JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **207,** 104–126 (1997)
ARTICLE NO. AY975270

Existence of Solutions for Quasilinear Elliptic Equations

João Marcos Bezerra do Ó*

Departamento de Matemática, Universidade Federal da Paraíba, 58059.900 João Pessoa, PB, Brazil

Submitted by John Lavery

Received December 19, 1994

Using variational methods we study the existence and multiplicity of solutions of the Dirichlet problem for the equation

$$-\operatorname{div}\Big(a\big(|\nabla u|^p\big)|\nabla u|^{p-2}\nabla u\Big) = f(x,u).$$

1. INTRODUCTION

In this paper, we study the existence of solutions for the following class of quasilinear elliptic problems:

$$\begin{aligned} -\operatorname{div}\Big(a\big(|\nabla u|^p\big)|\nabla u|^{p-2}\nabla u\Big) &= f(x,u) \qquad \text{in } \Omega,\\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $p > 1$, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, f: $\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with subcritical growth, that is,

$$|f(x,u)| \le c_0|u|^{r-1} + d_0, \qquad \forall u \in \mathbb{R}, \text{ a.e. } x \in \Omega, \tag{2}$$

for some constants $c_0, d_0 > 0$, $1 \le r < p^*$, where $p^* = Np/(N-p)$ if $N > p$; $p^* = +\infty$ if $1 \le N \le p$ and a: $\mathbb{R}^+ \to \mathbb{R}$ is a continuous function

* Work partially supported by CAPES/Brazil. E-mail: jmbo@terra.npd.ufpb.br.

0022-247X/97 $25.00

such that

$$\left|a(u^p)u^{p-1}\right| \le \eta_0|u|^{p-1} + \zeta_0, \qquad \forall u \in \mathbb{R}^+, \tag{3}$$

where $\eta_0, \zeta_0 > 0$ are constants.

Here we search weak solutions of the problem (1), i.e., functions $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega a\left(|\nabla u|^p\right)|\nabla u|^{p-2}\nabla u \cdot \nabla v\,dx = \int_\Omega f(x,u)v\,dx, \qquad \forall v \in W_0^{1,p}(\Omega). \tag{4}$$

Observe that with the above growth conditions the expression in (4) is well defined.

In order to use variational methods, we suppose additional conditions on the function a. We assume that the function $h: \mathbb{R} \to \mathbb{R}$, given by $h(u) = A(|u|^p)$, where $A(u)$ is the primitive of $a(u)$, is strictly convex and

$$h(u) \ge \beta|u|^p - \alpha, \qquad u \in \mathbb{R}, \tag{5}$$

where α, β are constants with $\beta > 0$. Note that from (3), there exist positive constants η, ζ such that

$$h(u) \le \eta|u|^p + \zeta, \qquad \forall u \in \mathbb{R}. \tag{6}$$

Under these assumptions the functional $J: W_0^{1,p}(\Omega) \to \mathbb{R}$, given by

$$J(u) = \frac{1}{p}\int_\Omega A\left(|\nabla u|^p\right)dx, \tag{7}$$

is well defined, weakly lower semicontinuous, Fréchet differentiable, and J' (the derivative of J) is continuous and belongs to the class $(S)_+$. That is, for any sequence (u_n) in $W_0^{1,p}(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{and} \quad \lim_{n\to\infty} \sup\langle J'(u_n), u_n - u\rangle \le 0, \tag{8}$$

it follows that $u_n \to u$ in $W_0^{1,p}(\Omega)$ (here $\rightharpoonup$ denotes weak convergence and $\to$ denotes strong convergence). This is a special case of a more general class studied by Browder (cf. [6, 7]) in the theory of mappings of class $(S)_+$ of elliptic operators in the generalized divergence form.

Remark. An important example of problem (1) is given by $a(u) \equiv 1$, which corresponds to the so-called p-Laplacian. Explicitly, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. In this case for the constants appearing in conditions (5) and (6) we have $\beta = \eta = 1$.

With all the conditions given above, which we shall assume throughout the paper, the functional $I: W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$I(u) = J(u) - \int_\Omega F(x,u)\,dx, \tag{9}$$

where F is the primitive of f, is weakly lower semicontinuous, and C^1 on $W_0^{1,p}(\Omega)$ with

$$I'(u)v = \int_\Omega a\left(|\nabla u|^p\right)|\nabla u|^{p-2}\nabla u \cdot \nabla v\, dx - \int_\Omega f(x,u)v\, dx, \qquad \forall v \in W_0^{1,p}(\Omega). \quad (10)$$

Consequently the differential equation (1) is precisely the Euler equation of the functional I and the weak solutions of (1) are critical points of I and conversely (cf. [15, 21]). To find the critical points of I we need some compactness condition of the Palais–Smale type (cf. [3, 4]). With this in mind we consider the following basic assumption on the nonlinearities f:

$(F_1^+)_\mu$
$$pF(x,u) \le uf(x,u) - b_1|u|^\mu + b_2, \qquad \forall u \in \mathbb{R}, \text{ a.e. } x \in \Omega,$$

or

$(F_1^-)_\mu$
$$pF(x,u) \ge uf(x,u) + b_1|u|^\mu + b_2, \qquad \forall u \in \mathbb{R}, \text{ a.e. } x \in \Omega,$$

for some constants μ, b_1, b_2 with $b_1 > 0$, and we assume that the function a satisfies

(A_1^+)
$$A(u) - a(u)u \ge b, \qquad \forall u \in \mathbb{R}^+,$$

or

(A_1^-)
$$A(u) - a(u)u \le b, \qquad \forall u \in \mathbb{R}^+,$$

for some constant b.

Remark. Requirements of type $(F_1^\pm)_\mu$ were introduced by Costa and Magalhães [11–13] to study semilinear elliptic equations and systems.

Now consider the following nonlinear eigenvalue problem

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \lambda |u|^{p-2}u, \qquad u \in W_0^{1,p}(\Omega). \quad (11)$$

It is well known (cf. [1, 2]) that there exists a smallest positive eigenvalue λ, which we denote by $\lambda_1(p)$, and an associated function $\psi_1 > 0$ in Ω that solves (11), and that $\lambda_1(p)$ is a simple eigenvalue, i.e., any two solutions u, v of (11) satisfy $u = cv$ for some constant c. We recall that we have the following variational characterization

$$\lambda_1(p) = \inf\left\{\int_\Omega |\nabla u|^p\, dx : u \in W_0^{1,p}(\Omega), \int_\Omega |u|^p\, dx = 1\right\}. \quad (12)$$

Moreover, $\lambda_1(p)$ is an isolated eigenvalue of problem (11). Thus, given $W \subset W_0^{1,p}(\Omega)$ a closed complementary subspace to the one-dimensional subspace span$\{\psi_1\}$, we have

$$\lambda_W = \inf\left\{ \int_\Omega |\nabla w|^p \, dx : w \in W, \int_\Omega |w|^p \, dx = 1 \right\} > \lambda_1(p).$$

We denote by $\nu(p)$ the supreme of the numbers λ_W for all such closed complementary subspaces $W \subset W_0^{1,p}(\Omega)$.

Remark. It is not known if $\nu(p)$ as defined above is the second eigenvalue of the problem (11). This is in fact a very interesting question.

The class of problems considered here has applications in the study of non-Newtonian fluids, nonlinear elasticity, and reaction–diffusion. It has been studied recently by several authors, such as Anane [1, 2], Hirano [17], Narukawa and Suzuki [20], and Ubilla [23] (see also references therein). Here we obtain for this more general class of operators, analogous results to those obtained in [14] for the p-Laplacian. In addition we achieve a multiplicity result for these problems (as that in [17, 23]) using a version of the three critical points theorem (see Theorem 5 in the next section). In the last section we give some examples in order to illustrate the degree of generality of the kind of operators studied here.

Now we present the main results of this paper. The first three theorem treat the situation where resonance at the first eigenvalue may occur.

THEOREM 1. *Assume* $(F_1^+)_\mu$ *and in addition suppose that*

$$(F_2) \qquad \limsup_{|u| \to \infty} \frac{pF(x,u)}{|u|^p} \le \beta\lambda_1(p), \qquad a.e.\ \text{uniformly on } x \in \Omega.$$

Then, problem (1) *has a weak solution.*

In our next result we denote by $\tilde{\lambda}_i$ the ith eigenvalue of $-\Delta$ on Ω with zero boundary conditions [which corresponds to problem (11) with $p = 2$].

THEOREM 2. *Assume* $(F_1^+)_\mu$ *and* (A_1^+). *Furthermore suppose* $p \ge 2$ *and*

$$(A_2) \qquad \alpha_1 + \beta_1 u^{p-2} \le a(u^p) u^{p-2} \le \alpha_2 + \beta_2 u^{p-2}, \qquad \forall u \in \mathbb{R}^+,$$

$$(\tilde{F}_2) \qquad \limsup_{|u| \to \infty} \frac{pF(x,u)}{|u|^p} \le (\alpha_1 \delta_2(p) + \beta_1)\lambda_1(p), \qquad a.e.\ \text{uniformly on } x \in \Omega,$$

$$(F_3) \qquad d_2 \frac{\tilde{\lambda}_i}{2} u^2 \leq F(x,u) \leq d_1 \frac{\tilde{\lambda}_{i+1}}{2} u^2, \qquad \forall |u| \leq \sigma, \ a.e.\ x \in \Omega.$$

for some positive constants $\alpha_1, \alpha_2, \beta_1, \beta_2, \sigma$, *where*

$$d_1 < \begin{cases} (\alpha_1 + \beta_1)\left(1 - \tilde{\lambda}_1 \tilde{\lambda}_{i+1}^{-1}\right), & d_2 = \alpha_2 + \beta_2, \quad \textit{and} \quad \delta_2(p) = 1, \quad \textit{if } p = 2, \\ \alpha_1, & d_2 > \alpha_2, \quad \textit{and} \quad \delta_2(p) = 0, \quad \textit{if } p > 2. \end{cases}$$

Then problem (1) *possesses at least two nontrivial weak solution.*

THEOREM 3. *Assume* $(F_1^-)_\mu$ *and* (A_1^-) *and in addition suppose that for some real number* Λ *one has*

$$(\hat{F}_2) \qquad \lambda_1(p) \leq \limsup_{|u| \to \infty} \frac{p}{\eta} \frac{F(x,u)}{|u|^p} \leq \Lambda < \frac{\beta}{\eta} \nu(p), \qquad a.e. \textit{ uniformly on } x \in \Omega.$$

Then problem (1) *has a nontrivial weak solution.*

Remarks. As we will see in the next section, a compactness condition of the Palais–Smale type is a consequence of the following assumptions: $(F_1^+)_\mu$, (A_1^+) [or $(F_1^-)_\mu$, (A_1^-)], and

$$(F_4) \qquad F(x,u) \leq c_0 |u|^q + d_0, \qquad \forall u \in \mathbb{R}, \text{ a.e. } x \in \Omega,$$

where q satisfies the restrictions $p < q$ and $q - p < \mu$ if $1 \leq N \leq p$ or $(N(q-p))/p < \mu$ if $p < N$. From the growth hypothesis (2), we easily see that (F_4) is always satisfied with $q = r$. However, here we are interested in the case where (F_4) is satisfied for smaller values of q. Note that conditions like (F_2) appearing in the above theorems imply (F_4) with $q < r$. In the next theorem, which treats the case of crossing of the first eigenvalue, we replace condition (F_2) by (F_4) or (as in [22]) by the assumption

$$(F_5) \quad \theta F(x,u) \leq f(x,u)u + c_0 |u|^q + d_0, \qquad \forall u \in \mathbb{R}, \text{ a.e. } x \in \Omega,$$

where $p < \theta$. In case that (F_5) is assumed we require an additional technical restriction on the operator, namely,

(A_3) There exist constants c, d with $c > 0$ such that for all $u \in \mathbb{R}^+$,

$$\theta A(u) - p a(u) u \geq c u + d.$$

We see clearly that this kind of assumption is a natural generalization of the usual Ambrosetti–Rabinowitz superlinearity condition (cf. [3]). Observe that if F satisfies (F_4) and $f(x,u)u \geq c_1|u|^q + d_0 \quad \forall u \in \mathbb{R}$, a.e. $x \in \Omega$, then F also satisfies (F_5). However (F_4) and (F_5) are distinct in general. Note also that if $(F_1^+)_\mu$ is satisfied with $\mu > N/p(r-1)$, then, using the growth hypothesis (2), we obtain (F_5) as a consequence.

THEOREM 4. *Suppose* $(F_1^+)_\mu$, (A_1^+) [*or* $(F_1^-)_\mu$, (A_1^-)], *and* (F_4) [*or* (F_5) *and* (A_3)]. *Furthermore assume that*

$$(A_4) \qquad \alpha + \beta u^{p-r} \leq a(u^p)u^{p-r}, \qquad \forall u \in \mathbb{R}^+,$$

$$(F_6) \qquad \limsup_{|u| \to 0} \frac{rF(x,u)}{|u|^r} \leq K < \big(\alpha + \beta\delta_r(p)\big)\lambda_1(r),$$

a.e. uniformly on $x \in \Omega$,

$$(F_7) \qquad \liminf_{u \to +\infty} \frac{pF(x,u)}{|u|^p} \geq L > \eta\lambda_1(p),$$

a.e. uniformly on $x \in \Omega$,

where α, β *are positive constants,* $1 < r \leq p$ *and* $\delta_r(p) = 1$ *if* $p = r$, $\delta_r(p) = 0$ *if* $p \neq r$. *Then problem* (1) *has a nontrivial weak solution provided that* $p < q$, $q - p < \mu$ *if* $1 \leq N \leq p$, *or* $(N(q-p))/p < \mu$ *if* $p < N$.

Remark. We note that the same arguments allow us to show analogous results with less restricted conditions, where the potential $F(x,u)$ interacts with the first eigenvalue of some eigenvalue problem to the p-Laplacian with weights (cf. [16]).

2. THE ABSTRACT FRAMEWORK

We shall use a version of the Palais–Smale condition known as Cerami's condition [denoted by (Ce)] (cf. [8]). Let $(E, \|\cdot\|)$ be a real Banach space and $I: E \to \mathbb{R}$ be a C^1 functional. We say that I satisfies condition (Ce) if any sequence $(u_n) \subset E$ for which

$$\text{(i) } I(u_n) \to c; \qquad \text{(ii) } \|I'(u_n)\|(1 + \|u_n\|) \to 0, \qquad \text{as } n \to \infty, \tag{13}$$

possesses a convergent subsequence.

LEMMA 1. *Let* $J: W_0^{1,p}(\Omega) \to \mathbb{R}$ *be a* C^1 *functional such that* J' *belongs to the class* $(S)_+$ *and in addition suppose that the function* f *satisfies the growth conditions* (2). *Then the functional* $I(u) = J(u) - \int_\Omega F(x,u)\,dx$ *satisfies condition* (Ce), *provided that every sequence* (u_n) *in* $W_0^{1,p}(\Omega)$ *satisfying* (13), *is bounded.*

Proof. Let (u_n) in $W_0^{1,p}(\Omega)$ satisfy (13). Since (u_n) is bounded, we can take a subsequence denoted again by (u_n), such that for some u in $W_0^{1,p}(\Omega)$ we have $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$. Now using (ii) in (13) we obtain

$$\left| J'(u_n)v - \int_\Omega f(x,u_n)v\,dx \right| \leq \epsilon_n \|v\|_{W_0^{1,p}} \qquad \forall v \in W_0^{1,p}(\Omega), \quad (14)$$

where $\epsilon_n \to 0$, as $n \to \infty$. Note that by growth condition (2), taking a subsequence if necessary, we have $\lim_{n\to\infty}\int_\Omega f(x,u_n)(u_n-u)\,dx = 0$. Then considering $v = u_n - u$ in (14) and using that J' belongs to the class $(S)_+$, the result follows. ∎

LEMMA 2. *Suppose* $(F_1^+)_\mu$, (A_1^+) [*or* $(F_1^-)_\mu$, (A_1^-)], *and* (F_4) [*or* (F_5) *and* (A_3)], *with* $p < q$, $q - p < \mu$ *if* $1 \leq N \leq p$, *or* $(N(q-p))/p < \mu$ *if* $p < N$. *Then the functional I satisfies* (Ce).

Proof. We assume $(F_1^+)_\mu$ and (A_1^+); the proof with $(F_1^-)_\mu$ and (A_1^-) is similar. Let (u_n) in $W_0^{1,p}(\Omega)$ satisfy (13). By Lemma 1 it is sufficient to verify that (u_n) is bounded. From $(F_1^+)_\mu$, (A_1^+), and (13), we have that

$$\begin{aligned} c + 1 &\geq I(u_n) - \frac{1}{p} I'(u_n)u_n \\ &= \frac{1}{p}\int_\Omega \left[A(|\nabla u_n|^p) - a(|\nabla u_n|^p)|\nabla u_n|^p \right] dx \\ &\quad + \int_\Omega \left[\frac{1}{p} f(x,u_n)u_n - F(x,u_n) \right] dx \\ &\leq b_1 \int_\Omega |u|^\mu\,dx + c_1, \end{aligned}$$

for sufficiently large n. Consequently there exists a positive constant c_2 such that

$$|u_n|_{L^\mu} \leq c_2. \quad (15)$$

Considering $p < N$ and taking $t \in [0,1]$ such that $1/q = (1-t)/\mu + t/p^*$, where $p^* = NP/(N-p)$, from the Sobolev imbedding and (15), we obtain

$$|u_n|_{L^q} \leq c_3 |u_n|_{L^\mu}^{1-t} \|u_n\|_{W_0^{1,p}}^t \leq c_4 \|u_n\|_{W_0^{1,p}}^t. \quad (16)$$

Let us suppose that F satisfies (F_4). Thus using (5) and (i) in (13),

$$\begin{aligned} \beta \|u_n\|_{W_0^{1,p}}^p - \alpha \leq J(u_n) &= I(u_n) + \int_\Omega F(x,u_n)\,dx \\ &\leq c_5 + c_0 |u_n|_q^q + d_0. \end{aligned}$$

So, from (16),

$$\|u_n\|^p_{W_0^{1,p}} \leq c_6\Big(1 + \|u_n\|^{tq}_{W_0^{1,p}}\Big).$$

Then (u_n) is a bounded sequence, since $(N(q-p))/p < \mu$ is equivalent to $tq < p$.

On the other hand, if f satisfies (F_5) and the function A satisfies (A_3), we have

$$\begin{aligned} c_1 &\geq I(u_n) - \frac{1}{\theta} I'(u_n)u_n \\ &= \int_\Omega \left[\frac{1}{p} A\big(|\nabla u_n|^p\big) - \frac{1}{\theta} a\big(|\nabla u_n|^p\big)|\nabla u_n|^p \right] dx \\ &\quad + \int_\Omega \left[\frac{1}{\theta} f(x, u_n)u_n - F(x, u_n) \right] dx \\ &\leq c_2 \|u_n\|^p_{W_0^{1,p}} - c_0 |u_n|^q_q - c_3 \\ &\leq c_2 \|u_n\|^p_{W_0^{1,p}} - c_4 \|u_n\|^{tq}_{W_0^{1,p}} - c_3. \end{aligned}$$

Since $tq < p$ we obtain that (u_n) is a bounded sequence. ∎

Now, we give a result on the multiplicity of critical points for functionals that satisfy condition (Ce). Let E be an infinite dimensional Banach space with a decomposition

$$E = E_1 \oplus E_2 \tag{17}$$

with $0 < \dim E_2 < \infty$. We write any $u \in E$ as $u = u_1 + u_2 = (\mathrm{Id} - P)u + Pu$, where P is the projection onto E_2 along E_1.

THEOREM 5. *Let F: $E \to R$ be a C^1 functional satisfying* (Ce). *Furthermore assume that F is bounded below and that for some $R > 0$,*

$$\begin{aligned} F(u) &\geq 0, \qquad \text{for } u \in E_1, \\ &\leq 0, \qquad \text{for } u \in E_2, \qquad \|u\| \leq R. \end{aligned}$$

Then F has at least two nonzero critical points.

This theorem is related to the so-called three critical points theorem of Chang [9, 10] and to later results of Liu and Li [18], Liu [19], and Brezis and Nirenberg [5]. The proof of Theorem 5 follows the same kind of ideas used in the proof of an analogous result with condition (PS) instead of Cerami conditions (cf. [5]). Nevertheless for completeness we shall give here a proof of Theorem 5. For that matter we start with the following lemmas, which are the steps where the proof is slightly different from the one in [5].

LEMMA 3. *Assume that $F \in C^1(E, R)$ is a lower semicontinuous function bounded from below and satisfies* (Ce). *Then every minimizing sequence for F has a convergent subsequence.*

The proof of Lemma 3 follows from Ekeland's principle (cf. [15]), which we shall use in the following form:

EKELAND'S PRINCIPLE. Let (M, d) be a complete metric space and let

$$\theta : M \to (-\infty, +\infty], \qquad \theta \not\equiv +\infty,$$

be a lower semicontinuous function which is bounded from below. Let $\epsilon > 0$ be given and $\hat{u} \in M$ be such that

$$\theta(\hat{u}) \le \inf_M \theta + \epsilon.$$

Then, for any $\lambda > 0$, there exists $u_\lambda \in M$ such that

$$\begin{aligned} \theta(u_\lambda) &\le \theta(\hat{u}) \\ &< \theta(u) + \frac{\epsilon}{\lambda} d(u, u_\lambda) \qquad \forall u \ne u_\lambda, \\ d(u_\lambda, \hat{u}) &\le \lambda. \end{aligned}$$

Proof of Lemma 3. Let $(u_n) \in E$ be a minimizing sequence for F, i.e.,

$$F(u_n) \to \inf_E F.$$

For a subsequence, still denoted by (u_n), we may suppose that

$$F(u_n) \le \inf_E F + \frac{1}{n^2}.$$

We claim that $\|u_n\| \le C$. Assume by contradiction that for a subsequence we have $\|u_n\| \to \infty$. By Ekeland's principle with $\epsilon = 1/n^2$ and $\lambda = (1/n)\|u_n\|$, there exists v_n in E such that

$$\begin{aligned} F(v_n) &\le F(u_n) \\ &< F(u) + \frac{1}{n\|u_n\|}\|u - v_n\|, \qquad \forall u \ne v_n, \\ \|u_n - v_n\| &\le \frac{1}{n}\|u_n\|. \end{aligned}$$

Then, $\|F'(v_n)\|\,\|u_n\| \leq 1/n$ and $\|F'(v_n)\|\,\|v_n\| \leq (1/n)(1/n + 1)$, which implies that

$$\|F'(v_n)\|\big(\|v_n\| + 1\big) \to 0.$$

Thus, from (Ce) we have a convergent subsequence (v_{n_k}). However, since $\|v_n\| \to \infty$, this is impossible. So, it follows that (u_n) is a bounded sequence. Now using again Ekeland's principle with $\epsilon = 1/n^2$ and $\lambda = 1/n$, there exists v_n in E such that

$$\begin{aligned} F(v_n) &\leq F(u_n) \\ &< F(u) + \frac{1}{n}\|u - v_n\|, \qquad \forall u \neq v_n, \\ \|u_n - v_n\| &\leq \frac{1}{n}. \end{aligned}$$

Thus, $F'(v_n) \to 0$. Consequently $\|F'(v_n)\|(\|v_n\| + 1) \to 0$, since (v_n) is a bounded sequence. Therefore, from (Ce) we have a convergent subsequence (v_{n_k}) and consequently (u_{n_k}) also converges. ∎

Let V be a pseudogradient for F (cf. [21]), i.e.,

$$V\colon \tilde{E} = \{u \in E\colon F'(u) \neq 0\} \to E$$

is a locally Lipschitz continuous map, such that for all $u \in \tilde{E}$ we have

$$\text{(i) } \|V(u)\| \leq 2\|F'(u)\|; \qquad \text{(ii) } F'(u)V(u) \geq \|F'(u)\|^2$$

and consider $W\colon \tilde{E} \to E$, $W(u) = V(u)(\|u\| + 1)$. Note that W is also a locally Lipschitz continuous map and for all $u \in \tilde{E}$ we have

$$\text{(i) } \|W(u)\| \leq 2\|F'(u)\|\big(\|u\| + 1\big);$$

$$\text{(ii) } F'(u)W(u) \geq \|F'(u)\|^2\big(\|u\| + 1\big).$$

LEMMA 4. *Let $F\colon E \to R$ be a C^1 functional and $c \in R$. Then for any given $\delta < 1/8$ there exists a continuous deformation $\eta\colon [0,1] \times E \to E$ such that*

1°. *$\eta(0, u) = u$ for all $u \in E$.*

2°. *$\eta(t, u)$ is a homeomorphism of E onto E for each $t \in [0,1]$.*

3°. *$\eta(t, u) = u$ for all $t \in [0,1]$ if $|F(u) - c| \geq 2\delta$ or if $\|F'(u)\|(\|u\| + 1) \leq \sqrt{\delta}$.*

4°. *$0 \leq F(u) - F(\eta(t, u)) \leq 4\delta$ for all $t \in [0,1]$ and $u \in E$.*

5° *$\|\eta(t, u) - u\| \leq 16\sqrt{\delta}$ for all $t \in [0, 1]$ and $u \in E$.*

6°. *If* $F(u) \leq c + \delta$ *then either*

(i) $F(\eta(1, u)) \leq c - \delta$ *or*

(ii) *for some* $t_1 \in [0, 1]$, *we have* $\|F'(\eta(t_1, u)\|(\|\eta(t_1, u)\| + 1) \leq 2\sqrt{\delta}$.

7°. *More generally, let* $\tau \in [0, 1]$, *be such that for all* $t \in [0, \tau]$, $\eta(t, u)$ *belongs to the set* $\tilde{N} = \{v \in E: |F(v) - c| \leq \delta$ *and* $\|F'(v)\|(\|v\| + 1) \geq 2\sqrt{\delta}\}$. *Then* $F(\eta(\tau, u)) \leq F(u) - \tau/4$.

Proof. Since $\tilde{N}$ and the complement of the set

$$N = \left\{u \in E: |F(u) - c| < 2\delta \text{ and } \|F'(u)\|(\|u\| + 1) > \sqrt{\delta}\right\}$$

are disjoint closed sets, there is a locally Lipschitz function $g: E \to [0, 1]$ such that $g = 1$ on $\tilde{N}$ and $g = 0$ outside of N. Now consider the vector field

$$\Phi(u) = \begin{cases} -g(u)\dfrac{W(u)}{\|W(u)\|^2}, & \text{on } N, \\ 0, & \text{outside of } N, \end{cases}$$

and let $\eta(t, u)$ be the flow defined by

$$\frac{d\eta}{dt} = \Phi(u), \qquad \eta(0, u) = u.$$

From elementary properties of flow we obtain 1°–3° and from the properties of W it follows that

$$\frac{d}{dt}F(\eta(t, u)) \leq -\frac{1}{4}g(\eta(t, u)) \qquad \text{for all } (t, u) \in [0, 1] \times E.$$

Consequently

$$\int_0^t g(\eta(s, u))\, ds \leq 4\big(F(u) - F(\eta(t, u))\big).$$

Therefore, if $\eta(t, u)$ belongs to set $\tilde{N}$ for all $t \in [0, \tau]$, then $g(\eta(t, u)) = 1$ and 7° holds.

To verify 4°, note first that if $|F(u) - c| \geq 2\delta$, then $\eta(t, u) = u$ and 4° follows. So we may assume that $|F(u) - c| < 2\delta$. In this case we conclude that $F(\eta(1, u)) \geq c - 2\delta$ and consequently $F(u) - F(\eta(t, u)) \leq F(u) - F(\eta(1, u)) \leq c + 2\delta - (c - 2\delta) = 4\delta$.

Finally we will verify 5°: Consider $I_t = \{s\colon 0 \le s \le t \text{ and } \eta(s,u) \in N\}$,

$$\begin{aligned}
\|\eta(t,u) - u\| &\le \int_0^t \left\| \frac{d\eta}{dt}(s,u) \right\| ds \\
&\le \int_{I_t} \frac{g(\eta(s,u))}{\|W(\eta(s,u))\|}\, ds \\
&\le \int_{I_t} \frac{g(\eta(s,u))}{\|F'(\eta(s,u))\|(\|\eta(s,u)\| + 1)}\, ds \\
&\le \frac{1}{\sqrt{\delta}} \int_0^t g(\eta(s,u))\, ds \\
&\le \frac{4}{\sqrt{\delta}} (F(u) - F(\eta(t,u))) \\
&\le 16\sqrt{\delta}\,.
\end{aligned}$$

Thus we have finished the proof. ▮

Let K be a compact metric space and let K^* be a nonempty closed subset $\neq K$ Let

$$\mathscr{A} = \{ p \in C(K;E) \colon p = p^* \text{ on } K^*\},$$

where p^* is a fixed continuous map on K^* and

$$c = \inf_{p \in \mathscr{A}} \max_{\xi \in K} F(p(\xi)).$$

LEMMA 5. *Let $F\colon E \to R$ be a C^1 functional satisfying* (Ce). *Suppose that for every $p \in \mathscr{A}$ there is some point $\xi \in K \setminus K^*$ such that $F(p(\xi)) \ge c$ and in addition assume that there exists a closed set $\Sigma \subset E$, disjoint from $p^*(K^*)$, on which $F \ge c$ and such that $\forall p \in \mathscr{A}$, $p(K) \cap \Sigma \neq \emptyset$. Then F has a critical point $u_0 \in \Sigma$, with $F(u_0) = c$.*

Proof. Given $0 < \delta < 1/8$ such that $32\sqrt{\delta} < \operatorname{dist}(\Sigma, p^*(K^*))$ let us consider η a continuous deformation given by Lemma 4 and $p \in \mathscr{A}$ such that

$$\max_{\xi \in K} F(p(\xi)) < c + \delta.$$

Let $\zeta\colon E \to [0,1]$ be a continuous function such that

$$\zeta(u) = \begin{cases} 1 & \text{if dist } (u\,\Sigma) \leq 16\sqrt{\delta}\,, \\ 0, & \text{if dist } (u, \Sigma) \geq 32\sqrt{\delta}\,. \end{cases}$$

Consider $q\colon K \to E$ given by $q(\xi) = \eta(\zeta)p(\xi)), p(\xi))$. Note that $q \in \mathscr{A}$. Since $\forall p \in \mathscr{A}$, $p(K) \cap \Sigma \neq \emptyset$, thus there exists $\xi_1 \in K$ such that $u_1 = \eta(\zeta(p(\xi_1)), p(\xi)) \in \Sigma$. From properties 5° of η we have

$$\|\eta(t, p(\xi_1)) - p(\xi_1)\| \leq 16\sqrt{\delta} \qquad \text{for all } t \in [0,1].$$

Thus $\zeta(p(\xi_1)) = 1$, $u_1 = \eta(1, p(\xi_1))$, and $c \leq F(\eta(t, p(\xi_1))) < c + \delta$ for all $t \in [0,1]$. So, from properties (ii) in 6°, we have $t_1 \in [0,1]$ such that $u_2 = \eta(t_1, p(\xi_1))$ satisfies

$$\|F'(u_2)\|\big(\|u_2\| + 1\big) \leq 2\sqrt{\delta}\,,$$

and from properties 5° we obtain $\|u_1 - u_2\| \leq 32\sqrt{\delta}\,.$

Now, taking $\delta = 1/n$ for sufficiently large n, we obtain a sequence $\{u_n\}$ in E satisfying

$$F(u_n) \to c, \qquad \|F'(u_n)\|\big(\|u_n\| + 1\big) \to 0 \quad \text{and} \quad \text{dist}(u_n, \Sigma) \to 0.$$

Finally using the condition (Ce) we complete the proof. ∎

LEMMA 6. *Let $F\colon E \to R$ be a C^1 functional satisfying* (Ce) *and in addition suppose that for some $u_0 \in E$,*

$$F(u) > F(u_0), \qquad \forall u \neq u_0.$$

Let $y \neq u_0$ be such that $F'(y) \neq 0$ and F has no critical value in $(F(u_0), F(y))$. Then the "negative flow" starting at y, defined by

$$\frac{dx}{dt} = -\frac{W(x)}{\|W(x)\|^2}, \qquad x(0) = y, \tag{18}$$

exists for a maximal finite time $0 \leq t \leq T(y)$ and $x(T(y)) = u_0$.

Proof. We may suppose $u_0 = 0$ and $F(u_0) = 0$. By the theorem of existence and uniqueness for ordinary differential equations, it follows that there exists a maximum time $T(y) > 0$ such that the solution of (18) is defined on the interval $[0, T(y))$. From the properties of W and (18), we obtain

$$\frac{d}{dt}F(x(t)) \leq -\frac{1}{4}, \qquad \forall t \in [0, T(y)).$$

Consequently

$$T(y) \leq 4F(y) \quad \text{and} \quad 0 < F(x(t)) < F(y), \qquad \forall T \in (0, T(y)).$$

We claim that

$$x(t) \to 0 \qquad \text{as } t \to T(y).$$

Case 1. There exists $\delta > 0$ such that

$$\|F'(x(t))\|(\|x(t)\| + 1) \geq \delta, \qquad \forall t \in (0, T(y)).$$

Therefore,

$$\|W(x(t))\| \geq \|F'(x(t))\|(\|x(t)\| + 1) \geq \delta, \qquad \forall t \in (0, T(y))$$

and

$$\int_0^{T(y)} \frac{d}{dt}x(t)\, dt \qquad \text{exists}.$$

Thus, there exists $w \in E$ such that $x(t) \to w$ as $t \to T(y)$. Then necessarily $w = 0$; otherwise the solution $x(t)$ could be defined on an interval $[0, s)$ with $T(y) < s$. However, this contradicts the definition of $T(y)$.

Case 2. There exists a sequence $t_k \to T(y)$ such that

$$\|F'(x(t_k))\|(\|x(t_k)\| + 1) \to 0. \tag{19}$$

Then, by (Ce) there exists a subsequence still denoted by $x(t_k)$ and $w \in E$ such that $x(t_k) \to w$. From (19) we have that $F'(w) = 0$. Thus necessarily $w = 0$. Therefore, $F(x(t)) \to 0$ as $t \to T(y)$. Finally using Lemma 3 we get $x(t) \to 0$ as $t \to T(y)$. ∎

LEMMA 7. *Let v be a fixed unit vector in E_1 and set*

$$K = \{u = sv + u_2;\ u_2 \in E_2, \|u\| \leq 1 \text{ and } s \geq 0\}.$$

Consider any continuous map p: $K \to E$ satisfying

$$p(u_2) = u_2, \qquad \text{if } u_2 \in E_2 \text{ and } \|u_2\| \leq 1,$$

$$\|p(u)\| \geq r > 0, \qquad \text{if } u \in K \text{ and } \|u\| = 1.$$

Then, for any $r > 0$, *the image* $p(\partial K)$ *"links" the set of points in* E_1 *with norm* $\rho < r$. *That is, for any* $0 < \rho < r$, *there exists* $\bar{u} \in K$ *such that*

$$Pp(\bar{u}) = 0 \quad \textit{and} \quad \|p(u)\| = \rho.$$

For the proof of Lemma 7 we refer to Brezis and Nirenberg [5].

Proof of Theorem 5. From Lemma 3, F achieves its minimum at some point u_0. Supposing 0 and u_0 are the only critical points of F, we will get a contradiction. Note that in this case we have necessarily $F(u_0) < 0$ and without loss of generality we may assume $R = 1 < \|u_0\|$.

Now taking $\epsilon > 0$ sufficiently small such that $F(u) < 0$ if $\|u - u_0\| < \epsilon$, and using again Lemma 3, we obtain $\delta > 0$ such that

$$\{u \in E: F(u) \leq F(u_0) + \delta\} \subset \{u \in E: \|u - u_0\| < \epsilon\}.$$

By choosing δ sufficiently small, there is a continuous real valued function τ defined on the set $\{y \in E_2: \|y\| = 1\}$ such that $F(x(\tau(y))) = F(u_0) + \delta$ [where $x(t)$ is the flow starting at y given by the Lemma 6 and defined on the maximal interval $[0, T(y))$ with $x(t) \to u_0$ as $t \to T(y)$].

Let K be defined as in Lemma 7 and let $u \in \partial K$ with $u \neq v$ and $\|u\| = 1$. Thus we have the unique representation $u = sv + \sigma y$ with $0 \leq s \leq 1$, $y \in E_2$, $\|y\| = 1$, $0 < \sigma \leq 1$, i.e., s, σ, y are unique. Now, using this representation, we consider the map $p^*: \partial K \to E$, given by

$$p^*(u) = \begin{cases} u_0, & \text{if } u = v, \\ u, & \text{if } u \in E_2 \text{ and } \|u\| \leq 1, \\ x(2s\tau(y)), & \text{if } u = sv + \sigma y \text{ and } 0 \leq s \leq 1/2, \\ (2s-1)u_0 + (2-2s)x(\tau(y)), & \text{if } u = sv + \sigma y \text{ and } 1/2 \leq s < 1. \end{cases}$$

Note that $p^*(u)$ is a continuous map and $F(p^*(u)) \leq 0$, $\forall u \in \partial K$. Moreover we see that

$$\|p^*(u)\| \geq r > 0 \qquad \text{if } \|u\| = 1.$$

From Lemma 7, for any $p \in \Gamma = \{p: K \to E$ continuous and $p = p^*$ on $\partial K\}$ and $\rho < r$, the image $p(\partial K)$ "links" the set $\Sigma = \{u \in E_1: \|u\| = \rho\}$. Thus, using Lemma 5, we conclude that the nonnegative number

$$c = \inf_{p \in \Gamma} \max_{u \in K} F(p(u))$$

is a critical value of F. If $c > 0$ we have a second nontrivial critical point and the proof is finished. If $c = 0$ we apply Lemma 5 to get a critical point on Σ where $f = 0$, and therefore is different from the origin and u_0. ∎

3. PROOFS OF EXISTENCE RESULTS

Proof of Theorem 1. We shall prove that the functional I is coercive. By $(F_1^+)_\mu$, there exists $R > 0$ such that

$$\frac{d}{du}\left(\frac{F(x,u)}{|u|^p}\right) = \frac{f(x,u)u - pF(x,u)}{|u|^{p+1}} \geq b_1|u|^{\mu-p-1},$$

$$\forall |u| \geq R, \text{ a.e. } x \in \Omega.$$

Without loss of generality we may assume $\mu < p$. Thus, integrating and using the hypotheses (F_2), we have

$$F(x,u) \leq \frac{\beta}{p}\lambda_1(p)|u|^p - \frac{pb_1}{p-u}|u|^u + c_1, \qquad \forall u \in \mathbb{R}, \text{ a.e. } x \in \Omega. \tag{20}$$

Note that from the estimate above and (5) we get easily that I is bounded below. Now let us suppose that I is not coercive. Then there exists a sequence $\{u_n\}$ in $W_0^{1,p}(\Omega)$ such that $|I(u_n)| \leq C$ and $\|u_n\|_{W_0^{1,p}} \to \infty$. Let $v_n = u_n/\|u_n\|_{W_0^{1,p}}$ and let us assume (taking a subsequence) that v_n converges weakly in $W_0^{1,p}(\Omega)$, strongly in L^p, and a.e. to a certain function v_0 in $W_0^{1,p}(\Omega)$. Thus using (5) and (20) we obtain

$$c_2 \geq \frac{\beta}{p}\int_\Omega |\nabla u_n|^p\,dx - \frac{\beta}{p}\lambda_1(p)\int_\Omega |u_n|^p\,dx + \frac{pb_1}{p-\mu}\int_\Omega |u_n|^\mu\,dx + c_3. \tag{21}$$

Dividing (21) by $\|u_n\|^p_{W_0^{1,p}}$, we have

$$\frac{c_2}{\|u_n\|^p_{W_0^{1,p}}} \geq \frac{\beta}{p}\int_\Omega |\nabla v_n|^p\,dx - \frac{\beta}{p}\lambda_1(p)\int_\Omega |v_n|^p\,dx$$
$$+ \frac{pb_1}{p-\mu}\int_\Omega \frac{|v_n|^\mu}{\|u_n\|^{p-\mu}_{W_0^{1,p}}}\,dx + \frac{c_3}{\|u_n\|^p_{W_0^{1,p}}}.$$

Since $\|v_n\|_{W_0^{1,p}} = 1$, passing to the limit we obtain that

$$0 \geq 1 - \lambda_1(p)\int_\Omega |v_0|^p\,dx,$$

which implies that $v_0 \not\equiv 0$. Once more from (21), using the variational characterization of the first eigenvalue and dividing this expression by $\|u_n\|^\mu_{W_0^{1,p}}$, we have

$$\frac{c_2}{\|u_n\|^\mu_{W_0^{1,p}}} \geq \frac{pb_1}{p-\mu}\int_\Omega |v_n|^\mu\,dx + \frac{c_3}{\|u_n\|^\mu_{W_0^{1,p}}},$$

which, by passing to the limit gives

$$0 \geq \frac{pb_1}{p-\mu} \int_\Omega |\nabla v_0|^\mu \, dx;$$

thus $v_0 \equiv 0$, which is a contradiction.

Finally taking $R > 0$ such that $I(u) \geq 0$ if $\|u\|_{W_0^{1,p}} \geq R$ and using that I is weakly lower semicontinuous we obtain $u_0 \in W_0^{1,p}(\Omega)$ such that $\|u_0\|_{W_0^{1,p}} \leq R$ and $I(u_0) = \inf\{I(u) \colon \|u\|_{W_0^{1,p}} \leq R\} = \inf_{W_0^{1,p}} I$. Therefore u_0 is a critical point of I. ∎

Proof of Theorem 2. Here we apply the three critical points theorem with condition (Ce) (see Theorem 5). As in the proof of Theorem 1, we obtain that the functional I is bounded from below and Lemma 2 gives that I satisfies (Ce). So it only remains to prove the local linking condition. We shall do it in the sequel.

We recall that in this theorem we are supposing that $p \geq 2$. We denote by H_k the finite dimensional subspace of $W_0^{1,p}(\Omega)$ generated by the eigenfunctions of $(-\Delta, H_0^1(\Omega))$ corresponding to the eigenvalues $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_k$ and $W_k = W_0^{1,p}(\Omega) \cap H_k^\perp$, where $H_k^\perp$ denotes the orthogonal subspace of H_k in $H_0^1(\Omega)$. Thus we have

$$\begin{aligned} W_0^{1,p}(\Omega) &= H_k \oplus W_k, \\ \|u\|_{H_0^1}^2 &\geq \tilde{\lambda}_{k+1} \|u\|_{L^2}^2, \qquad \forall u \in W_k, \\ &\leq \tilde{\lambda}_k \|u\|_{L^2}^2, \qquad \forall u \in H_k. \end{aligned}$$

LEMMA 8 (local linking). *There exists a positive constant ρ such that*

$$\begin{aligned} I(u) &\leq 0, \qquad \forall u \in H_i, \|u\|_{W_0^{1,p}} \leq \rho, \\ &\geq 0, \qquad \forall u \in W_i, \|u\|_{W_0^{1,p}} \leq \rho. \end{aligned}$$

Proof. Using (A_2), (F_3) and the fact that in H_i the norms $\| \ \|_{W_0^{1,p}}$ and $\| \ \|_\infty$ are equivalent, we obtain $\rho_1 > 0$ such that

$$I(u) \leq \frac{\alpha_2}{2} \int_\Omega |\nabla u|^2 \, dx + \frac{\beta_2}{p} \int_\Omega |\nabla u|^p \, dx - d_2 \frac{\tilde{\lambda}_i}{2} \int_\Omega u^2 \, dx,$$

for every $u \in H_i$ with $\|u\|_{W_0^{1,p}} \leq \rho_1$. Therefore, from the above estimate we get $\rho \leq \rho_1$ such that

$$I(u) \leq 0 \qquad \text{if } u \in H_i \text{ and } \|u\|_{W_0^{1,p}} \leq \rho,$$

since $2 \le p$ and

$$d_2 = \alpha_2 + \beta_2, \qquad \text{if } p = 2$$
$$> \alpha_2, \qquad \text{if } p > 2.$$

To prove the second assertion we use $(F_1^+)_\mu$, $(\tilde{F}_2)$, and (F_3), to obtain constants $c > 0$ and $2 < r_0 < p^*$ such that

$$F(u) \le d_1 \tilde{\lambda}_{i+1} \frac{|u|^2}{2} + \frac{(\alpha_1 \delta_2(p) + \beta_1)\lambda_1(p)}{p}|u|^p + c|u|^{r_0}, \qquad \forall u \in \mathbb{R}.$$

Consequently from (A_2) we obtain

$$\begin{aligned} I(u) \ge{}& \frac{\alpha_1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\beta_1}{p}\int_\Omega |\nabla u|^p\,dx \\ &- d_1 \frac{\tilde{\lambda}_{i+1}}{2}\int_\Omega |u|^2\,dx - \frac{(\alpha_1\delta_2(p) + \beta_1)\lambda_1(p)}{p} \\ &\times \int_\Omega |u|^p\,dx - c\int_\Omega |u|^{r_0}\,dx. \end{aligned}$$

Case 1: $p = 2$. For all $u \in W_i$,

$$\begin{aligned} I(u) \ge{}& \frac{\alpha_1 + \beta_1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{(\alpha_1 + \beta_1)\tilde{\lambda}_1}{2}\int_\Omega |u|^2\,dx \\ &- d_1 \frac{\tilde{\lambda}_{i+1}}{2}\int_\Omega |u|^2\,dx - c\int_\Omega |u|^{r_0}\,dx \\ \ge{}& \frac{1}{2}\Big[(\alpha_1 + \beta_1)\big(1 - \tilde{\lambda}_1\tilde{\lambda}_{i+1}^{-1}\big) - d_1\Big]\|u\|_{H_0^1}^2 - c_1\|u\|_{H_0^1}^{r_0}, \end{aligned}$$

where in the last inequality we have used that $\|u\|_{H_0^1}^2 \ge \tilde{\lambda}_{i+1}\|u\|_{L^2}^2$, $\forall u \in W_i$, and the Sobolev imbedding $H_0^1(\Omega) \hookrightarrow L^{r_0}(\Omega)$. Now using that $d_1 < (\alpha_1 + \beta_1)(1 - \tilde{\lambda}_1\tilde{\lambda}_{i+1}^{-1})$ and $2 < r_0$, we an choose $\rho > 0$ such that

$$I(u) > 0, \qquad \forall u \in W_i; \qquad 0 < \|u\|_{W_0^{1,p}} \le \rho.$$

Case 2: $p > 2$. For all $u \in W_i$,

$$
\begin{aligned}
I(u) \geq\ & \frac{\alpha_1}{2}\int_\Omega |\nabla u|^2\,dx - d_1\frac{\tilde{\lambda}_{i+1}}{2}\int_\Omega |u|^2\,dx \\
& + \frac{\beta_1}{p}\int_\Omega |\nabla u|^p\,dx - \frac{\beta_1\tilde{\lambda}_1(p)}{p}\int_\Omega |u|^p\,dx - c\int_\Omega |u|^{r_0}\,dx \\
\geq\ & \frac{\alpha_1 - d_1}{2}\|u\|^2_{H_0^1} - c_1\|u\|^{r_0}_{H_0^1}.
\end{aligned}
$$

Finally, using that $d_1 < \alpha_1$, $2 < r_0$, and the continuous imbedding $W_0^{1,p}(\Omega) \hookrightarrow H_0^1(\Omega)$, we complete the proof of the lemma. ∎

Proof of Theorem 3.

LEMMA 9. *Suppose* $(F_1^-)_\mu$ *and* $(\hat{F}_2)$. *Then* $I(t\psi_1) \to -\infty$ *as* $t \to \pm\infty$, *where* ψ_1 *is the* $\lambda_1(p)$ *eigenfunction with* $\|\psi_1\|_{W_0^{1,p}} = 1$. *Moreover there exists* $W \subset W_0^{1,p}(\Omega)$ *a closed complementary subspace to one dimensional subspace* $V = \operatorname{span}(\psi_1)$, *such that I is bounded below in W*.

Proof. Using $(F_1^-)_\mu$ and the first inequality of $(\hat{F}_2)$ we obtain

$$
F(x,u) \geq \frac{\eta\lambda_1(p)}{p}|u|^p + c|u|^\mu + d,
$$

where c and d are constants with $c > 0$. Thus from (6) we have the estimate

$$
\begin{aligned}
I(t\psi_1) \leq\ & |t|^p\frac{\eta}{p}\int_\Omega |\nabla\psi_1|^p\,dx - |t|^p\frac{\eta\lambda_1(p)}{p}\int_\Omega |\psi_1|^p\,dx \\
& - c|t|^\mu\int_\Omega |\psi_1|^\mu\,dx - d_1 \\
=\ & -c|t|^\mu\int_\Omega |\psi_1|^\mu\,dx - d_1.
\end{aligned}
$$

Consequently we have $I(t\psi_1) \to -\infty$ as $T \to \pm\infty$.

Now taking $\epsilon > 0$ such that $\eta(\Lambda + \epsilon) < \beta\nu(p)$, then there exists $W \subset W_0^{1,p}(\Omega)$ a closed complementary subspace to one dimensional subspace $\operatorname{span}\{\psi_1\}$, such that

$$
\frac{\eta(\Lambda + \epsilon)}{\beta} \leq \inf_{0 \neq w \in W}\frac{\int_\Omega |\nabla w|^p\,dx}{\int_\Omega |w|^p\,dx}.
$$

Thus, using (5) and $(\hat{F}_2)$ we obtain for all $w \in W$,

$$I(w) \geq \frac{\beta}{p}\int_\Omega |\nabla w|^p \, dx - \frac{\eta(\Lambda + \epsilon)}{p}\int_\Omega |w|^p \, dx - c \geq -c.$$

Therefore I is bounded below in W. ▮

Finally to get a nontrivial solution to problem (1), we use Lemmas 2 and 9 and apply the mountain-pass theorem with condition (Ce) instead of condition (PS). To prove this mountain-pass theorem version with condition (Ce) we use a deformation theorem like Lemma 4 and proceed as in [21], where the standard version is proved.

Proof of Theorem 4. Here we apply again the mountain-pass theorem [21] with condition (Ce). In order to get the geometrical conditions we need the following

LEMMA 10. *Suppose* (A_4) *with* $1 < r \leq p$, (F_6), *and* (F_7). *Then there exist* $\delta, \rho > 0$ *such that* $I(u) \geq \delta$ *if* $\|u\|_{W_0^{1,p}} = \rho$. *Moreover*, $I(s\psi_1) \to -\infty$ *as* $s \to +\infty$, *where* $\psi_1 > 0$ *is the* $\lambda_1(p)$ *eigenfunction with* $\|\psi_1\|_{W_0^{1,p}} = 1$.

Proof. Using growth condition (2) and (F_6), there exists $r_0 \in (p, p^*)$ such that

$$\begin{aligned}\int_\Omega F(x,u)\, dx &\leq \frac{K+\epsilon}{r}\int_\Omega |u|^r \, dx + c_1\int_\Omega |u|^{r_0}\, dx \\ &\leq \frac{K+\epsilon}{r\lambda_1(r)}\|u\|^r_{W_0^{1,r}} + c_2\|u\|^{r_0}_{W_0^{1,r}},\end{aligned}$$

where in the last inequality we have used the variational characterization of the first eigenvalue and the Sobolev imbedding. From (A_4) we get

$$I(u) \geq \frac{\alpha}{r}\|u\|^r_{W_0^{1,r}} + \frac{\beta}{p}\|u\|^p_{W_0^{1,p}} - \frac{K+\epsilon}{r\lambda_1(r)}\|u\|^r_{W_0^{1,r}} - c_2\|u\|^{r_0}_{W_0^{1,p}}.$$

Since $r \leq p < r_0$, $W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,r}(\Omega)$, taking $\epsilon > 0$ such that

$$K + \epsilon < \big(\alpha + \beta\delta_r(p)\big)\lambda_1(r),$$

we can fix $\rho > 0$ and $\delta > 0$ such that $I(u) \geq \delta$ if $\|u\|_{W_0^{1,p}} = \rho$.

Now we will prove the second assertion. Choose $\sigma > 0$ such that $L - \sigma > \eta\lambda_1(p)$. From condition (F_7) we obtain

$$F(x,u) \geq \frac{1}{p}(L - \sigma)u^p - c_4, \qquad \forall u \geq 0.$$

Finally, using (6) we have the estimate

$$\begin{aligned} I(s\psi_1) &\le \eta \frac{s^p}{p} \int_\Omega |\nabla \psi_1|^p \, dx - \frac{s^p}{p}(L - \sigma) \int_\Omega \psi_1^p \, dx + c \\ &\le \eta \frac{s^p}{p} \left[1 - \frac{L - \sigma}{\eta \lambda_1(p)} \right] \| \psi_1 \|_{W_0^{1,p}}^p + c. \end{aligned}$$

Clearly we obtain $I(s\psi_1) \to -\infty$ as $s \to +\infty$. Thus we obtain that the assertions hold. ∎

4. SOME EXAMPLES

EXAMPLE 1. Let $a\colon \mathbb{R}^+ \to \mathbb{R}$ satisfy the following conditions:

(a_1) there exist constants $b_1, b_2 > 0$ such that $b_1 \le a(u) \le b_2$;

(a_2) the function $k\colon \mathbb{R} \to \mathbb{R}$, $k(u) = a(|u|^p)|u|^{p-2}, u$ is strictly increasing.

Note that the function a satisfies the conditions (3), (5), and $h(u) = A(|u|^p)$ is strictly convex. Suppose that $f(x, u) = g(u) + h(x)$, where $h \in L^\infty(\Omega)$ and g satisfies $\lim_{|u| \to \infty}((pG(u))/|u|^p) = \lambda_1(p) b_1$ and $(G_1^+)_\mu$ with $\mu = 1$ and $\|h\|_{L^\infty} < b_1(p-1)$ or $\mu > 1$. As a consequence of Theorem 1, problem (1) has a weak solution.

EXAMPLE 2. Let $a(u) = \beta + \eta/((1+u)^p)$, where α, β are positive constants. Consider $f(x, u) = g(u) + h(x)$ with $h \in L^\infty(\Omega)$, such that g satisfies both $\lim_{|u| \to \infty}((pG(u))/|u|^p) = \beta \lambda_1(p)$ and $(G_1^-)_\mu$, where either $\mu = 1$ and $\|h\|_\infty < b_1(p-1)$ or $\mu > 1$. As a consequence of Theorem 3, we have a weak solution for the problem

$$\begin{aligned} -\operatorname{div}\left\{ \left(\eta + \frac{\beta}{(1 + |\nabla u|^p)^p} \right) |\nabla u|^{p-2} \nabla u \right\} &= f(x, u), \qquad \text{in } \Omega, \\ u = 0, \qquad \text{on } \partial\Omega&. \end{aligned}$$

EXAMPLE 3. Consider the function $a(u) = \beta + \eta u^{(2-p)/p}$, where $2 \le p$, $0 \le \beta, \eta$ are constants. By Theorem 2, the problem

$$\begin{aligned} -\beta \Delta_p u - \eta \Delta u &= f(u) \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

possesses at least two nontrivial weak solutions provided that f satisfies $(F_1^+)_\mu$, $(\tilde{F}_2)$, and (F_3).

EXAMPLE 4. Let $a(t) = t/\sqrt{1+t^2}$, which corresponds to the so-called modified capillary surface equation (cf. [20]). Explicitly,

$$-\operatorname{div}\left(\frac{|\nabla u|^{2p-1}\nabla u}{\sqrt{1+|\nabla u|^{2p}}}\right) = f(x,u) \qquad \text{in } \Omega,$$

$$u = 0 \qquad \text{on } \partial\Omega$$

Suppose that $f(x,u) = \lambda_1(p)|u|^{p-2}u + g(u) + h(x)$, where $h \in L^\infty(\Omega)$ and that g satisfies $\lim_{|u|\to\infty}((G(u)/|u|^p) \leq 0$ and $(G_1^-)_\mu$ with $\mu = 1$ and $\|h\|_\infty < b_1/(p-1)$ or $\mu > 1$. As a consequence of Theorem 3, we have a nontrivial weak solution for this problem.

EXAMPLE 5. From Theorem 4 we have a nontrivial weak solution for the problem

$$-\Delta_p u - \Delta_r u = F'(u) \qquad \text{in } \Omega; \qquad u = 0 \qquad \text{on } \partial\Omega,$$

where $1 < r < p$ and $F \in C^1(\mathbb{R},\mathbb{R})$ is such that $F(u) = |u|^p \ln|u|$ if $|u| \geq 2R > 0$ and $F(u) = \lambda|u|^r$ if $|u| \leq R$, with $\lambda < \lambda_1(r)$.

ACKNOWLEDGMENTS

The author wishes to thank Professors Djairo G. de Figueiredo and Elves A. de Barros e Silva for several suggestions.

REFERENCES

1. A. Anane, Simplicité et isolation de la premére valeur propre du p-Laplacien avec poids, *C. R. Acad. Sci. Paris* **305** (1987), 725–728.
2. A. Anane, "Étude des Valeurs Propres et de la Resonance pour l'Operaeur p-Laplacien," Thèse de Doctorat, Université Libre de Bruxelles, 1988.
3. A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* **14** (1973), 349–381.
4. P. Bartolo, V. Benci, and Fortunato, Abstract critical point theorems and applications to some nonlinear problems with "strong resonance" at infinity, *Nonlinear Anal.* **7** (1983), 981–1012.
5. H. Brezis and L. Nirenberg, Remarks on finding critical points, *Comm. Pure Appl. Math.* **44** (1991), 939–963.
6. F. Browder, Existence theorems for nonlinear partial differential equations, *Proc. Sympos. Pure Math.* **16** (1970), 1–60.
7. F. Browder, Fixed point theory and nonlinear problems, *Bull. Amer. Math. Soc.* **9** (1983), 1–39.
8. G. Cerami, Un cirterio di esistenza per i punti critici su varietà illimitate, *Rend. Accad. Sci. Let. Ist. Lombardo* **112** (1978), 332–336.

9. K. C. Chang, Solutions of asymptotically linear operator equations via Morse theory, *Comm. Pure Appl. Math.* **34** (1981), 693–712.
10. K. C. Chang, ''Infinite Dimensional Morse Theory and its Applications,'' Séminaire de Mathématiques Supérieures, Vol. 97, Presses Univ. Montréal, 1985.
11. D. G. Costa and C. A. Magalhães, Variational elliptic problems which are nonquadratic at infinity, *Trabalhos de Matemática* **260** (1991).
12. D. G. Costa and C. A. Magalhães, A variational approach to subquadratic perturbations of elliptic systems, preprint.
13. D. G. Costa and C. A. Magalhães, Un problème elliptique non-quadratique à l'infini, *C. R. Acad. Sci. Paris*, t. 315, Série I, (1992), 1059–1062.
14. D. G. Costa and C. A. Magalhães, Existence results for perturbations of p-Laplacian, preprint.
15. D. G. de Figueiredo, Lectures notes on the Ekeland variational principle with applications and detours, *Tata Inst. Fund. Res. Lectures on Math. Phys.* **81** (1989).
16. D. G. de Figueiredo and I. Massabó, Semilinear elliptic equations with the primitive of the nonlinearity interacting with the first eigenvalue, *J. Math. Anal. Appl.* **156** (1991), 381–394.
17. N. Hirano, Multiple solutions for quasilinear elliptic equations, *Nonlinear Anal.* **15** (1990), 625–638.
18. J. Q. Liu and S. J. Li, Existence theorems of multiple critical points and their applications, *Kexue Tongbao* **17** (1984), 1025–1027.
19. J. Q. Liu, Morse index of a saddle point, *J. Systems Sci. Math. Sci.* **2** (1989), 42–49.
20. K. Narukawa and T. Suzuki, Nonlinear eigenvalue problem for a modified capillary surface equation, *Funkcial. Ekva.* **37** (1994), 81–100.
21. P. H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, *CBMS Regional Conf. Ser. in Math.* **65** (1986).
22. E. A. de B. e Silva, Existence and multiplicity of solutions for semilinear elliptic systems, *Nonlinear Differential Equations Appl.*, to appear.
23. P. Ubilla, ''Alguns Resultados de Multiplicidade de Soluções para Equações Elíticas quasilineares,'' Doctoral dissertation, Unicamp, 1992.
24. P. Ubilla, Multiplicity results for quasilinear elliptic equations, *in* ''Proceedings of the 36th Seminário Brasileiro de Análise, 1992.''