0,1) \times E;\ \text{there exists } x_0 \in E \\ \text{such that } h(1,x) = x_0 \text{ for each } x \in E \text{ and that } h(t,x) \to x_0 \text{ as } t \to 1 \text{ uniformly} \\ \text{on bounded subsets of } E \end{array} \right\}.$$

Definition 4.3. A subset A of E links a subset B of E if $A \cap B = \emptyset$ and, for each $h \in \varPhi$, there is $t \in [0,1]$ such that $h(t,A) \cap B \neq \emptyset$.

Finally, we define

$$\varPhi^* := \{h \in \varPhi\colon h(t, \mathcal{D}_\epsilon) \subset \mathcal{D}_\epsilon\}.$$

Notice that $h(t,x) = (1-t)x \in \varPhi^*$.

We will achieved the existence of a sign-changing solution by using the following linking theorem (see [27, Theorem 2.1]).

Theorem 4.4. *Suppose that* (a_1) *holds. Assume that a compact subset A of E links a closed subset B of $\mathcal{S}_\epsilon$ and*

$$a_0 := \sup_A I_\lambda \leqslant b_0 := \inf_B I_\lambda.$$

If I_λ satisfies the (PS) *condition for all*

$$c \in \Big[b_0, \sup_{(t,u)\in[0,1]\times A} I_\lambda\big((1-t)u\big)\Big],$$

then $\mathcal{K}[a^* - \epsilon, a^* + \epsilon] \cap (E \setminus (P^- \cup P^+)) \neq \emptyset$ for all ϵ small, where

$$a^* := \inf_{h\in\Phi^*} \sup_{h([0,1]\times A)\cap\mathcal{S}_\epsilon} I_\lambda(u).$$

In order to apply Theorem 4.4, we fix $m > k + 2$ and let us define

$$B_m = (N_k \oplus N_{k+1} \oplus \cdots \oplus N_m) \cap \overline{B}_{\rho_0},$$

where ρ_0 is given in Lemma 2.3. Take $\rho_0 < R$ and consider the set

$$A = \big\{u = v + sw\colon\ v \in E_{k-1},\ s \geqslant 0,\ w \in N_k,\ \|w\| = 1,\ \|u\| = R\big\} \cup W,$$

where $W = E_{k-1} \cap B_R$. Then A and B_m link each other (see [27]). Note still that A is independent of m for m large.

Let $P_m^\pm = P^\pm \cap E_m$ be the positive and negative cone in E_m. Since all elements in E_m change sign, we have $P_m^\pm \cap B_m = \emptyset$. This together with the compactness of B_m imply the existence of $\epsilon_m > 0$ such that

$$\operatorname{dist}\big(B_m, P_m^\pm\big) = \epsilon_m > 0.$$

Now, define

$$\mathcal{D}_\epsilon^\pm(m) = \big\{u \in E_m\colon \operatorname{dist}\big(u, P_m^\pm\big) < \epsilon\big\}.$$

Taking $G_m = G_{E_m}$, we have

$$G'_m(u) = u - \operatorname*{Proj}_m(K_G u), \quad u \in E_m,$$

where Proj_m denotes the projection of E onto E_m. In order to verify the condition (a_1) in Theorem 4.4, we need the following lemma which can be found in [27, Lemma 3.11], see also [4, Lemma 3.1].

Lemma 4.5. *There exists $\epsilon_0 \in (0, \epsilon_m)$ such that*

$$\operatorname*{Proj}_m K_G\big(\mathcal{D}_{\epsilon_0}^\pm(m)\big) \subset \mathcal{D}_{\epsilon_0/2}^\pm(m).$$

In particular, the condition (a_1) *is satisfied.*

Proof. The proof is a straightforward adaptation of the proof of [27, Lemma 3.11]. □

Remark 4.6. We can see that condition (a_1) holds if $|g(u)| \leqslant d|u| + C(d)|u|^{p-1}$ for $u \in \mathbb{R}$, where $2d = \inf_{x\in\mathbb{R}^N} V(x)$ (see [4, Lemma 3.1]).

Proposition 3. *The problem* (2.3) *has a sign-changing solution u_λ. Furthermore, we have the following estimate*:

$$|u_\lambda|_\infty \leqslant \delta \tag{4.21}$$

for λ sufficiently large.

Proof. We adapt the proof of Theorem 3.1 in [27]. For each integer $m > k + 1$, let

$$\mathcal{D}(m) := \mathcal{D}_{\epsilon_0}^+(m) \cup \mathcal{D}_{\epsilon_0}^-(m) \quad \text{and} \quad \mathcal{S}_m := E_m \setminus \mathcal{D}(m).$$

By Theorem 4.4, there exists $(u_m) \subset E_m \setminus P_m^\pm$ such that $I'_\lambda(u_m) = 0$ and

$$I_\lambda(u_m) \in \Big[b_0 - \bar\epsilon, \sup_{(t,u)\in[0,1]\times A} I_\lambda\big((1-t)u\big) + \bar\epsilon\Big]$$

for all $\bar{\epsilon}$ small enough. This implies that (u_m) is a (PS) sequence. Since I_λ satisfies the (PS) condition, using the notation $u_m^\pm := \max\{\pm u_m, 0\}$, we get

$$\|u_m^\pm\|^2 = \lambda \int_{\mathbb{R}^N} g(u_m^\pm) u_m^\pm \, dx.$$

This together with inequality (2.4) imply

$$\|u_m^\pm\|^2 \leqslant \lambda C |u_m^\pm|_p^p.$$

Since $2 < p < 2^*$, by the Sobolev embedding $E \hookrightarrow L^p(\mathbb{R}^N)$ we get

$$\|u_m^\pm\| \geqslant C > 0.$$

Thus, the limit u_λ of a subsequence of (u_m) is a sign-changing solution of (2.3).

In view of Lemma 2.6, we have

$$\left(\frac{\alpha-2}{2\alpha}\right)\|u_\lambda\|^2 \leqslant \liminf \left(\frac{\alpha-2}{2\alpha}\right)\|u_m\|^2 \leqslant \liminf I_\lambda(u_m).$$

Also notice that

$$I_\lambda(u_m) \leqslant a^* = \inf_{h\in\Phi^*} \sup_{h([0,1]\times A)\cap\mathcal{S}} I_\lambda(u_\lambda) + \bar{\epsilon} \leqslant \sup_{(t,u)\in[0,1]\times A} I_\lambda\big((1-t)u_\lambda\big) + \bar{\epsilon}.$$

Since A is independent of m for m large ($m > k+1$), we deduce from Lemma 4.2 that

$$\sup_{(t,u)\in[0,1]\times A} I_\lambda\big((1-t)u_\lambda\big) \leqslant b_{k,\lambda} \leqslant C_* C_k \lambda^{-2/(q-2)}.$$

Hence, we obtain

$$\|u_\lambda\|^2 \leqslant C\lambda^{-2/(q-2)} + \bar{\epsilon},$$

which together with Lemma 2.4 imply the required result. □

Proof of Theorem 1.1. This proof is a immediate consequence of Propositions 1 and 3. □

Acknowledgments

Part of this work was done while the first author was visiting the University of British Columbia. He thanks Professor Nassif Ghoussoub and all the faculty and staff of Department of Mathematics and PIMS at UBC for their support and kind hospitality.

References

[1] A. Ambrosetti, P.H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14 (1973) 349–381.
[2] T. Bartsch, Critical point theory on partially ordered hilbert spaces, J. Funct. Anal. 186 (2001) 117–152.
[3] T. Bartsch, Z. Liu, Multiple sign changing solutions of a quasilinear elliptic eigenvalue problem involving the p-laplacian, Commun. Contemp. Math. 6 (2004) 245–258.
[4] T. Bartsch, Z. Liu, T. Weth, Sign changing solutions of superlinear Schrödinger equations, Comm. Partial Differential Equations 29 (2004) 25–42.
[5] T. Bartsch, A. Pankov, Z.Q. Wang, Nonlinear Schrödinger equations with steep potential well, Commun. Contemp. Math. 3 (2001) 549–569.
[6] T. Bartsch, Z.Q. Wang, Existence and multiplicity results for some superlinear elliptic problems on $\mathbb{R}^n$, Comm. Partial Differential Equations 20 (1995) 1725–1741.
[7] T. Bartsch, Z.Q. Wang, Sign changing solutions of nonlinear Schrödinger equations, Topol. Methods Nonlinear Anal. 13 (1999) 191–198.
[8] H. Berestycki, P.-L. Lions, Nonlinear scalar field equations. I, Arch. Ration. Mech. Anal. 82 (1983) 313–345.
[9] H. Berestycki, P.-L. Lions, Nonlinear scalar field equations. II, Arch. Ration. Mech. Anal. 82 (1983) 347–375.
[10] J. Chabrowski, Variational Methods for Potential Operator Equations with Applications to Nonlinear Elliptic Equations, de Gruyter Stud. Math., vol. 24, Walter de Gruyter & Co., Berlin, 1997.
[11] S. Chen, S. Li, On a nonlinear elliptic eigenvalue problem, J. Math. Anal. Appl. 307 (2005) 691–698.
[12] D.G. Costa, On a class of elliptic systems in $\mathbb{R}^n$, Electron. J. Differential Equations 1994 (1994) 1–14.
[13] D.G. Costa, Z.Q. Wang, Multiplicity results for a class of superlinear elliptic problems, Proc. Amer. Math. Soc. 133 (2005) 787–794.

[14] M.G. Crandall, P.H. Rabinowitz, Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems, Arch. Ration. Mech. Anal. 58 (1975) 207–218.
[15] E.N. Dancer, Z. Zhang, Fucik spectrum, sign-changing, and multiple solutions for semilinear elliptic boundary value problems with resonance at infinity, Arch. Ration. Mech. Anal. 250 (2000) 449–464.
[16] D. Gilbarg, N. Trudinger, Elliptic Partial Differential Equations of Second Order, Classics Math., vol. 24, Springer-Verlag, Berlin, 1998.
[17] V. Kondrat'ev, M. Shubin, Discreteness of spectrum for the Schrödinger operators on manifolds of bounded geometry, Oper. Theory Adv. Appl. 110 (1999) 185–226.
[18] S. Li, Z.Q. Wang, Ljusternik–Schnirelman theory in partially ordered Hilbert spaces, Trans. Amer. Math. Soc. 354 (2002) 3207–3227.
[19] Z. Liu, J. Sun, Invariant sets of descending flow in critical point theory with applications to nonlinear differential equations, J. Differential Equations 172 (2001) 257–299.
[20] Z. Liu, Z.Q. Wang, T. Weth, Multiple solutions of nonlinear Schrödinger equations via flow invariance and morse theory, Proc. Roy. Soc. Edinburgh Sect. A 136 (2006) 945–969.
[21] W. Omana, M. Willem, Homoclinic orbits for a class of Hamiltonian systems, Differential Integral Equations 5 (1992) 1115–1120.
[22] A. Qian, S. Li, Multiple sign-changing solutions of an elliptic eigenvalue problem, Discrete Contin. Dyn. Syst. 12 (2005) 737–746.
[23] P.H. Rabinowitz, Variational methods for nonlinear elliptic eigenvalue problems, Indiana Univ. Math. J. 23 (1973) 729–754.
[24] P.H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, CBMS Reg. Conf. Ser. Math., vol. 65, Amer. Math. Soc., Providence, RI, 1986.
[25] P.H. Rabinowitz, On a class of nonlinear Schrödinger equations, Z. Angew. Math. Phys. 43 (1992) 272–291.
[26] M. Schechter, Z.Q. Wang, W. Zou, New linking theorem and sign-changing solutions, Comm. Partial Differential Equations 29 (2004) 471–488.
[27] M. Schechter, W. Zou, Sign-changing critical points from linking type theorems, Trans. Amer. Math. Soc. 358 (2006) 5293–5318.
[28] W.A. Strauss, Mathematical Aspects of Classical Nonlinear Field Equations, Lecture Notes in Phys., vol. 98, Springer-Verlag, Berlin, 1979.
[29] C.A. Stuart, Self-trapping of an electromagnetic field and bifurcation from the essential spectrum, Arch. Ration. Mech. Anal. 113 (1991) 65–96.
[30] C.A. Stuart, H.S. Zhou, A variational problem related to self-trapping of an electromagnetic field, Math. Methods Appl. Sci. 113 (1996) 1397–1407.
[31] M. Willem, Minimax Theorems, Progr. Nonlinear Differential Equations Appl., vol. 24, Birkhäuser Boston, Boston, 1996.
[32] Z. Zhang, K. Perera, Sign changing solutions of Kirchhoff type problems via invariant sets of descent flow, J. Math. Anal. Appl. 317 (2006) 456–463.
[33] W. Zou, On finding sign-changing solutions, J. Funct. Anal. 234 (2006) 364–419.