Available online at www.sciencedirect.com
SCIENCE DIRECT®

Journal of Functional Analysis 224 (2005) 471–496

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

An Orlicz-space approach to superlinear elliptic systems ☆

Djairo G. de Figueiredo[a,*], João Marcos do Ó[b], Bernhard Ruf[c]

[a]*IMECC, Univ. Estadual di Campinas, 13081-970 Campinas, SP, Brazil*
[b]*Dep. de Matemática, Univ. Federal Paraíba, 58059-900 João Pessoa, PB, Brazil*
[c]*Dip. di Matematica, Università degli Studi, Via Saldini 50, 20133 Milano, Italy*

Received 27 July 2004; accepted 30 September 2004
Communicated by H. Brezis
Available online 8 December 2004

Abstract

In this paper we study superlinear elliptic systems in Hamiltonian form. Using an Orlicz-space setting, we extend the notion of critical growth to superlinear nonlinearities which do not have a polynomial growth. Existence of nontrivial solutions is proved for superlinear nonlinearities which are subcritical in this generalized sense.

MSC: 35J55; 35J20

Keywords: Elliptic system; Critical growth; Orlicz space; Variational methods

☆ Research partially supported by CNPq, PRONEX-MCT/Brazil and Millennium Institute for the Global Advancement of Brazilian Mathematics—IM-AGIMB.

* Corresponding author.

E-mail addresses: djairo@ime.unicamp.br (D.G. de Figueiredo), jmbo@mat.ufpb.br (J. Marcos do Ó), ruf@mat.unimi.it (B. Ruf).

0022-1236/$ - see front matter
doi:10.1016/j.jfa.2004.09.008

1. Introduction

In this paper, we study nonlinear elliptic systems in Hamiltonian form

$$\begin{cases} -\Delta u = g(v) & \text{in } \Omega, \\ -\Delta v = f(u) & \text{in } \Omega, \\ v > 0, \ u > 0 & \text{in } \Omega, \\ u = 0, \ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N (N \geqslant 3)$, with smooth boundary $\partial\Omega$ and Δ is the Laplace operator.

For the scalar equation $-\Delta u = f(u)$ critical growth is given by $f(s) \sim s^{\frac{N+2}{N-2}}$. This is obtained by considering the related functional $\frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(u)$, where $F(s) = \int_0^s f(t)\,dt$. The natural space for the first term is $H_0^1(\Omega)$, and then the maximal growth allowed for $F(s)$ is given by the Sobolev embedding $H_0^1(\Omega) \subset L^{2^*}(\Omega)$. For the system (1.1) the associated functional is

$$\int_\Omega \nabla u \nabla v\,dx - \int_\Omega (F(u) + G(v))\,\,dx. \tag{1.2}$$

If we consider this functional on $H_0^1(\Omega) \times H_0^1(\Omega)$, then we find again the maximal growths $F(s) \sim |s|^{2^*}$ and $G(s) \sim |s|^{2^*}$. However, in interesting papers by Hulshof–Van der Vorst [5] and Felmer–de Figueiredo [2] the use of Sobolev spaces of fractional order has been proposed. Roughly speaking, these spaces, denoted by $H^s(\Omega), s > 0$, consist of the functions whose derivative of order s is in $L^2(\Omega)$ (these spaces can be defined via interpolation or via Fourier expansion). Introducing suitable self-adjoint operators $A^s : H^s(\Omega) \to L^2(\Omega)$, the first term in the functional (1.2) can be substituted by

$$\int_\Omega A^s u A^t v \quad \text{with } s + t = 2.$$

The maximal growth condition on $F(s) \sim |s|^{p+1}$ and $G(s) \sim |s|^{q+1}$ is then given by the largest values p and q such that $H^s \subset L^{p+1}$ and $H^t \subset L^{q+1}$. This yields the so-called *critical hyperbola*

$$\frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{N}. \tag{1.3}$$

One notes that now one of the nonlinearities may have a larger growth than $|s|^{2^*}$ provided the other nonlinearity has a suitably lower growth.

We propose here another approach: in order to have the term $\int_\Omega \nabla u \nabla v$ well-defined, we can use Hölder's inequality to estimate

$$|\int_\Omega \nabla u \nabla v| \leqslant \|u\|_{W_0^{1,\alpha}} \|v\|_{W_0^{1,\beta}} \quad \text{with } \frac{1}{\alpha} + \frac{1}{\beta} = 1 \tag{1.4}$$

and hence the first term in (1.2) can be considered on the space $W_0^{1,\alpha}(\Omega) \times W_0^{1,\beta}(\Omega)$. Again, the maximal growth for F and G is then given by the embeddings

$$W_0^{1,\alpha}(\Omega) \subset L^{p+1}(\Omega) \quad \text{and} \quad W_0^{1,\beta}(\Omega) \subset L^{q+1}(\Omega)$$

with

$$\frac{1}{p+1} = \frac{1}{\alpha} - \frac{1}{N} \quad \text{and} \quad \frac{1}{q+1} = \frac{1}{\beta} - \frac{1}{N}; \tag{1.5}$$

thus, this approach yields the same critical hyperbola.

Consequently, the same equations studied in [2,5] can be treated also by this approach. However, the second approach has the advantage that it can be generalized to more general situations by using an *Orlicz-space* setting. Namely, we can replace the spaces $W_0^{1,\alpha}$ and $W_0^{1,\beta}$ by Sobolev–Orlicz spaces $W_0^1 L_A$ and $W_0^1 L_{\tilde{A}}$; here $W_0^1 L_A$ is given by the functions u such that $\int_\Omega A(|\nabla u|) < \infty$, where A is a so-called N-function, and u vanishes in a certain sense at the boundary. The function $\tilde{A}$ is the Young-conjugate function to A (see below). Then we have again a Hölder-type inequality

$$|\int_\Omega \nabla u \nabla v| \leqslant 2\|u\|_{W_0^1 L_A} \|v\|_{W_0^1 L_{\tilde{A}}}.$$

The maximal growth for F and G are then determined by the Orlicz-space embeddings

$$W_0^1 L_A(\Omega) \subset L_\Phi(\Omega) \quad \text{and} \quad W_0^1 L_{\tilde{A}}(\Omega) \subset L_\Psi(\Omega), \tag{1.6}$$

where L_Φ and L_Ψ are Orlicz spaces with suitable N-functions Φ and Ψ. In this way, we can treat nonlinearities f and g in Eq. (1.1) that cannot be handled by the Hilbert space approach of Hulshof and van der Vorst [5] and de Figueiredo and Felmer [2].

Let us call two N-functions (Φ, Ψ) a *critical Orlicz pair*, if they have the maximal possible growth in the above embeddings (1.6), for suitable N-functions A and $\widetilde{A}$. For example, $\Phi(s) = |s|^{p+1}$ and $\Psi(s) = |s|^{q+1}$ with $p, q \in (1, +\infty)$ satisfying (1.3) constitute a critical Orlicz pair, with $A(s) = |s|^\alpha$ and $\widetilde{A}(s) = |s|^\beta$, where α and β are given by (1.5) (see Example 2.1).

We will prove

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded, smooth domain. Let $\Phi \in C^1$ be a given N-function, and set $\varphi(t) = \Phi'(t)$. Assume that*

$$\lim_{s\to\infty} \frac{\varphi(s)\, s}{\Phi(s)} = \theta_\Phi > \frac{N}{N-2}. \tag{1.7}$$

Then there exists an associate N-function Ψ such that (Φ, Ψ) form a critical Orlicz pair. Furthermore, the limit

$$\lim_{s\to\infty} \frac{s\Psi'(s)}{\Psi(s)} = \theta_\Psi$$

exists, and

$$\frac{1}{\theta_\Phi} + \frac{1}{\theta_\Psi} = 1 - \frac{2}{N}. \tag{1.8}$$

We give some examples of critical Orlicz pairs:

Example 1.1. Let

$$\Phi(s) \sim |s|^{p+1}(\log|s|)^{\alpha} \quad \text{and} \quad \Psi(s) \sim |s|^{q+1}(\log|s|)^{-\alpha\frac{q+1}{p+1}}$$

with $\alpha > 0$ and $p, q \in (1, +\infty)$ satisfying (1.3). Then Φ and Ψ satisfy the two above limits, with $\theta_\Phi = p+1$ and $\theta_\Psi = q+1$, respectively, and (Φ, Ψ) form a critical Orlicz-pair (see the proof in *Section* 6).

Remark 1.2. The restriction $\theta_\Phi > N/(N-2)$ is necessary in order to obtain a Ψ which is θ-regular, in the sense of *Definition* 2.10. Also in the polynomial case such a restriction, which here is $p+1 > N/(N-2)$, is necessary in order to obtain $q > 1$.

Based on this characterization, we will prove an existence theorem for superlinear nonlinearities, which have subcritical growth with respect to the above specified Orlicz criticality.

We make the following hypotheses

(H1) let $f, g \in C(\mathbb{R})$ and let F, G denote their primitives;
(H2) there exist constants $\theta > 2$ and $t_0 > 0$ such that, for all $t \geqslant t_0$,

$$0 < \theta F(t) \leqslant t f(t) \quad \text{and} \quad 0 < \theta G(t) \leqslant t g(t).$$

(H3) F and G are uniformly superquadratic near zero (for the definition, see Definition 5.1 below).

Then we prove

Theorem 1.3. *Suppose that* (Φ, Ψ) *is a critical Orlicz pair. Suppose that* F *and* G *satisfy* (H1)–(H3), *and that* F *and* G *have an essentially slower growth than* Φ *and* Ψ, *respectively* (*see Section* 2 *below*). *Suppose also that*

$$\lim_{t\to 0} \frac{F(t)}{\Phi(t)} = C_F < \infty \quad \textit{and} \quad \lim_{t\to 0} \frac{G(t)}{\Psi(t)} = C_G < \infty.$$

Then system (1.1) *has a nontrivial solution.*

Example 1.2. Let (Φ, Ψ) denote the critical Orlicz pair given in Example 1.1. Suppose that $F(s) \sim s^{p+1}(\log s)^{\beta}$ and $G(s) \sim s^{q+1}(\log s)^{-\gamma}$, for s positive and large with $\beta < \alpha$ and $\gamma > \alpha \frac{q+1}{p+1}$. Then F and G have essentially slower growth than Φ and Ψ, respectively.

2. Orlicz spaces

Here we recall some basic facts about Orlicz spaces, for more details see for instance [1,6,9]. Let M be a N-function, that is, $M : \mathbb{R} \to [0, +\infty)$ is continuous, convex, even, $M(t) = 0$ if and only if $t = 0$,

$$M(t)/t \to 0 \text{ as } t \to 0 \text{ and } M(t)/t \to +\infty \text{ as } t \to +\infty.$$

We say that an N-function A *dominates* an N-function B (near infinity) if, for some positive constant k, $B(x) \leqslant A(kx)$ (for $x \geqslant x_0$), and write $B \prec A$. A and B are *equivalent* if A dominates B and B dominates A; then we write $A \sim B$. Finally, we say that B *increases essentially more slowly* than A if $\lim_{t\to\infty} \frac{B(kt)}{A(t)} = 0$, for all $k > 0$; in this case we write $B \prec\prec A$.

Associated to the N-function M we have the following class of functions.

Definition 2.1 (*Orlicz class*). The Orlicz class is defined by

$$K_M(\Omega) := \{u : \Omega \to \mathbb{R} : u \text{ measurable and} \int_{\Omega} M(u(x))\, dx < \infty\}.$$

Orlicz classes are convex sets, but in general not linear spaces. One then defines

Definition 2.2 (*Orlicz space*). The vector space $L_M(\Omega)$ generated by $K_M(\Omega)$ is called Orlicz space.

Fact 1. The Orlicz class $K_M(\Omega)$ is a vector space, and hence equal to $L_M(\Omega)$ if and only if M satisfies the following

Definition 2.3 (Δ_2-*condition*). There exist numbers $k > 1$ and $t_0 \geqslant 0$ such that

$$M(2t) \leqslant kM(t) \quad \text{for } t \geqslant t_0.$$

Furthermore, we define

Definition 2.4 (∇_2-*condition*). There exist numbers $h > 1$ and $t_1 \geqslant 0$ such that

$$M(t) \leqslant \frac{1}{2h} M(ht) \quad \text{for } t \geqslant t_1.$$

We call a function satisfying the Δ_2- and the ∇_2-condition *Δ-regular*.

We remark that the Orlicz class depends only on the asymptotic growth of the function *M*; therefore, also the Δ_2-condition and the ∇_2-condition need to be satisfied only near infinity.

We define the following norm on $L_M(\Omega)$:

Definition 2.5 (*Luxemburg norm*).

$$\|u\|_{(M)} = \inf\{\, \lambda > 0 : \int_\Omega M(\frac{|u|}{\lambda}) \leqslant 1\}$$

Fact 2. $(L_M, \|\cdot\|_{(M)})$ is a Banach space.

Definition 2.6 (*Conjugate function*). *Let*

$$\widetilde{M}(x) = \sup_{y>0} \{xy - M(y)\}.$$

$\widetilde{M}$ is called the Young conjugate function of M.

It is clear that $\widetilde{\widetilde{M}} = M$, and M and $\widetilde{M}$ satisfy the Young inequality:

$$st \leqslant M(t) + \widetilde{M}(s) \quad \forall s, \quad t \in \mathbb{R},$$

with *equality* when $s = M'(t)$ or $t = \widetilde{M}'(s)$. In the spaces L_M and $L_{\widetilde{M}}$ the *Hölder inequality* holds

$$|\int_\Omega u(x)v(x)\,dx| \leqslant 2\|u\|_{(M)}\|v\|_{(\widetilde{M})}.$$

Hence, for every $\widetilde{u} \in L_{\widetilde{M}}$ we can define a continuous linear functional $l_{\tilde{u}} v := \int_\Omega \tilde{u} v \, dx$ and $l_{\tilde{u}} \in (L_M)^*$. Then we can define

$$\|\tilde{u}\|_{\widetilde{M}} = \|l_{\tilde{u}}\| = \sup_{\|v\|_{(M)} \leqslant 1} \int_\Omega \tilde{u}(x) v(x) \, dx$$

Definition 2.7. $\|\tilde{u}\|_{\widetilde{M}}$ is called the *Orlicz norm* on the space $L_{\widetilde{M}}$, and analogously one defines the Orlicz norm $\|u\|_M$ on L_M.

Thus, we have two different norms on L_M, the Luxemburg (or gauge) norm $\|\cdot\|_{(M)}$ and the Orlicz norm $\|\cdot\|_M$; they are equivalent, and satisfy

$$\|u\|_{(M)} \leqslant \|u\|_M \leqslant 2\|u\|_{(M)}. \tag{2.1}$$

In order to be precise about which norm is considered in the spaces, we are going to use from now on the following notations:

$$\big(L_M, \|\cdot\|_M\big) := L_M \quad \text{and} \quad \big(L_M, \|\cdot\|_{(M)}\big) := L_{(M)}$$

and similarly for $\widetilde{M}$.

Fact 3. It follows from the definition of Orlicz norm that, if $u \in L_M$ and $\widetilde{w} \in L_{\widetilde{M}}$, then one has the following Hölder inequality:

$$\Big|\int_\Omega u\widetilde{w} \, dx\Big| \leqslant \|u\|_M \|\widetilde{w}\|_{(\widetilde{M})}. \tag{2.2}$$

Fact 4. L_M is *reflexive* if and only if M and $\widetilde{M}$ satisfy the Δ_2-condition, and then

$$(L_{(M)})^* = L_{\widetilde{M}} \quad \text{and} \quad (L_{(\widetilde{M})})^* = L_M$$

(see [4,9, p. 111]).

Fact 5. If Φ is Δ-regular, then there exists a $\Phi_1 \sim \Phi$ such that $L_\Phi = L_{\Phi_1}$ as sets, and their Luxemburg norms (respectively, Orlicz norms) are equivalent, with the following additional structure:

(a) L_Φ and L_{Φ_1} are isomorphic, and both are reflexive spaces,
(b) L_{Φ_1} is uniformly convex and uniformly smooth (see [9, Theorem 2, p. 297]).

Next, we define the *Orlicz–Sobolev* spaces: Let A be a N-function. Then set

Definition 2.8.

$$W^1L_A = \left\{u : \Omega \to \mathbb{R};\ \max_{|\alpha|\in\{0,1\}} \int_\Omega A(|D^\alpha u|) < +\infty\right\}$$

with Luxemburg norm

$$\|u\|_{W^1L_A} := \max\left\{\|D^\alpha u\|_{(M)} : |\alpha| \in \{0,1\}\right\}.$$

On the space $W_0^1L_{(A)}$, i.e. the space of functions in W^1L_A which vanish on the boundary, an equivalent Luxemburg norm is given by

$$\|u\|_{1,(A)} = \|\nabla u\|_{(A)} = \inf\left\{\lambda > 0 : \int_\Omega A\left(\frac{|\nabla u|}{\lambda}\right) \leqslant 1\right\}.$$

The equivalence of these two norms is a consequence of the Poincaré inequality,

$$\|u\|_{(M)} \leqslant C \sum_{i=1}^n \|D_i u\|_{(M)} \quad \forall u \in W_0^1L_M(\Omega),$$

(see [4]). In analogy with the above definition of the Orlicz norm in L_M we can define an Orlicz norm in $W_0^1L_{(A)}$ by

$$\|u\|_{1,A} := \sup\left\{\int_\Omega \nabla u \nabla \widetilde{w}\, dx : \widetilde{w} \in W_0^1L_{(\widetilde{A})}, \|\widetilde{w}\|_{1,(\widetilde{A})} \leqslant 1\right\}.$$

The space $W_0^1L_{(A)}$ endowed with this new norm is denoted by $W_0^1L_A$.

Definition 2.9 (*Sobolev conjugate (Adams [1], p. 248)*). Suppose that $\int_1^\infty \frac{A^{-1}(t)}{t^{1+1/n}}\, dt = +\infty$. Then the Sobolev conjugate function $\Phi(t)$ is given by

$$\Phi^{-1}(t) = \int_0^t \frac{A^{-1}(\tau)}{\tau^{1+1/n}}\, d\tau, \quad t \geqslant 0.$$

Fact 6. Let Ω be bounded, and satisfying the cone property. Then

$$W^1L_A(\Omega) \hookrightarrow L_\Phi(\Omega) \text{ continuously}$$

and compactly into $L_G(\Omega)$, where G is any N-function increasing *essentially more slowly* than Φ, i.e. $\lim_{t\to\infty} \frac{G(kt)}{\Phi(t)} = 0$, for all $k > 0$.

Example 2.1. One easily checks that for $\Phi(s) = s^{p+1}$ the above formula yields $A(s) = cs^{\alpha}$, with α satisfying $\frac{1}{\alpha} = \frac{1}{p+1} + \frac{1}{N}$, i.e. we have the classical Sobolev imbedding theorem for $W^{1,\alpha}(\Omega) \hookrightarrow L^{p+1}(\Omega)$.

Next, we make the following:

Definition 2.10. Let $g \in C(\mathbb{R})$ be an N-function, and G its primitive. Then we say that G is θ-regular, if there exists a constant $\theta_G > 1$ such that

$$\lim_{s\to\infty} \frac{sg(s)}{G(s)} = \theta_G. \tag{2.3}$$

Let $F(t) = G^{-1}(t)$, and $f(t) = F'(t)$. Then the above condition is equivalent to

$$\lim_{t\to\infty} \frac{tf(t)}{F(t)} = \frac{1}{\theta_G}. \tag{2.4}$$

Indeed, we have $G(s) = t \Leftrightarrow F(t) = s$, and $f(t) = \frac{d}{dt}[G^{-1}(t)] = \frac{1}{g(s)}$.

Note that by Rao-Ren [9, p. 26] we have

Fact 7. If G is θ-regular, then G is Δ-regular, i.e. $G \in \Delta_2 \cap \nabla_2$.

3. Orlicz-space criticality

Definition 3.1 (*Critical Orlicz pair*). Let Φ and Ψ be Δ-regular N-functions. Then (Φ, Ψ) are a *critical Orlicz pair* if there exist Δ-regular and conjugate N-functions A and $\widetilde{A}$ such that L_Φ and L_Ψ are the smallest Orlicz spaces with

$$W^1 L_A \hookrightarrow L_\Phi, \quad W^1 L_{\widetilde{A}} \hookrightarrow L_\Psi.$$

Consider the following example:

Example 3.1. In Example 2.1 we saw that to $\Phi(s) = s^{p+1}$ corresponds the inverse Sobolev conjugate $A(s) = cs^{\alpha}$, with

$$\frac{1}{p+1} + \frac{1}{N} = \frac{1}{\alpha}.$$

The conjugate function $\widetilde{A}$ to A is given by $\widetilde{A}(s) = cs^{\beta}$, with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, which in turn has as Sobolev conjugate $\Psi(s) = s^{q+1}$, with

$$\frac{1}{q+1} + \frac{1}{N} = \frac{1}{\beta}.$$

Adding the two equation yields

$$\frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{N}.$$

This is the critical hyperbola, see [2,5]. Thus, $(|s|^{p+1}, |s|^{q+1})$ are a critical Orlicz pair, and so the above theorem contains the critical hyperbola as a special case. We remark that the proof given here is also new in the polynomial case; in [2,5] fractional Sobolev spaces H^s were used in order to conserve the Hilbert space structure.

3.1. Proof of Theorem 1.1

(1) Hypothesis (1.7) expresses the fact that the function Φ is θ-regular with $\theta_\Phi > N/(N-2)$. Let A be the inverse Sobolev conjugate of Φ, see Definition 2.9. Note that W^1L_A is the largest Orlicz–Sobolev space that embeds into L_Φ.

Claim 1. *A is θ-regular, with* $\theta_A = \frac{N\theta_\Phi}{N+\theta_\Phi} > 1$.

Indeed, let $F(s) = \Phi^{-1}(s)$ and $B(t) = A^{-1}(t)$. Then $F(s) = \int_0^s \frac{B(t)}{t^{1+1/N}}\,dt$, and hence

$$f(s) = \frac{B(s)}{s^{1+1/N}}.$$

Then we have by (2.4)

$$\frac{1}{\theta_\Phi} = \lim_{s\to\infty} \frac{f(s)s}{F(s)} = \lim_{s\to\infty} \frac{B(s)\ s^{-1/N}}{F(s)}.$$

Then, by l'Hospital's rule

$$\lim_{s\to\infty} \frac{B(s)\ s^{-1/N}}{F(s)} = \lim_{s\to\infty} \frac{b(s)\ s^{-1/N} - \frac{1}{N}\ s^{-\frac{1}{N}-1}\ B(s)}{\frac{B(s)}{s^{1+1/N}}} = \lim_{s\to\infty} \frac{b(s)s}{B(s)} - \frac{1}{N}.$$

We conclude that

$$\frac{1}{\theta_\Phi} = \lim_{s\to\infty} \frac{b(s)s}{B(s)} - \frac{1}{N}$$

and thus

$$\lim_{s\to\infty} \frac{b(s)s}{B(s)} = \frac{1}{\theta_\Phi} + \frac{1}{N} < 1.$$

This implies that A is θ-regular, with $\theta_A = \frac{N\theta_\Phi}{N+\theta_\Phi} > 1$.

(2) Next, let $\widetilde{A}$ be the conjugate function of A, given by Definition 2.6. $\widetilde{A}$ is a N-function, and Δ-regular, see [9, Corollary 4, p. 26].

In the following, suppose that $s = A'(t)$ (iff $t = \widetilde{A}'(s)$); note that $t \to \infty$ iff $s \to \infty$. Then

$$\frac{1}{\theta_A} = \lim_{t\to\infty} \frac{A(t)}{tA'(t)} = \lim_{t\to\infty} \frac{A(t)}{ts} = \lim_{s\to\infty} \frac{st - \widetilde{A}(s)}{st} = 1 - \lim_{s\to\infty} \frac{\widetilde{A}(s)}{s\widetilde{A}'(s)} = 1 - \frac{1}{\theta_{\widetilde{A}}}.$$

Thus, $\widetilde{A}$ is θ-regular, with $\theta_{\widetilde{A}} > 1$.

We can now define the corresponding Orlicz–Sobolev space $W^1L_{\widetilde{A}}$.

(3) Next, use Definition 2.9 again to define the function Ψ; by Adams [1, p. 248], Ψ is an N-function.

Claim 2. *Ψ is θ-regular, with* $\theta_\Psi = \frac{N\theta_{\widetilde{A}}}{N-\theta_{\widetilde{A}}}$.

This follows similarly as in Claim 1, reversing the direction in the arguments.

Finally, L_Ψ is the smallest Orlicz space into which $W^1L_{\widetilde{A}}$ imbeds continuously.

Thus, we have shown that (Φ, Ψ) is a critical Orlicz pair.

Finally, we have

$$\frac{1}{\theta_\Phi} + \frac{1}{\theta_\Psi} = \frac{N-\theta_A}{N\theta_A} + \frac{N-\theta_{\widetilde{A}}}{N\widetilde{A}} = \frac{1}{\theta_A} - \frac{1}{N} + \frac{1}{\theta_{\widetilde{A}}} - \frac{1}{N} = 1 - \frac{2}{N}. \qquad \square$$

4. The tilde-map

In this section we define a map from $W_0^1L_A$ to the space $W_0^1L_{(\widetilde{A})}$, where $\widetilde{A}$ is the Young conjugate of A.

Theorem 4.1. *For each $u \in W_0^1L_A$ consider*

$$S := \sup\left\{ \int_\Omega \nabla u(x)\nabla\widetilde{w}(x)\,dx : \widetilde{w} \in W_0^1L_{(\widetilde{A})},\ \|\widetilde{w}\|_{1,(\widetilde{A})} = \|u\|_{1,A} \right\}. \tag{4.1}$$

Then there exists a unique $\widetilde{u} \in W_0^1L_{(\widetilde{A})}$ such that

$$\|\widetilde{u}\|_{1,(\widetilde{A})} = \|u\|_{1,A} \quad \text{and} \quad S = \int_\Omega \nabla u(x)\nabla\tilde{u}(x)\,dx = \|u\|_{1,A}\|\widetilde{u}\|_{1,(\widetilde{A})}$$

Furthermore, $\tilde{u}$ depends continuously (but nonlinearly) on u.

Proof. We first remark that by Fact 5 we may assume that all the spaces W^1L_A, $W^1L_{(A)}$, $W^1L_{\widetilde{A}}$ and $W^1L_{(\widetilde{A})}$ are reflexive and uniformly convex. Observe also that by (2.2)

$$S \leqslant \sup \|\widetilde{w}\|_{1,(\widetilde{A})}\|u\|_{1,A} = \|u\|_{1,A}^2.$$

• *Existence*: Let $\{\widetilde{v}_n\}$ be a maximizing sequence for (4.1). Since the sequence is bounded, we have by reflexivity that (for a subsequence) $\tilde{u} \rightharpoonup \tilde{v}$ weakly in $W_0^1 L_{(\widetilde{A})}$. So $\int_\Omega \nabla u \nabla \widetilde{v}_n \to S$, and consequently $\int_\Omega \nabla u \nabla \widetilde{u} = S$, that is, the supremum is attained. It remains to prove that $\|\widetilde{u}\|_{1,(\widetilde{A})} = \|u\|_{1,A}$. Suppose by contradiction that $\|\widetilde{u}\|_{1,(\widetilde{A})} = k < \|u\|_{1,A}$. Take $\widetilde{w} = (\widetilde{u}/k)\|u\|_{1,A}$. Then for this $\widetilde{w}$ we have

$$S \geqslant \int_\Omega \nabla u \nabla \widetilde{w} = \int_\Omega \nabla u \nabla \widetilde{u} \, \frac{\|u\|_{1,A}}{k} > S,$$

which is impossible.

• *Uniqueness*: by the uniform convexity of $W_0^1 L_A$, $\tilde{u}$ is unique.

• *Continuity*: Let $u_n \to u \neq 0$ in $W_0^1 L_A$. By the above we have $\|\widetilde{u}_n\|_{1,(\widetilde{A})} = \|u_n\|_{1,A}$ and $\|\widetilde{u}\|_{1,(\widetilde{A})} = \|u\|_{1,A}$. Consequently $\|\widetilde{u}_n\|_{1,(\widetilde{A})} \to \|\widetilde{u}\|_{1,A}$. So we have that, for some subsequence, $\widetilde{u}_n \rightharpoonup \widetilde{v}$ in $W_0^1 L_{(\widetilde{A})}$. If we prove that $\widetilde{v} = \widetilde{u}$, then in fact we are concluding that the whole sequence $\widetilde{u}_n$ converges strongly to $\widetilde{u}$ in $W_0^1 L_{(\widetilde{A})}$. To this end observe that

$$\int_\Omega \nabla u_n \nabla \widetilde{u}_n = \|\widetilde{u}_n\|_{1,(\widetilde{A})} \|u_n\|_{1,A}$$

implies

$$\int_\Omega \nabla u \nabla \widetilde{v} = \|\widetilde{u}\|_{1,(\widetilde{A})} \|u\|_{1,A}.$$

We claim that $\|\widetilde{v}\|_{1,(\widetilde{A})} = \|u\|_{1,A}$, and then by the uniqueness of $\widetilde{u}$ it follows that $\widetilde{v} = \widetilde{u}$ and the proof is complete. The claim is proved by contradiction as we did in the existence above assuming that $\|\widetilde{v}\|_{1,(\widetilde{A})} = k < \|u\|_{1,A}$. □

Using the previous theorem we now define the "*tilde-map*"

$$\begin{array}{rccc} \sim: & W_0^1 L_A & \longrightarrow & W_0^1 L_{(\widetilde{A})}, \\ & u & \longmapsto & \widetilde{u}, \end{array}$$

which is continuous.

Remark 4.2. It follows from the construction that the tilde-map is positively homogeneous, i.e.

$$\widetilde{\rho u} = \rho \widetilde{u} \quad \forall u \in W^1 L_A \quad \forall \rho \geqslant 0$$

With the help of the tilde-map, we define two continuous sub-manifolds of

$$E := W_0^1 L_A \times W_0^1 L_{(\widetilde{A})}$$

by

$$E^+ = \{(u, \widetilde{u});\ u \in W_0^1 L_A\} \quad \text{and} \quad E^- = \{(u, -\widetilde{u});\ u \in W_0^1 L_A\}.$$

We remark that E^+ and E^- are nonlinear submanifolds of E when regarded with respect to the standard vector space structure of E. Surprisingly, E^+ and E^- turn out to be *linear* with respect to the following notion of *tilde-sum*:

Definition 4.3 (*Tilde sum*). Given elements $(u, \widetilde{v}) \in E$ and $(y, \widetilde{z}) \in E$, we set

$$(u, \widetilde{v})\ \widetilde{+}\ (y, \widetilde{z}) := (u + y, \widetilde{v + z}).$$

Indeed, with this notion we can prove

Lemma 4.4. (1) *Let* $(u, \widetilde{u}) \in E^+$ *and* $(v, \widetilde{v}) \in E^+$; *then, for all* $\alpha, \beta \in \mathbb{R}$

$$\alpha(u, \widetilde{u})\ \widetilde{+}\ \beta(v, \widetilde{v}) \in E^+ \quad \textit{and} \quad \alpha(u, \widetilde{u})\ \widetilde{+}\ \beta(v, \widetilde{v}) = (\alpha u + \beta v, \widetilde{\alpha u + \beta v}).$$

(2) *For every* $(y, \widetilde{z}) \in E$ *there exist unique elements* $(u, \widetilde{u}) \in E^+$ *and* $(v, -\widetilde{v}) \in E^-$ *such that*

$$(y, \tilde{z}) = (u, \tilde{u})\ \widetilde{+}\ (v, -\tilde{v}),$$

i.e. we can write

$$E = E^+\ \widetilde{\oplus}\ E^-.$$

Proof. (1) We have

$$\begin{aligned}\alpha(u, \tilde{v})\ \widetilde{+}\ \beta(y, \tilde{z}) &= (\alpha u, \alpha\tilde{v})\ \widetilde{+}\ (\beta y, \beta\tilde{z}) \\ &= (\alpha u, \widetilde{\alpha v})\ \widetilde{+}\ (\beta y, \widetilde{\beta z}) = (\alpha u + \beta y, \widetilde{\alpha v + \beta z}).\end{aligned}$$

(2) • *Uniqueness*: Suppose that $(u + v, \widetilde{u - v}) = (u_1 + v_1, \widetilde{u_1 - v_1})$; then $u + v = u_1 + v_1$ and $\widetilde{u - v} = \widetilde{u_1 - v_1}$, which implies $u = u_1$ and $v = v_1$.

• *Existence*: We look for elements u and v in $W_0^1 L_A$ such that

$$(y, \tilde{z}) = (u, \tilde{u})\ \widetilde{+}\ (v, -\tilde{v}) = (u + v, \widetilde{u - v}).$$

That is, $y = u + v$ and $z = u - v$, and hence $u = \frac{y+z}{2}$ and $v = \frac{y-z}{2}$. $\square$

5. An existence theorem

In this section we prove the existence of a nontrivial solution for system (1.1), in the case of superlinear nonlinearities which have subcritical growth with respect to a given critical Orlicz pair.

5.1. The functional

In this section, we define the framework for the functional I associated to problem (1.1) and given in (1.2).

We first give the precise definition of *uniformly superquadratic near zero*.

Definition 5.1. A continuous function $H : \mathbb{R} \to \mathbb{R}$ is *uniformly superquadratic near zero*, if there exist numbers $\sigma > 2$ and $c \geqslant 1$ such that

$$H(st) \leqslant cs^{\sigma}\, H(t) \quad \forall t > 0 \quad \forall s \in [0, 1]$$

Note that if $H(t) = t^p$ with $p > 2$, then H satisfies the definition with $\sigma = p$.

In Section 3 we have proved the existence of critical Orlicz pairs (Φ, Ψ). As specified in Theorem 1.3, we assume that the functions F and G grow essentially more slowly than Φ and Ψ, respectively. Since we are interested in positive solutions we redefine F and G to be zero on $(-\infty, 0]$.

Consider the functional $I : W_0^1 L_A(\Omega) \times W_0^1 L_{(\widetilde{A})}(\Omega) \to \mathbb{R}$ given by

$$I(u, \widetilde{v}) = \int_{\Omega} \nabla u \nabla \widetilde{v}\, dx - \int_{\Omega} [F(u) + G(\widetilde{v})]\, dx \tag{5.1}$$

Here $\tilde{v} \in W_0^1 L_{(\widetilde{A})}(\Omega)$ is an independent variable; we write $\tilde{v}$ to emphasize that $\tilde{v}$ belongs to the space $W_0^1 L_{(\widetilde{A})}(\Omega)$.

The functional I is well defined and belongs to the class C^1 with

$$I'(u, \widetilde{v})(\eta, \widetilde{\xi}) = \int_{\Omega} [\nabla u \nabla \widetilde{\xi} + \nabla \widetilde{v} \nabla \eta]\, dx - \int_{\Omega} [f(u)\eta + g(\widetilde{v})\widetilde{\xi}\ ]\, dx, \tag{5.2}$$

for all $(\eta, \widetilde{\xi}) \in W_0^1 L_A(\Omega) \times W_0^1 L_{(\widetilde{A})}(\Omega)$. Consequently, critical points of the functional I correspond to the weak solutions of (1.1).

5.2. The geometry of the linking theorem

We first prove that the functional I given in (5.1) has the geometry of the linking theorem.

Let E^+ and E^- be as in Section 4.

Lemma 5.2. *There exist* $\rho_0,\ \sigma_0 > 0$ *such that* $I(z) \geqslant \sigma_0$, *for all* $z \in \partial B_{\rho_0} \cap E^+$.

Proof.

$$I(u, \widetilde{u}) = \int_\Omega \nabla u \nabla \widetilde{u}\, dx - \int_\Omega F(u)\, dx - \int_\Omega G(\widetilde{u})\, dx.$$

Now, using that

$$\int_\Omega \nabla u \nabla \widetilde{u}\, dx = \|u\|_{1,A}\ \|\widetilde{u}\|_{1,(\widetilde{A})} = \|u\|^2_{1,A} = \|\widetilde{u}\|^2_{1,(\widetilde{A})}$$

and by (2.1)

$$\|u\|_{1,A} \geqslant \|u\|_{1,(A)}$$

we obtain that

$$I(u, \widetilde{u}) \geqslant \frac{1}{2}\ \|u\|^2_{1,(A)} - \int_\Omega F(u)\, dx + \frac{1}{2}\ \|\widetilde{u}\|^2_{1,(\widetilde{A})} - \int_\Omega G(\widetilde{u})\, dx.$$

Assume that $\rho \in (0, 1),\ u \in W^1_0 L_A(\Omega)$ and $\|u\|_{1,(A)} = c_1^{-1}$, where $c_1 > 0$ is such that

$$\|u\|_{(\Phi)} \leqslant c_1 \|u\|_{1,(A)}.$$

Since

$$\|u\|_{(\Phi)} = \inf\{\ \lambda > 0 : \int_\Omega \Phi(\frac{|u|}{\lambda}) \leqslant 1\} \leqslant 1,$$

it follows that $\int_\Omega \Phi(|u|) \leqslant 1$, and thus $\int_\Omega F(|u|) \leqslant c$. By hypothesis (H3) we get for $0 \leqslant \rho \leqslant 1$

$$\int_\Omega F(\rho u)\, dx \leqslant c\rho^\sigma \int_\Omega F(u)\, dx \leqslant c\rho^\sigma.$$

Hence we obtain that

$$\frac{1}{2}\ \|\rho u\|^2_{1,(A)} - \int_\Omega F(\rho u)\, dx \geqslant \frac{1}{2}\ \rho^2\ \|u\|^2_{1,(A)} - c\rho^\sigma.$$

Arguing similarly for G and $\tilde{u}$ we get

$$\frac{1}{2}\ \|\rho\widetilde{u}\|^2_{1,(\widetilde{A})} - \int_\Omega G(\rho\widetilde{u})\,dx\ \geqslant \frac{1}{2}\ \rho^2\ \|\tilde{u}\|^2_{1,(\tilde{A})} - c\rho^{\sigma_1}.$$

By joining the two estimates we can find a $\rho_0 > 0$ such that

$$I(u,\tilde{u}) \geqslant \sigma_0 > 0 \quad \text{for } \|(u,\tilde{u})\| = \rho_0 > 0$$

This concludes the proof. □

Let e_1 denote the first eigenfunction of the Laplacian, with $\|(e_1,\widetilde{e_1})\| = 1$ and set

$$Q = \{r(e_1,\widetilde{e_1})\ \widetilde{+}\ w : w \in E^-, \|w\| \leqslant R_0 \text{ and } 0 \leqslant r \leqslant R_1\}.$$

Lemma 5.3. *There exist positive constants R_0, R_1 such that $I(z) \leqslant 0$ for all $z \in \partial Q$.*

Proof. Notice that the boundary ∂Q of the set Q is taken in the set $\mathbb{R}(e_1,\widetilde{e_1})\ \widetilde{+}\ E^-$, and consists of three parts.

(i) If $z \in \partial Q \cap E^-$ we have $I(z) \leqslant 0$ because, for all $z = (\omega, -\widetilde{\omega}) \in E^-$,

$$I(z) = -\|\omega\|^2_{1,(A)} - \int_\Omega [F(\omega) + G(-\widetilde{\omega})]\,dx \leqslant 0.$$

(ii) If $z = r(e_1,\widetilde{e_1})\ \widetilde{+}\ (\omega, -\widetilde{\omega}) = (re_1 + \omega, \widetilde{re_1 - \omega}) \in \partial Q$ with $\|(\omega, -\widetilde{\omega})\| = R_0$ and $0 \leqslant r \leqslant R_1$, we proceed as follows:

First step: Assume that $R_1 = 1$:

$$\begin{aligned}
I(z) &\leqslant \int_\Omega \nabla(re_1 + \omega)\nabla(\widetilde{re_1 - \omega})\,dx \\
&= -\int_\Omega \nabla(\omega - re_1)\nabla(\widetilde{\omega - re_1})\,dx - 2r\int_\Omega \nabla e_1 \nabla(\widetilde{\omega - re_1}) \\
&\leqslant -\|\omega - re_1\|^2_{1,A} + 2\|re_1\|_{1,A}\|\widetilde{\omega - re_1}\|_{1,(\widetilde{A})} \\
&\leqslant -\|\omega - re_1\|^2_{1,A} + 2\|re_1\|_{1,A}\|\omega - re_1\|_{1,A} \\
&\leqslant -\|\omega\|^2_{1,A} - \|re_1\|^2_{1,A} + 2\|re_1\|_{1,A}\|\omega\|_{1,A} \\
&\quad +2\|re_1\|_{1,A}\left(\|\omega\|_{1,A} + \|re_1\|_{1,A}\right) \\
&\leqslant -\|\omega\|^2_{1,A} + 4r\|\omega\|_{1,A} + r^2.
\end{aligned}$$

Since $2\|\omega\|^2_{1,A} \geqslant \|\omega\|^2_{1,A} + \|\widetilde{\omega}\|^2_{1,(\widetilde{A})} = R_0^2$, we conclude that the last expression is $\leqslant 0$, for R_0 sufficiently large.

Second step: Observe that using homogeneity this now holds for all $\rho \geqslant 1$ with $0 \leqslant r \leqslant \rho$ and $\|(\omega, -\widetilde{\omega})\| = \rho R_0$.

(iii) Finally, let $z = \rho(e_1, \widetilde{e}_1) \,\widetilde{+}\, \rho(\omega, -\widetilde{\omega}) = (\rho e_1 + \rho\omega, \widetilde{\rho e_1 - \rho\omega}) \in \partial Q$ with $\|e_1\|_{1,(A)} = \frac{1}{2}$ and $\|(\omega, -\widetilde{\omega})\| \leqslant R_0$.

We show: there exists $R_1 > 0$ sufficiently large such that for all $\rho \geqslant R_1$, we have $I(z) \leqslant 0$. We use that $W_0^1 L_A(\Omega) \hookrightarrow L_\Phi(\Omega)$, $W_0^1 L_{\widetilde{A}}(\Omega) \hookrightarrow L_\Psi(\Omega)$, and that by assumption (H2): $F(s) \geqslant c|s|^\theta - c_1$ and $G(s) \geqslant c|s|^\theta - c_1$, for some $\theta > 2$; then

$$
\begin{aligned}
I(z) &= \int_\Omega \nabla(\rho e_1 + \rho\omega)\nabla(\widetilde{\rho e_1 - \rho\omega})\, dx - \int_\Omega [F(\rho e_1 + \rho\omega) + G(\widetilde{\rho e_1 - \rho\omega})]\, dx \\
&\leqslant \rho^2 \|e_1 + \omega\|_{1,A} \|\widetilde{e_1 - \omega}\|_{1,(\widetilde{A})} - c\int_\Omega |\rho e_1 + \rho\omega|^\theta\, dx + c_1 \\
&\quad -c\int_\Omega |\widetilde{\rho e_1 - \rho\omega}|^\theta\, dx + c_1 \\
&\leqslant \rho^2 [\|e_1\|_{1,A} + \|\omega\|_{1,A}]^2 - c\rho^\theta \left\{ \int_\Omega |e_1 + \omega|^\theta\, dx + \int_\Omega |\widetilde{e_1 - \omega}|^\theta\, dx \right\} + 2c_1.
\end{aligned}
$$

Thus

$$
I(z) \leqslant \rho^2 (1 + R_0)^2 - c\rho^\theta \delta_0 + 2c_1, \tag{5.3}
$$

where

$$
\delta_0 := \inf_{\|(\omega, -\widetilde{\omega})\| \leqslant R_0} \left\{ \int_\Omega |e_1 + \omega|^\theta\, dx + \int_\Omega |\widetilde{e_1 - \omega}|^\theta\, dx \right\} > 0.
$$

Indeed, suppose by contradiction that there exists a sequence $(\omega_n) \subset W_0^1 L_A(\Omega)$ such that $\|(\omega_n, -\widetilde{\omega}_n)\| \leqslant R_0$ and

$$
\lim_{n\to\infty} \left\{ \int_\Omega |e_1 + \omega_n|^\theta\, dx + \int_\Omega |\widetilde{e_1 - \omega_n}|^\theta\, dx \right\} = 0.
$$

Taking a subsequence, we may assume that $\omega_n \to \omega \in L^\theta$ (since $W^1 L_A \subset\subset L_F \subset L^\theta$) which implies that $e_1 + \omega_n \to e_1 + \omega$ and $\widetilde{e_1 - \omega_n} \to \widetilde{e_1 - \omega}$ in L^θ, where we have used the continuity of the tilde mapping. Thus, taking the limit we see that

$$
\int_\Omega |e_1 + \omega|^\theta\, dx + \int_\Omega |\widetilde{e_1 - \omega}|^\theta\, dx = 0
$$

which implies that $e_1 + \omega = \widetilde{e_1 - \omega} = 0$. So, $e_1 = \omega = 0$, which is a contradiction.

Finally, using (5.3) we can find $R_1 > 0$ such that $I(z) \leqslant 0$ for all $\rho \geqslant R_1$, and hence, the geometry of the linking theorem holds. □

5.3. On Palais–Smale sequences

Proposition 5.4. *Let* $(u_m, \widetilde{v}_m) \in E$ *such that*

(I_1) $I(u_m, \widetilde{v}_m) = c + \delta_m$, *where* $\delta_m \to 0$ *as* $m \to +\infty$;
(I_2) $|I'(u_m, \widetilde{v}_m)(\eta, \widetilde{\xi})| \leqslant \varepsilon_m \|(\eta, \widetilde{\xi})\|$, *for* $\eta,\ \xi \in \{u_m, v_m\}$, *where* $\varepsilon_m \to 0$ *as* $m \to +\infty$,

then

$$\begin{array}{ll} \|u_m\|_{1,A} \leqslant C, & \|\widetilde{v}_m\|_{1,(\widetilde{A})} \leqslant C, \\ \int_\Omega f(u_m)u_m\, dx \leqslant C, & \int_\Omega g(\widetilde{v}_m)\widetilde{v}_m\, dx \leqslant C, \\ \int_\Omega F(u_m)\, dx \leqslant C, & \int_\Omega G(\widetilde{v}_m)\, dx \leqslant C. \end{array}$$

Proof. From (I_1) we have

$$\int_\Omega \nabla u_m \nabla \widetilde{v}_m\, dx - \int_\Omega F(u_m)\, dx - \int_\Omega G(\widetilde{v}_m)\, dx = c + \delta_m. \tag{5.4}$$

Taking $(\eta, \widetilde{\xi}) = (u_m, \widetilde{v}_m)$ in (I_2) we have

$$\left|\, 2\int_\Omega \nabla u_m \nabla \widetilde{v}_m\, dx - \int_\Omega f(u_m)u_m\, dx - \int_\Omega g(\widetilde{v}_m)\widetilde{v}_m\, dx \,\right| \leqslant \varepsilon_m \|(u_m, \widetilde{v}_m)\|, \tag{5.5}$$

which together with (I_1) and (H_3) implies that

$$(\theta - 2)\int_\Omega [F(u_m) + G(\widetilde{v}_m)]\, dx \leqslant 2c + 2\delta_m + \varepsilon_m \|(u_m, \widetilde{v}_m)\|.$$

Thus

$$\int_\Omega F(u_m)\, dx \leqslant c(1 + \delta_m + \varepsilon_m \|(u_m, v_m)\|),$$

$$\int_\Omega G(\widetilde{v}_m)\, dx \leqslant c(1 + \delta_m + \varepsilon_m \|(u_m, v_m)\|)$$

and then by (5.4)

$$\left|\int_\Omega \nabla u_m \nabla \widetilde{v}_m\, dx\right| \leqslant c(1 + \delta_m + \varepsilon_m \|(u_m, v_m)\|)$$

and finally by (5.5) also

$$\int_\Omega f(u_m)u_m\,dx \leqslant c(1+\delta_m+\varepsilon_m\|(u_m,v_m)\|),$$

$$\int_\Omega g(\widetilde{v}_m)\widetilde{v}_m\,dx \leqslant c(1+\delta_m+\varepsilon_m\|(u_m,v_m)\|).$$

Taking $(\eta,\widetilde{\xi})=(0,\widetilde{u}_m)$ in (I_2) we have

$$\left|\int_\Omega \nabla u_m \nabla \widetilde{u}_m\,dx - \int_\Omega g(\widetilde{v}_m)\widetilde{u}_m\,dx\right| \leqslant \varepsilon_m\|(0,\widetilde{u}_m)\| = \varepsilon_m\|\widetilde{u}_m\|_{1,(\widetilde{A})},$$

thus,

$$\|u_m\|_{1,A}^2 - \int_\Omega g(\widetilde{v}_m)\widetilde{u}_m\,dx \leqslant \varepsilon_m\|\widetilde{u}_m\|_{1,(\widetilde{A})}. \tag{5.6}$$

Setting $\widetilde{U}_m=\widetilde{u}_m/C_0\|\widetilde{u}_m\|_{1,(\widetilde{A})}$ and $V_m=v_m/C_1\|v_m\|_{1,A}$ we have $\|\widetilde{U}_m\|_{1,(\widetilde{A})}=1/C_0$ and $\|\widetilde{U}_m\|_{(\Psi)}\leqslant C_0\|\widetilde{U}_m\|_{1,(\widetilde{A})}\leqslant 1$ and thus by (5.6)

$$\|u_m\|_{1,A} \leqslant C_0\int_\Omega g(\widetilde{v}_m)\widetilde{U}_m + \varepsilon_m. \tag{5.7}$$

Note also that

$$\frac{1}{C}\int_\Omega G(\widetilde{U}_m) \leqslant \int_\Omega \Psi(\widetilde{U}_m)\,dx \leqslant 1$$

since $\|\widetilde{U}_m\|_{(\Psi)}=\inf\{\lambda\colon \int_\Omega \Psi(\frac{\widetilde{U}_m}{\lambda})\leqslant 1\}$.

We now rely on the following elementary inequalities

$$xy \leqslant F(x)+f^{-1}(y)y \quad \text{and} \quad xy \leqslant G(x)+g^{-1}(y)y. \tag{5.8}$$

Applying (5.8) to the first term on the right-hand side in (5.7), with $y=g(\widetilde{v}_m)$ and $x=\widetilde{U}_m$ yields

$$\begin{aligned}\int_\Omega g(\widetilde{v}_m)\widetilde{U}_m\,dx &\leqslant \int_\Omega G(\widetilde{U}_m)+\int_\Omega g(\widetilde{v}_m)\widetilde{v}_m\,dx\\ &\leqslant C(1+\delta_m+\varepsilon_m\|(u_m,\widetilde{v}_m)\|).\end{aligned}$$

Now using (5.7), we get

$$\|u_m\|_{1,A} \leqslant \varepsilon_m + C(1 + \delta_m + \varepsilon_m \|(u_m, \widetilde{v}_m)\|).$$

Arguing similarly, choosing $(\eta, \tilde{\xi}) = (v_m, 0)$, yields

$$\|\widetilde{v}_m\|_{1,(\widetilde{A})} \leqslant \varepsilon_m + C(1 + \delta_m + \varepsilon_m \|(u_m, \widetilde{v}_m)\|).$$

Joining the two estimates yields the claim. □

5.4. Approximation by finite-dimensional problem

The functional I given by (1.2) is strongly indefinite near zero, since the first term is positive on the submanifold E^+, and negative on the submanifold E^-. Since both E^+ and E^- are infinite dimensional, the standard linking theorems do not apply. We overcome this difficulty by using a finite-dimensional approximation. Denoting by $e_1, e_2, \ldots$ an orthonormal basis of eigenfunctions associated to the eigenvalues $\lambda_1, \lambda_2, \ldots$ of the Laplacian (with Dirichlet boundary conditions), we set $E_n = \operatorname{span}\{e_1, e_2, \ldots, e_n\}$. Let

$$E_n^+ := \{(z, \tilde{z});\ z \in E_n\}, \quad E_n^- := \{(z, -\tilde{z});\ z \in E_n\}.$$

Setting $\widetilde{E}_n = \{\widetilde{v} \mid v \in E_n\}$, one shows exactly as in Lemma 4.4 that

$$E_n \times \widetilde{E}_n = E_n^+ \ \widetilde{\oplus}\ E_n^-.$$

We recall once more that E^+ and E^- are *linear* with respect to the "tilde-sum". Thus, we can define the following "projections":

$$\begin{aligned} P_n^- &: E_n^+ \ \widetilde{\oplus}\ E_n^- \to E_n^-, \quad P_n^-((u, \widetilde{v})) = \left(\tfrac{u-v}{2}, -\widetilde{\tfrac{u-v}{2}}\right), \\ P_n^+ &: E_n^+ \ \widetilde{\oplus}\ E_n^- \to E_n^+, \quad P_n^+((u, \widetilde{v})) = \left(\tfrac{u+v}{2}, \widetilde{\tfrac{u+v}{2}}\right), \end{aligned}$$

which are clearly continuous mappings.

We now restrict the functional I to $E_n \times \widetilde{E}_n = E_n^+ \ \widetilde{\oplus}\ E_n^-$. Consider the set

$$Q_n := \{w \ \widetilde{+}\ r(e_1, \widetilde{e}_1);\ w \in E_n^-,\ \|w\| \leqslant R_0 \quad \text{and} \quad 0 \leqslant r \leqslant R_1\} \subset E_n^+ \ \widetilde{\oplus}\ E_n^-,$$

where R_0 and R_1 are as in Lemma 5.3. Furthermore, define the class of mappings

$$H_n = \{h \in C(Q_n, E_n^+ \widetilde{\oplus} E_n^-);\ h(z) = z \text{ on } \partial Q_n\},$$

where ∂Q_n is the boundary of Q_n relative to $E_n^+ \widetilde{\oplus} E_n^-$. Finally, set

$$c_n = \inf_{h \in H_n} \max_{z \in Q_n} I(h(z)).$$

We show

Lemma 5.5. *The sets Q_n and $\partial B_{\rho_0} \cap E_n^+$ link, i.e.*

$$h(Q_n) \cap (\partial B_{\rho_0} \cap E_n^+) \neq 0 \quad \forall h \in H_n. \tag{5.9}$$

Proof. The statement (5.9) is equivalent to saying that

$$\exists (u, \widetilde{v}) \in Q_n \text{ such that } \|h(u, \widetilde{v})\| = \rho_0 \quad \text{and} \quad P_n^- h(u, \widetilde{v}) = 0. \tag{5.10}$$

Let $(u, \tilde{v}) = w \widetilde{+} s(e_1, \widetilde{e}_1) \in Q_n$. Define the continuous maps (here we use Remark 4.2)

$$\begin{aligned}
&\psi_t : Q_n \to E_n^- \widetilde{+} [(e_1, \widetilde{e}_1)],\\
&\psi_t(w \widetilde{+} s(e_1, \widetilde{e}_1))\\
&\quad = t P_n^- h((u, \tilde{v})) \widetilde{+} (1-t)w \widetilde{+} [t\|P_n^+ h((u, \tilde{v}))\| + (1-t)s - \rho_0](e_1, \widetilde{e}_1).
\end{aligned}$$

Note that for $(u, \widetilde{v}) = w \widetilde{+} t(e_1, \widetilde{e}_1) \in \partial Q_n$, we have

$$\psi_t(w \widetilde{+} s(e_1, \widetilde{e}_1)) = w \widetilde{+} (s - \rho_0)(e_1, \widetilde{e}_1) \neq (0, 0) \quad \forall t \in [0, 1]$$

and hence

$$\psi_0(w \widetilde{+} s(e_1, \widetilde{e}_1)) = w \widetilde{+} (s - \rho)(e_1, \widetilde{e}_1)$$

is homotopic to

$$\psi_1(w \widetilde{+} s(e_1, \widetilde{e}_1)) = P_n^- h((u, \tilde{v})) \widetilde{+} (\|P_n^+ h((u, \tilde{v}))\| - \rho_0)(e_1, \widetilde{e}_1).$$

By the properties of the topological degree on oriented manifolds (see [7]) we have that the degree of the maps ψ_t with respect to Q_n and $(0, 0)$ is well defined, and that

$$deg(\psi_1, Q_n, (0, 0)) = deg(\psi_0, Q_n, (0, 0)) = 1.$$

Hence, there exists an element $(u, \widetilde{v}) \in Q_n$ such that $\psi_1(u, \widetilde{v}) = (0, 0)$, and hence satisfying (5.10). □

Choosing ρ_0 as in Lemma 5.2, we now conclude that

$$c_n \geqslant \sigma_0 > 0 \quad \text{for all } n \in \mathbb{N}.$$

Furthermore, since $id_{E_n^+ \widetilde{\oplus} E_n^-} \in \Gamma_n$, we have for $z = r(e_1, \widetilde{e}_1) \widetilde{+} (u, -\tilde{u}) \in Q_n$

$$c_n \leqslant \max_{z \in Q_n} I(z) \leqslant R_1^2 \; \|e_1\|^2 \leqslant c \; R_1^2.$$

Thus, by the linking theorem (see [8]), we obtain a PS-sequence, which is bounded in view of Proposition 5.4. Since $E_n^+ \widetilde{\oplus} E_n^-$ is finite dimensional, we therefore get that c_n is a critical level of $I|_{E_n^+ \widetilde{\oplus} E_n^-}$, for each $n \in \mathbb{N}$, with a corresponding sequence of critical points $z_n \in E_n^+ \widetilde{\oplus} E_n^-$ with $\|z_n\| \leqslant c$, where c does not depend on n.

5.5. *Limit for $n \to \infty$*

By the last subsection we have a sequence $z_n = (u_n, \widetilde{v}_n) \in E_n \times E_n$ with

$$I(z_n) = c_n \in [\sigma_0, c R_1^2] \quad \text{and} \quad I'(z_n) = 0. \tag{5.11}$$

By Proposition 5.4 we have $\|z_n\| \leqslant c$ and hence, for a subsequence, $z_n = (u_n, \widetilde{v}_n) \rightharpoonup z = (u, \tilde{v})$ in $E = W_0^1 L_A \times W_0^1 L_{(\widetilde{A})}$. Again by Proposition 5.4 we have $\int_\Omega F(u_n)\, dx \leqslant c$, $\int_\Omega G(\widetilde{v}_n)\, dx \leqslant c$ and $\int_\Omega f(u_n) u_n\, dx \leqslant c$, $\int_\Omega g(u_n) u_n \leqslant c$. Using Lemma 2.1 in [3] we conclude that

$$f(u_n) \to f(u) \quad \text{and} \quad g(\widetilde{v}_n) \to g(\widetilde{v}) \text{ in } L^1(\Omega)$$

Taking arbitrary test functions $(0, \widetilde{\eta})$ and $(\zeta, 0)$ in $E_n \times \widetilde{E}_n$ we get

$$\int_\Omega \nabla u_n \nabla \widetilde{\eta}\, dx = \int_\Omega g(\widetilde{v}_n) \widetilde{\eta}\, dx, \;\; \int_\Omega \nabla \widetilde{v}_n \nabla \zeta\, dx = \int_\Omega f(u_n) \zeta\, dx \quad \forall \eta, \quad \zeta \in E_n. \tag{5.12}$$

Using the fact that $\cup_{n \in \mathbb{N}} (E_n \times \widetilde{E}_n)$ is dense in E, we obtain by taking the limit $n \to \infty$,

$$\int_\Omega \nabla u \nabla \widetilde{\eta}\, dx = \int_\Omega g(\widetilde{v}) \widetilde{\eta}\, dx \quad \forall \widetilde{\eta} \in W^1 L_{\widetilde{A}},$$

$$\int_\Omega \nabla \widetilde{v} \nabla \zeta\, dx = \int_\Omega f(u) \zeta\, dx \quad \forall \zeta \in W_0^1 L_A \;.$$

Thus, $(u, \widetilde{v}) \in W_0^1 L_A \times W_0^1 L_{(\widetilde{A})}$ is a weak solution of problem (1.1).

It remains to show that $(u, \tilde{v})$ is nontrivial; assume by contradiction that $u = 0$, then by the Eq. (1.1) also $\tilde{v} = 0$. Note that we can find a suitable Δ-regular N-function F_1 with $F_1 \prec\prec \Phi$ and the properties $F(x) \leqslant F_1(x),\ f(x) \leqslant f_1(x) \quad \forall x \in \mathbb{R}^+$. Thus

$$\|u_n\|_{(F_1)} \to 0, \text{ i.e. } \inf\{\lambda > 0 \ ; \ \int_\Omega F_1(\frac{u_n}{\lambda}) \leqslant 1\} =: \lambda_n \to 0.$$

Since, for $\lambda_n < 1$ holds $\frac{1}{\lambda_n} \int_\Omega F_1(u_n) \leqslant \int_\Omega F_1(\frac{u_n}{\lambda_n}) \leqslant 1$, we conclude that

$$\int_\Omega F(u_n) \leqslant \int_\Omega F_1(u_n) \leqslant \lambda_n \to 0.$$

Since F_1 is Δ-regular, we have $x f_1(x) \leqslant c F_1(x)$, for some $c > 1$, and hence

$$0 \leqslant \int_\Omega f(u_n) u_n \leqslant \int_\Omega f_1(u_n) u_n \leqslant c \int_\Omega F_1(u_n)\, dx \to 0 \ . \tag{5.13}$$

This implies now by (5.12), choosing $\zeta = u_n$, that $\int_\Omega \nabla u_n \nabla \tilde{v}_n \, dx \to 0$, and thus also $I(u_n, \tilde{v}_n) \to 0$. But this contradicts that $I(u_n, \tilde{v}_n) \geqslant \sigma_0 > 0$, for all $n \in \mathbb{N}$.

This concludes the proof of Theorem 1.3. $\square$

6. Critical Orlicz pairs near the critical hyperbola

In this section, we consider N-functions of the (asymptotic) type

$$\Phi(s) \sim s^{p+1} (\log(1+s))^\alpha \tag{6.1}$$

with $p > 1$ and $\alpha > 0$.

It is natural to expect that the critical Orlicz associate Ψ (i.e. such that (Φ, Ψ) form a critical Orlicz pair) will be given by a N-function Ψ of the asymptotic form

$$\Psi(s) \sim s^{q+1} (\log(1+s))^{-\beta}, \tag{6.2}$$

where q satisfies $\frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{N}$, and some relation between α and β. This is indeed so, and the relation between α and β will be given in Proposition 6.3 below.

We begin by showing that functions of type (6.1) and (6.2) satisfy the hypotheses of Theorem 1.3.

Lemma 6.1. *Suppose that $\Phi(t)$ is of the form*

$$\Phi(t) = t^{p+1}\, g(t),$$

where $p > 1$, *and* $g \in C(\mathbb{R}^+, \mathbb{R}^+)$ *satisfying one of the following conditions*:

(1) $g(t)$ *increasing*,
(2) $g(t) \searrow 0$ *and* $g(t)\,t^{\varepsilon}$ *increasing for large* t, *for any* $\varepsilon > 0$.

Then Φ *is uniformly superquadratic near zero* (*see Definition* 5.1).

Proof. Let $0 < s \leqslant 1$ and $t > 0$.

(1) We have, using that g is increasing,

$$\Phi(st) = (st)^{p+1} g(st) = s^{p+1} t^{p+1} g(t) \frac{g(st)}{g(t)} = s^{p+1} \frac{g(st)}{g(t)} \Phi(t) \leqslant s^{p+1} \Phi(t).$$

(2) We have, for some $0 < \delta < p-1$

$$\Phi(st) = (st)^{p+1} g(st) = s^{p+1} \frac{g(st)}{g(t)} \Phi(t) = s^{2+\delta} \Phi(t)\; s^{p-1-\delta} \frac{g(st)}{g(t)} \leqslant s^{2+\delta} \Phi(t),$$

indeed, let $\varepsilon = p-1-\delta$, and suppose that $t^{\varepsilon} g(t)$ is increasing for $t \geqslant t_0$. Then we have, since $0 \leqslant s \leqslant 1$

$$s^{\varepsilon} \frac{g(st)}{g(t)} \leqslant \max_{0 \leqslant t \leqslant t_0} \frac{g(st)}{g(t)} + \max_{t \geqslant t_0} \frac{(st)^{\varepsilon} g(st)}{t^{\varepsilon} g(t)} \leqslant c\ . \qquad \square$$

Lemma 6.2. *Suppose that* Φ *is of class* C^1, *and* (*asymptotically*) *of the form*

$$\Phi(s) \sim c s^{p+1} g(s) \quad with\ \ p + 1 > \frac{N}{N-2}$$

and

$$\lim_{s \to \infty} \frac{g'(s)}{g(s)} = 0.$$

Then Φ *is* θ*-regular, with* $\theta = p + 1$ (*see Definition* 2.11).

Proof. Indeed, we have

$$\lim_{s \to \infty} \frac{s\varphi(s)}{\Phi(s)} = \lim_{s \to \infty} \frac{(p+1) s^{p+1} g(s) + s^{p+1} g'(s)}{s^{p+1} g(s)} = p + 1. \qquad \square$$

Proposition 6.3. *Suppose that Φ is (asymptotically) of the form*

$$\Phi(s) = cs^{p+1}(\log s)^{\alpha} \quad with \ p+1 > \frac{N}{N-2}.$$

Then the associate critical Orlicz function Ψ is (asymptotically) given by

$$\Psi(s) = ds^{q+1}(\log s)^{-\alpha\frac{q+1}{p+1}}$$

with

$$\frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{N}. \tag{6.3}$$

Proof. It is easy to check that (asymptotically)

(1) $\Phi^{-1}(t) \sim c_1 t^{\frac{1}{p+1}}(\log t)^{-\frac{\alpha}{p+1}}$.
(2) $(\Phi^{-1})'(t) \sim c_2 t^{\frac{1}{p+1}-1}(\log t)^{-\frac{\alpha}{p+1}}$.
(3) Using Definition 2.9:

$$A^{-1}(t) \sim c_3 t^{\frac{N+(p+1)}{N(p+1)}}(\log t)^{-\frac{\alpha}{p+1}}.$$

(4) $A(s) \sim c_4 s^{\frac{N(p+1)}{N+(p+1)}}(\log s)^{\alpha\frac{N}{N+(p+1)}}$.
(5) $\widetilde{A}(s) \sim c_5 s^{\frac{N(p+1)}{Np-(p+1)}}(\log s)^{-\alpha\frac{N}{Np-(p+1)}}$.
(6) $\widetilde{A}^{-1}(t) \sim c_6 t^{\frac{Np-(p+1)}{N(p+1)}}(\log t)^{\frac{\alpha}{p+1}}$.
(7) $\frac{\widetilde{A}^{-1}(t)}{t^{1+\frac{1}{N}}} \sim c_6 t^{\frac{-2p-1}{Np}}(\log t)^{\frac{\alpha}{p+1}}$.
(8) Using again Definition 2.9:

$$\Psi^{-1}(t) \sim c_7 t^{\frac{(N-2)(p+1)-N}{N(p+1)}}(\log s)^{\frac{\alpha}{p+1}}.$$

(9) $\Psi(s) \sim c_8 s^{\frac{N(p+1)}{(N-2)(p+1)-N}}(\log s)^{-\frac{\alpha}{p+1}\frac{N(p+1)}{(N-2)(p+1)-N}}$. Setting $q+1 := \frac{N(p+1)}{(N-2)(p+1)-N}$, once checks that (6.3) holds, and thus finally
(10) $\Psi(s) \sim ds^{q+1}(\log s)^{-\alpha\frac{q+1}{p+1}}$. □

We remark that M.A. Krasnoselskǐ and J.B. Rutickiĭ, in their book on Orlicz spaces [6, Chapter I, Section 7], consider the class of N-functions

$$\Phi(s) = cs^{p+1}(\log s)^{\alpha_1}(\log\log s)^{\alpha_2}\dots(\log\log\dots\log s)^{\alpha_k}, \quad \alpha_i \in \mathbb{R}.$$

Repeating the above calculations one shows that the critical Orlicz associates to these functions are given by

$$\Psi(s) = ds^{q+1}(\log s)^{-\beta_1}(\log \log s)^{-\beta_2} \dots (\log \log \dots \log s)^{-\beta_k}$$

with

$$\frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{N} \quad \text{and} \quad \beta_i = \alpha_i \frac{q+1}{p+1}, \quad i = 1, \dots, k.$$

References

[1] R.A. Adams, Sobolev Spaces, Pure and Applied Mathematics, vol. 65, Academic Press, New York, London, 1975.

[2] D.G. de Figueiredo, P.L. Felmer, On superquadratic elliptic systems, Trans. Amer. Math. Soc. 343 (1994) 99–116.

[3] D.G. de Figueiredo, O. Miyagaki, B. Ruf, Elliptic equations in R^2 with nonlinearities in the critical growth range, Calculus Variations Partial Differential Equations 4 (1996).

[4] J.P. Gossez, Orlicz Spaces, Orlicz–Sobolev Spaces and Strongly Nonlinear Elliptic Problems, Trabalho de Matemática, No. 103, Departamento de Matematica, Universidade de Brasilia, 1976.

[5] J. Hulshof, R. van der Vorst, Differential systems with strongly indefinite variational structure, J. Funct. Anal. 114 (1993) 32–58.

[6] M.A. Krasnoselskiĭ, J.B. Rutickiĭ, Convex functions and Orlicz Spaces, Transl. first Russian edition by Leo F. Boron. P. Noordhoff Ltd., Groningen, 1961.

[7] L. Nirenberg, Topics in Nonlinear Functional Analysis, Courant Institute, New York University, 1974.

[8] P.H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, CBMS Regional Conference Series in Mathematics, vol. 65, American Mathematical Society, Providence, RI, 1986.

[9] M.M. Rao, Z.D. Ren, Theory of Orlicz Spaces, Monographs and Textbooks in Pure and Applied Mathematics, vol. 146. Marcel Dekker, Inc., New York, 1991.