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Multiparameter Elliptic Equations in Annular Domains

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Dedicated to Djairo G. de Figueiredo on the occasion of his 70th birthday

Abstract. Using fixed point theorems of cone expansion/compression type, the lower and upper solution method and degree arguments, we study existence, non-existence and multiplicity of positive solutions for a class of second-order ordinary differential equations with multiparameters. We apply our results to semilinear elliptic equations in bounded annular domains with non–homogeneous Dirichlet boundary conditions. More precisely, we apply the main results to equations of the form

$$\begin{array}{rcllc} -\Delta u & = & \lambda\, f(|x|, u) & \text{in} & r_1 < |x| < r_2\,, \\ u(x) & = & a & \text{on} & |x| = r_1\,, \\ u(x) & = & b & \text{on} & |x| = r_2\,, \end{array}$$

where a, b and λ are nonnegative parameters. One feature of the hypotheses on the nonlinearities which we consider in this paper is that they have some sort of local character.

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1. Introduction

We establish existence, non-existence, and multiplicity of positive solutions for the second-order ordinary differential equation

$$\begin{array}{rcl} -u'' & = & \lambda g(t,u(t),a,b) \quad \text{in} \quad (0,1)\,, \\ u(0) & = & u(1)=0, \end{array} \qquad (P_{a,b,\lambda})$$

where a, b and λ are nonnegative parameters, and $g\in C([0,1]\times[0,+\infty)^3,[0,+\infty))$ is a nondecreasing function in the last three variables.

Our first result treats the case where $\lambda = 1$ and the function g has a local superlinear growth at infinity. The behavior at zero of the function g may change according to the parameters a, b considered. (See assumptions (H_1) and (H_2) below.) We show that there exists a continuous curve Γ which splits the positive quadrant of the $(a,b)-$plane into two disjoint sets, say $\mathcal{S}$ and $\mathcal{R}$, so that $(P_{a,b,1})$ has at least two positive solutions in $\mathcal{S}$; at least one positive solution on the boundary of $\mathcal{S}$; and no positive solutions in $\mathcal{R}$. (See Theorem 1.1 below.)

Our second result treats the case where the function g has sublinear growth at infinity. Again, the behavior at zero of the function g may change according to the parameters a, b considered. We show that $(P_{a,b,\lambda})$ has at least one positive solution, for all $a, b, \lambda > 0$. Further, we show that there exists $\rho > 0$ such that $(P_{a,b,\lambda})$ has at least three positive solutions, for all $0 < |(a,b)| < \rho$ and λ sufficiently large. (See Theorem 1.2 below.)

We subsequently give applications of our main results to semilinear elliptic equations in annular domains.

The approach taken to prove our main results is based on a well known fixed point theorem of cone expansion and compression type, the lower and upper solution method and some topological degree arguments.

We will assume the following six basic hypotheses:

(H_0) $g \in C([0,1]\times[0,+\infty)^3,[0,+\infty))$ is a nondecreasing function in the last three variables. In other words,

$$g(t,u_1,a_1,b_2) \le g(t,u_2,a_2,b_2)$$

whenever $(u_1,a_1,b_1) \le (u_2,a_2,b_2)$. The above inequality is understood inside every component. Furthermore, there exist constants $0<\delta_0<\varepsilon_0<1$ such that, for all $t\in[\delta_0,\varepsilon_0]$, we have $g(t,0,a,b)>0$ whenever $a+b>0$.

(H_1) There exist constants $0 < \delta_1 < \varepsilon_1 < 1$ such that, for all $(a,b) \in [0,+\infty)^2\setminus\{(0,0)\}$, we have

$$\lim_{u\to 0}\frac{g(t,u,a,b)}{u} = +\infty \quad \text{uniformly in } t\in[\delta_1,\varepsilon_1]\,.$$

(H_2) $\displaystyle\lim_{|(u,a,b)|\to 0}\frac{g(t,u,a,b)}{|(u,a,b)|} = 0$ uniformly in $t \in [0,1]$. Here we use the notation $|(z_1,z_2,z_3)| = (z_1^2+z_2^2+z_3^2)^{1/2}$.

(H_3) There exist constants $0 < \delta_2 < \varepsilon_2 < 1$ such that

$$\lim_{u \to +\infty} \frac{g(t,u,0,0)}{u} = +\infty \quad \text{uniformly in } t \in [\delta_2, \varepsilon_2] .$$

(H_4) There exist constants $0 < \delta_3 < \varepsilon_3 < 1$ such that

$$\lim_{|(a,b)| \to +\infty} g(t,0,a,b) = +\infty \quad \text{uniformly in } t \in [\delta_3, \varepsilon_3] .$$

(H_5) For all $(a,b) \in [0,+\infty)^2$, we have

$$\lim_{u \to +\infty} \frac{g(t,u,a,b)}{u} = 0 \quad \text{uniformly in } t \in [0,1] .$$

(H_6) There exist constants $R > 0$ and $0 < \delta_4 < \varepsilon_4 < 1$ such that

$$0 < g(t,u,0,0) , \text{ for all } 0 < u < R \text{ and } t \in [\delta_4, \varepsilon_4] .$$

Our main results are the following.

Theorem 1.1 (Superlinear case at $+\infty$). *Suppose that $\lambda = 1$ and that $g(t,u,a,b)$ satisfies assumptions (H_0) through (H_4). Then there exist $\overline{a} > 0$ and a non-increasing continuous function $\Gamma : [0,\overline{a}] \to [0,+\infty)$ so that, for all $a \in [0,\overline{a}]$, we have:*

(i) $(P_{a,b,1})$ *has at least one positive solution if* $0 \le b \le \Gamma(a)$.
(ii) $(P_{a,b,1})$ *has no solution if* $b > \Gamma(a)$.
(iii) $(P_{a,b,1})$ *has a second positive solution if* $0 < b < \Gamma(a)$.

Theorem 1.2 (Sublinear case at $+\infty$). *Suppose that $g(t,u,a,b)$ satisfies assumptions (H_0) through (H_2), as well as assumptions (H_5) and (H_6). Then:*

(i) *$(P_{a,b,\lambda})$ has at least one positive solution for all $a,b,\lambda > 0$.*
(ii) *There exists a positive constant ρ sufficiently small such that, for all $0 < |(a,b)| < \rho$, $(P_{a,b,\lambda})$ has at least three positive solutions for λ sufficiently large.*

Remark 1.3. We would like to call attention to the local character of assumptions (H_1), (H_3), (H_4), and (H_6) on the nonlinearity g in the variable t. More precisely, in this paper some sort of sublinearity and some sort of superlinearity is required to hold uniformly in t only on open sub-intervals of $(0,1)$ which may be small and possibly disjoints.

Our main results may be applied to several classes of elliptic problems. For example, we may apply our results to the semilinear elliptic equation

$$\begin{array}{rcll} -\Delta u &=& \lambda \hat{f}(|x|,u) & \text{in} \quad r_1 < |x| < r_2 , \\ u(x) &=& a & \text{on} \quad |x| = r_1 , \\ u(x) &=& b & \text{on} \quad |x| = r_2 , \end{array} \qquad (Q_{a,b,\lambda})$$

where $0 < r_1 < r_2$ and $N \ge 3$. For instance, in the case $\hat{f}(|x|,u) = c(|x|)f(u)$, where $c : [r_1,r_2] \to [0,+\infty)$ is a nonnegative, non-trivial continuous function and the nonlinearity f is a superlinear continuous function both at zero and infinity, we may apply Theorem 1.1. Note that a simpler model is given by $f(u) = u^p$, with $p > 1$. The case $f(u) = u^{(N+2)/(N-2)}$, $c \equiv 1$, and $a = 0$ was studied by C. Bandle and L. A. Peletier [1]. This result was subsequently improved by M. G. Lee and S. S. Lin [8]. In fact, using Shooting Methods, the results of [1] were extended by Lee and Lin to nonlinearities f that are convex and superlinear at both zero and infinity. Using degree arguments and the lower and upper solution method, D. D. Hai extends and complements some of the results of [1, 8] to locally Lipschitz continuous nonlinearities. (See [5, Theorem 3.7].)

Our multiplicity result is an improvement because $(P_{a,b,\lambda})$ is not necessarily autonomous, and we do not impose either local Lipschitz continuity assumptions or convexity on the nonlinearity f. In addition, by Theorem 1.2, we obtain the existence of three positive solutions of $(Q_{a,b,\lambda})$, a type of result not yet found in the literature. As an application of Theorem 1.2, a simple model is given by $f(u) = u^p/(1+u^q)$, with $\max\{1, q\} < p < q+1$.

The paper is organized as follows. Section 2 contains preliminary results. Sections 3, 4 are devoted to proving Theorems 1.1, 1.2, respectively. Finally, in Section 5 we give more examples and remarks.

Notation. Here is a brief summary of the notation we make use of.
We denote the closed ball of radius R centered at the point $p \in X$ by $B[p, R] = \{x \in X : |x| \leq R\}$, and denote the open ball with radius R centered at the point $p \in X$ by $B(p, R)$. The *mapping degree* for the equation $F(x) = y$, $x \in A$, is denoted by $deg(F, A, y)$.

2. Preliminary results

In the next section using the lower and upper solution method and fixed point techniques we will prove Theorem 1.1. For this purpose we observe that if u is a solution of $(P_{a,b,\lambda})$, then for all $t \in [0, 1]$,

$$u(t) = (1-t)\lambda \int_0^1 \tau g(\tau, u(\tau), a, b)\ d\tau + \lambda \int_t^1 (t-\tau) g(\tau, u(\tau), a, b)\ d\tau,$$

or, equivalently,

$$u(t) = \lambda \int_0^1 K(t, \tau) g(\tau, u(\tau), a, b)\ d\tau$$

where

$$K(t, \tau) = \begin{cases} (1-t)\tau, & \tau < t, \\ (1-\tau)t, & \tau \geq t. \end{cases}$$

Thus, solutions of $(P_{a,b,\lambda})$ correspond to the fixed points of the operator

$$Tu(t) = \lambda \int_0^1 K(t, \tau) g(\tau, u(\tau), a, b)\ d\tau \tag{2.1}$$

defined in the Banach space $X = C([0,1],\mathbb{R})$ endowed with the usual norm $\|u\|_\infty := \sup_{t\in[0,1]} |u(t)|$.

The following fixed point theorem in cones is due to Krasnoselskii (see [2, 3, 4, 7]).

Lemma 2.1. *Let X be a Banach space with norm $|\cdot|$, and let $C \subset X$ be a cone in X. For $R > 0$, define $C_R = C \cap B[0,R]$. Assume that $F : C_R \to C$ is a completely continuous map and that there exists $0 < r < R$ such that*

$$|Fx| < |x|,\ x \in \partial C_r \quad \textit{and} \quad |Fx| > |x|,\ x \in \partial C_R,\ \textit{or}$$
$$|Fx| > |x|,\ x \in \partial C_r \quad \textit{and} \quad |Fx| < |x|,\ x \in \partial C_R,$$

where $\partial C_R = \{x \in C : |x| = R\}$. Then F has a fixed point $u \in C$ with $r < |u| < R$.

Let C be the cone defined by

$$C = \{u \in C[0,1] \ : \ u \text{ is concave and } u(0) = u(1) = 0\}.$$

Using the concavity of the function $u \in C$ it is not difficult to obtain the following result.

Lemma 2.2. *For each $u \in C$ and $\alpha, \beta \in (0,1)$ with $\alpha < \beta$, we have*

$$\inf_{t\in[\alpha,\beta]} u(t) \geq \alpha(1-\beta)\|u\|_\infty.$$

Remark 2.3. In this work we mainly use fixed points in Cones and Topological Degree. In this context Lemma 2.2 is crucial in order to obtain estimates of expansion/compression type as well as when we want to establish a priori bounds.

Lemma 2.4. *$T : X \to X$ is completely continuous and $T(C) \subset C$.*

Proof. The proof of this lemma is standard and we include here only the main ideas for completeness. The complete continuity of T follows from The Arzela-Ascoli theorem. It is easy to see that Tu is twice differentiable on $(0,1)$ with the second derivative negative. This implies that $T(C) \subset C$. □

3. Proof of Theorem 1.1 (Superlinear case at $+\infty$)

In this section we combine the fixed point theorem, lower and upper solution method and degree arguments to prove Theorem 1.1. We recall that through this section $\lambda = 1$.

3.1. The first positive solution for Problem $(P_{a,b,1})$

Lemma 3.1. *If $g(t,u,a,b)$ satisfies (H_0), (H_1) and (H_2), then there exist positive parameters a_0 and b_0 such that $(P_{a_0,b_0,1})$ has at least one positive solution.*

Proof. Let $u \in C$ with $||u||_\infty = R > 0$. In view of assumption (H_0), for all $t \in [0,1]$ we have

$$Tu(t) = \int_0^1 K(t,\tau) g(\tau, u(\tau), a, b) d\tau \leq \max_{(t,\tau)\in[0,1]^2} K(t,\tau) \max_{\tau\in[0,1]} g(\tau, R, a, b).$$

Hence, using condition (H_2), we can take $a_0, b_0, R > 0$ sufficiently small such that

$$||Tu||_\infty < ||u||_\infty \text{ if } ||u||_\infty = R. \tag{3.2}$$

Next, using assumption (H_1), given $M > 0$ there exist $r_1 \in (0, R)$ such that,

$$g(t, u, a_0, b_0) \geq Mu, \text{ for all } (\tau, u) \in [\delta_1, \varepsilon_1] \times [0, r_1]. \tag{3.3}$$

From Lemma 2.2, for all $u \in C$ we have

$$u(t) \geq (1 - \varepsilon_1)\delta_1 \, ||u||_\infty, \text{ for all } t \in [\delta_1, \varepsilon_1]. \tag{3.4}$$

This estimate in combination with (3.3), and taking M sufficiently large we have

$$||Tu||_\infty > ||u||_\infty \text{ if } ||u||_\infty = r_1. \tag{3.5}$$

Therefore, in view of estimates (3.2) and (3.5), we can apply Lemma 2.1 to get a fixed point $u \in C$ with $r_1 < ||u|| < R$. Finally, using the maximum principle we obtain that u is positive. $\square$

The following lemma corresponds to a nonexistence result.

Lemma 3.2. *If $g(t, u, a, b)$ satisfies (H_3) and (H_4), then there exists $c_0 > 0$ such that for all $(a, b) \in [0, +\infty)^2$ with $|(a, b)| > c_0$, $(P_{a,b,1})$ has no positive solutions.*

Proof. Assume by contradiction that there exists a sequence (a_n, b_n) with $|(a_n, b_n)| \to +\infty$ such that for each n $(P_{a_n,b_n,1})$ possesses a positive solution $(u_n) \in C$. By assumption (H_4), given $M > 0$, there exists $c_0 > 0$ such that for all $(a, b) \in [0, +\infty)^2$ with $|(a, b)| \geq c_0$, we have

$$g(t, u, a, b) \geq M, \text{ for all } t \in [\delta_3, \varepsilon_3] \text{ and } u \geq 0. \tag{3.6}$$

Thus,

$$\begin{aligned} u_n(t) &= \int_0^1 K(t,\tau) g(\tau, u_n(\tau), a_n, b_n) \, d\tau \\ &\geq \int_{\delta_3}^{\varepsilon_3} K(t,\tau) g(\tau, u_n(\tau), a_n, b_n) \, d\tau, \end{aligned}$$

which implies that, for n sufficiently large,

$$u_n(t) \geq M \int_{\delta_3}^{\varepsilon_3} K(t,\tau) \, d\tau.$$

Hence

$$||u_n||_\infty \geq M \max_{t\in[0,1]} \int_{\delta_3}^{\varepsilon_3} K(t,\tau) \, d\tau.$$

Since in (3.6) we may choose an arbitrary constant M, we have that (u_n) is an unbounded sequence in X.

On the other hand, by using assumption (H_3), we have that given $M > 0$ there exits $R > 0$ such that for all $t \in [\delta_2, \varepsilon_2]$ and $a, b \geq 0$,

$$g(t, u, a, b) \geq Mu, \text{ for all } u \geq R. \tag{3.7}$$

Using Lemma 2.2, for n sufficiently large, we get

$$u_n(t) \geq M(1-\varepsilon_2)\delta_2 ||u_n||_\infty \int_{\delta_2}^{\varepsilon_2} K(t, \tau)\, d\tau.$$

Hence

$$1 \geq M(1-\varepsilon_2)\delta_2 \max_{t\in[0,1]} \int_{\delta_2}^{\varepsilon_2} K(t, \tau)\, d\tau,$$

which is a contradiction with the fact that M can be chosen arbitrarily large. The proof of Lemma 3.2 is now complete. □

Remark 3.3. As an immediate consequence of Lemma 3.2, we have a priori estimate for positive solutions of $(P_{a,b,1})$, more precisely, there exists $k_0 > 0$ independent of (a, b) such that $\|u\|_\infty \leq k_0$, for all $u \in X$ positive solutions of $(P_{a,b,1})$.

Next, using the lower and upper solution method we may establish the following result.

Lemma 3.4. *If $g(t, u, a, b)$ satisfies (H_0) and $(P_{a,b,1})$ has a positive solution, then for all $(0, 0) \leq (c, d) \leq (a, b)$, $(P_{c,d,1})$ has a positive solution provided that $c+d > 0$.*

Proof. Since the function $g(t, u, a, b)$ is nondecreasing in the last two variables we have that the solution u of $(P_{a,b,1})$ is a upper-solution of $(P_{c,d,1})$, while the null function is a lower solution for this problem. Therefore, using the classical lower and upper solution method we have that $(P_{c,d})$ has a positive solution. □

Let us define

$$\overline{a} := \sup\{a > 0 :\ (P_{a,b,1}) \text{ has a positive solution for some } b > 0\}.$$

From Lemma 3.2 it follows immediately that $0 < \overline{a} < +\infty$. It is easy to see, using the lower and upper solution method that for all $a \in (0, \overline{a})$ there exists $b > 0$ such that $(P_{a,b,1})$ has a positive solution. Thus we may define the function $\Gamma : [0, \overline{a}] \to [0, +\infty)$ given by

$$\Gamma(a) := \sup\{b > 0\ :\ (P_{a,b,1}) \text{ has a positive solution}\}.$$

As a consequence of Lemma 3.4, we obtain that Γ is a continuous and nonincreasing function. Therefore, it is easy to see by the definition of the function Γ that $(P_{a,b,1})$ has at least one positive solution if $0 \leq b \leq \Gamma(a)$ and it has no positive solutions when $b > \Gamma(a)$.

3.2. The second positive solution for Problem $(P_{a,b,1})$

Now, we are working to prove the existence of a second positive solution of $(P_{a,b,1})$ when $0 < b < \Gamma(a)$. In this case, according to conclusions above we have positive solutions u_1 and $\overline{u}$ of $(P_{a,b,1})$ and $(P_{a,\Gamma(a),1})$ respectively. Using a combination of maximum principle and the monotonicity of the function $g(t,u,a,b)$ in the second variable we may suppose that

$$0 < u_1 < \overline{u}, \quad 0 < u_1'(0) < \overline{u}'(0) \text{ and } u_1'(1) < \overline{u}'(1) < 0.$$

Now we consider the Banach space X_1 given by

$$X_1 := \{u \in C^1[0,1] \ : \ u(0) = u(1) = 0\},$$

endowed with the norm $||u||_1 := ||u||_\infty + ||u'||_\infty$. Moreover, we consider the following open subset of X_1 given by

$$\mathcal{A} := \{u \in X_1 : \ 0 < u < \overline{u},\ 0 < u'(0) < \overline{u}'(0),\ u'(1) < \overline{u}'(1) < 0 \text{ and } ||u||_1 < R_1\},$$

where R_1 is chosen such that $||u_1||_1 < R_1$.

Let us consider the operator $\mathcal{S}_{(a,b)} : X_1 \to X_1$ given by

$$\mathcal{S}_{(a,b)}u(t) = \int_0^1 K(t,\tau) g(\tau, u(\tau), a, b)\ d\tau.$$

We notice that if there exists a fixed point of $\mathcal{S}_{(a,b)}$ on $\partial\mathcal{A}$, then we have a second positive solution of $(P_{a,b,1})$, otherwise we will obtain the existence of our second positive solution as a consequence of the following result.

Lemma 3.5. *Suppose that $\mathcal{S}_{(a,b)}$ has no fixed point on $\partial\mathcal{A}$ and assume that $0 < b < \Gamma(a)$. By using the notation above, we have:*

(i) *$deg(Id - \mathcal{S}_{(a,b)}, \mathcal{A}, 0) = 1$*

(ii) *There exists $\overline{R} > R_1$ such that $deg(Id - \mathcal{S}_{(a,b)}, B_{X_1}(0, \overline{R}), 0) = 0$.*

Proof. Let us define

$$\overline{g}(t,v,a,b) := \begin{cases} g(t, \overline{u}(t), a, b) & \text{if } \ \overline{u}(t) < v, \\ g(t,v,a,b) & \text{if } \ 0 \le v \le \overline{u}(t), \\ 0 & \text{if } \ v < 0, \end{cases}$$

and $\overline{\mathcal{S}}_{(a,b)} : X_1 \to X_1$ given by

$$(\overline{\mathcal{S}}_{(a,b)}u)(t) = \int_0^1 K(t,\tau)\overline{g}(\tau, u(\tau), a, b)\ d\tau.$$

It is easy to see that this operator $\overline{\mathcal{S}}_{(a,b)}$ satisfies the following properties:

(a) $\overline{\mathcal{S}}_{(a,b)}$ is a completely continuous operator;

(b) if u is a fixed point of $\overline{\mathcal{S}}_{(a,b)}$, then u is a fixed point of $\mathcal{S}_{(a,b)}$ with $0 \le u \le \overline{u}$;

(c) If $u = \lambda\overline{\mathcal{S}}_{(a,b)}u$ with $0 \le \lambda \le 1$ then $||u||_1 \le C_3$, where C_3 does not depend on λ and $u \in X_1$.

Using the a priori estimate property established in assertion (c), we have that there exists $R_2 > R_1$ such that

$$deg(Id - \overline{\mathcal{S}}_{(a,b)}, B_{X_1}(0, R_2), 0) = 1. \tag{3.8}$$

Now by the Maximum Principle, the operator $\overline{\mathcal{S}}_{(a,b)}$ has no fixed point in $\overline{B(0,R_2)} \setminus \mathcal{A}$. By hypothesis $\overline{\mathcal{S}}_{(a,b)}$ has no a fixed point on $\partial\mathcal{A}$, thus we have that the topological degree of Leray-Schauder is defined for the equation $(Id - \overline{\mathcal{S}}_{(a,b)})(x) = 0,\ x \in \mathcal{A}$. Then by using (3.8) and the excision property of mapping degree we have

$$deg(Id - \overline{\mathcal{S}}_{(a,b)}, \mathcal{A}, 0) = 1.$$

Since $S_{(a,b)}(u) = \overline{\mathcal{S}}_{(a,b)}(u),\ u \in \partial\mathcal{A}$ the part (i) of Lemma 3.5 is proved.

Next, using (3.4) and assumption (H_3) (see also (3.7)) we obtain an a priori estimate $\overline{R}$ which can be taken bigger than R_1 for solutions of the equation

$$u = S_{(a,b)}u,\ u \in X_1, \tag{3.9}$$

which does not depend on the parameters a and b. Let $(\overline{a}, \overline{b})$ such that $|(\overline{a}, \overline{b})|$ is sufficiently large such that $(P_{a,b,1})$ has no positive solutions (see Lemma 3.2). Thus

$$deg(Id - \mathcal{S}_{(\overline{a},\overline{b})}, B(0, \overline{R}), 0) = 0.$$

Hence, by the homotopy invariance property of the mapping degree we have

$$deg(Id - \mathcal{S}_{(a,b)}, B(0, \overline{R}), 0) = 0.$$

The proof of Lemma 3.5 is now complete. □

Finally, the Lemma 3.5 and the excision property of the topological degree imply

$$deg(Id - \mathcal{S}_{(a,b)}, B(0, \overline{R}) \setminus \overline{\mathcal{A}}, 0) = -1,$$

hence we have a second solution of $(P_{a,b,1})$. The proof of Theorem 1.1 is complete.

4. Proof of Theorem 1.2

In this section we apply Lemma 2.1 to get three solutions of $(P_{a,b,\lambda})$ when $g(t,u,a,b)$ is sublinear at infinity.

Lemma 4.1. *Assume that hypothesis (H_1) holds, then given $(a,b) \in [0,+\infty)^2 \setminus \{(0,0)\}$ there exists $R_1 > 0$ small enough such that for all $u \in \partial C_{R_1}$,*

$$\|Tu\|_\infty > \|u\|_\infty.$$

Proof. By using hypothesis (H_1) we have that for each $M > 0$, there exists $R_1 > 0$ such that for all $t \in [\delta_1, \varepsilon_1]$

$$g(t,u,a,b) \geq Mu, \quad \text{for each } u \in [0, R_1].$$

Therefore, for all $u \in C_{R_1}$,

$$\begin{aligned} \|Tu\|_\infty &\geq \int_0^1 K(1/2,\tau)g(\tau,u(\tau),a,b)\ d\tau \\ &\geq M\int_{\delta_1}^{\varepsilon_1} K(1/2,\tau)u(\tau)\ d\tau \\ &\geq \delta_1(1-\varepsilon_1)M\|u\|_\infty \int_{\alpha_1}^{\beta_1} K(1/2,\tau)\ d\tau. \end{aligned}$$

Finally, taking $M > 0$ sufficiently large we conclude the proof of Lemma 4.1. □

Lemma 4.2. *Assume condition (H_5), then given $(a,b) \in [0,+\infty)^2$ and $R_1 > 0$ there exists $R_2 > R_1$ such that for all $u \in \partial C_{R_2}$,*

$$\|Tu\|_\infty < \|u\|_\infty.$$

Proof. Let $(a,b) \in [0,+\infty)^2$. From assumption (H_5), given $\varepsilon > 0$, there exists $R_2 > R_1$ such that for all $u \geq R_2$,

$$g(t,u,a,b) \leq \varepsilon u.$$

Thus

$$\begin{aligned} (Tu)(t) &= \int_0^1 K(t,\tau)g(\tau,u(\tau),a,b)\ d\tau \\ &\leq \int_0^1 K(t,\tau)g(\tau,\|u\|_\infty,a,b)\ d\tau \\ &\leq \varepsilon\|u\|_\infty \int_0^1 K(t,\tau)\ d\tau, \end{aligned}$$

which, taking $\varepsilon > 0$ sufficiently small, proves the Lemma 4.2. □

In view of Lemmas 4.1 and 4.2, as a direct consequence of Lemma 2.1 we have the proof of the first part of Theorem 1.2.

On the other hand, by using (H_2) we have that there exist positive constants small enough ρ, R_3 such that for all $0 < |(a,b)| < \rho$,

$$\|Tu\|_\infty \quad < \quad \|u\|_\infty,\ u \in \partial C_{R_3}.$$

Now, according to hypotheses (H_0) and (H_6) we have that for all $u \in \partial C_R$,

$$\begin{aligned} (Tu)(t) &= \lambda\int_0^1 K(t,\tau)g(\tau,u(\tau),a,b)d\tau \\ &\geq \lambda\int_{\delta_4}^{\varepsilon_4} K(t,\tau)g(\tau,\|u\|_\infty(1-\varepsilon_4)\delta_4,a,b)d\tau \\ &\geq \lambda\int_{\delta_4}^{\varepsilon_4} K(t,\tau)g(\tau,R(1-\varepsilon_4)\delta_4,0,0)d\tau. \\ &\geq \lambda C_R. \end{aligned}$$

where C_R depends only on R. Thus, there exists $\lambda_1 > 0$ sufficiently large such that for all $\lambda > \lambda_1$, we have

$$\|Tu\|_\infty > \|u\|_\infty, \text{ for all } u \in \partial C_R, \text{ and } a, b \geq 0 .$$

We may choose the constants R_1, R_2 and R_3 such that $R_1 < R_3 < R < R_2$. Therefore, we may apply the Lemma 2.1, to obtain three fixed points of F in C satisfying

$$R_1 < \|u_1\|_\infty < R_3 < \|u_2\|_\infty < R < \|u_3\|_\infty < R_2,$$

and the proof of Theorem 1.2 is now complete.

5. Applications

In this section we will state some applications of Theorems 1.1 and 1.2 . Indeed, let us consider the following examples in annular domains. Through this section, we assume that $N \geq 3$.

Example 5.1. We consider the problem

$$\begin{array}{rcll} -\Delta u &=& \alpha c_1(|x|) + c_2(|x|)(\beta + u^p)\exp(\zeta u^q) & \text{in} \quad r_1 < |x| < r_2, \\ u(x) &=& 0 & \text{on} \quad |x| = r_1, \\ u(x) &=& 0 & \text{on} \quad |x| = r_2, \end{array} \tag{5.10}$$

where c_1, c_2 are nonnegative continuous functions, $0 < r_1 < r_2$, $\alpha, \beta \geq 0$; $p > 1$; $q \geq 0$ and $\zeta > 0$. Moreover, we suppose that there exists $t_0 \in (r_1, r_2)$ such that $c_1(t_0)$ and $c_2(t_0)$ are positive real numbers. Performing the change of variable $t = a(r)$ with

$$a(r) = -\frac{A}{r^{N-2}} + B,$$

where

$$A = \frac{(r_1 r_2)^{N-2}}{r_2^{N-2} - r_1^{N-2}} \text{ and } B = \frac{{r_2}^{N-2}}{r_2^{N-2} - r_1^{N-2}},$$

we obtain the equivalent problem

$$\begin{array}{rcl} -u'' &=& g(t, u(t), a, b) \quad \text{in} \quad (0,1) \\ u(0) &=& u(1) = 0 \end{array} \tag{5.11}$$

where $g(t, u, a, b) = a^p d_1(t) + d_2(t)(b^p + u^p)\exp(\zeta u^q)$, $\alpha = a^p$, $\beta = b^p$ and

$$d_i(t) = (1-N)^2 \frac{A^{2/(N-2)}}{(B-t)^{2(N-1)/(N-2)}} c_i\Big(\big(\frac{A}{B-t}\big)^{1/(N-2)}\Big), \quad \text{for } i = 1, 2.$$

It is not difficult to verify that (5.11) satisfies the hypotheses of Theorem 1.1. Hence, we may conclude that there exists $\overline{\alpha} > 0$ and a function $\Gamma : [0, \overline{\alpha}] \to [0, +\infty)$ satisfying

(i) If $\beta = 0$ or $\beta = \Gamma(\alpha)$, (5.11) has at least one positive solution.
(ii) If $0 < \beta < \Gamma(\alpha)$, (5.11) has at least two positive solutions.
(iii) If $\beta > \Gamma(\alpha)$, (5.11) has no positive solutions.

Example 5.2. We consider the problem

$$\begin{array}{rcllc} -\Delta u & = & \lambda\, c(|x|) f(u) & \text{in} & r_1 < |x| < r_2, \\ u(x) & = & a & \text{on} & |x| = r_1, \\ u(x) & = & b & \text{on} & |x| = r_2, \end{array} \tag{5.12}$$

where a, b are nonnegative parameters, $0 < r_1 < r_2,\ c : [0,+\infty) \to [0,+\infty)$ is continuous function and the nonlinearity f is a nondecreasing continuous function satisfying

(i) $\lim_{u\to 0} \frac{f(u)}{u} = 0$.

(ii) $\lim_{u\to +\infty} \frac{f(u)}{u} = +\infty$.

Then the conclusions of Theorem 1.1 are true. Indeed, coming from a similar way to the previous example, it is possible to verify that equation (5.12) is equivalent to

$$\begin{array}{rcll} -u'' & = & d(t)\, f(u + (1-t)a + tb) & \text{in} \quad (0,1) \\ u(0) & = & u(1) = 0 & \end{array} \tag{5.13}$$

where $g(t, u, a, b) = d(t)\, f(u + (1-t)a + tb)$ verifies the hypothesis of Theorem 1.1 with

$$d(t) = (1-N)^2 \frac{A^{2/(N-2)}}{(B-t)^{2(N-1)/(N-2)}} c\Big(\big(\frac{A}{B-t}\big)^{1/(N-2)}\Big).$$

We observe that Theorem 1.2 may be applied in Equation (5.12) assuming the hypothesis (i) above and moreover assuming the following sub-linear hypothesis at infinity

(iii) $\lim_{u\to +\infty} \frac{f(u)}{u} = 0$.

Finally, we notice that Theorem 1.2 may be applied to establish the existence and multiplicity (three) of solutions for the following two equations below.

Example 5.3. We consider the problem

$$\begin{array}{rcllc} -\Delta u & = & \lambda\, (c_1(|x|)u^{p_1} + 1)\Phi(c_2(|x|)u^{p_2}) & \text{in} & r_1 < |x| < r_2, \\ u(x) & = & a & \text{on} & |x| = r_1, \\ u(x) & = & b & \text{on} & |x| = r_2, \end{array} \tag{5.14}$$

where $p_1 < 1 < p_2,\ c_i : [r_1, r_2] \longrightarrow [0,+\infty)$ for $i = 1, 2$ are nontrivial and nonnegative continuous functions. Furthermore, it is assumed that $\Phi : [0,+\infty) \longrightarrow [0,+\infty)$ is a nondecreasing continuous function satisfying

$$\lim_{u\to 0} \frac{\Phi(u)}{u} = \hat{c}_1 \geq 0 \quad \text{and} \quad \lim_{u\to +\infty} \Phi(u) = \hat{c}_2 > 0.$$

Example 5.4. We consider the problem

$$\begin{array}{rcllc} -\Delta u & = & \lambda\, \frac{c_1(|x|)u^{p_3}}{1+c_2(|x|)u^{p_4}} & \text{in} & r_1 < |x| < r_2, \\ u(x) & = & a & \text{on} & |x| = r_1, \\ u(x) & = & b & \text{on} & |x| = r_2, \end{array} \tag{5.15}$$

where $1 < p_3 < 1 + p_4$ and the function $c_i(|x|)$ are like in the example above, verifying in addition that the intersection of its supports is not empty.

Remark 5.1. We note that 5.12 belongs to the frame of autonomous elliptic equations perturbed by a weight $c(|x|)$. When the weight is nonnegative and nontrivial on any compact subinterval in $(0,1)$, this type of problems has been considered in the literature by several authors (see for example [6] and [9]). We note that here the weight may vanish in parts of the annulus. In addition, Equations (5.10), (5.14) and (5.15) correspond to elliptic equations strongly non-autonomous. Finally, we notice that another novelty here is the multiplicity result of three positive solutions for the semi-linear elliptic equations in bounded annular domains with nonhomogeneous boundary conditions.

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