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On existence and concentration of positive bound states of p-Laplacian equations in $\mathbb{R}^N$ involving critical growth

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Abstract

This paper deals with the study of the quasilinear critical problem

$$\begin{cases} -\varepsilon^p \Delta_p u + V(z)u^{p-1} = f(u) + u^{p^*-1} & \text{in } \mathbb{R}^N, \\ u \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N), \ u > 0 & \text{in } \mathbb{R}^N, \end{cases} \qquad (P_\varepsilon)$$

where ε is a small positive parameter; f is a subcritical nonlinearity; $p^* = pN/(N-p)$, $1 < p < N$, is the critical Sobolev exponent; and $V : \mathbb{R}^N \to \mathbb{R}$ is a function which is bounded from below away from zero such that $\inf_{\partial\Omega} V > \inf_{\Omega} V$ for some open bounded subset Ω of $\mathbb{R}^N$. We study whether we can find solutions of (P_ε) which concentrate around a local minima of V, not necessarily nondegenerate. The proof of this result is variational based on the local mountain-pass theorem.

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Keywords: Elliptic problems; p-Laplacian; Critical growth; Variational methods; Mountain pass theorem

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1. Introduction

The main purpose of this paper is to study the existence and concentration behavior of *bound state* for the quasilinear critical problem of the form

$$\begin{cases} -\varepsilon^p \Delta_p u + V(z)u^{p-1} = f(u) + u^{p^*-1} & \text{in } \mathbb{R}^N, \\ u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N), u > 0 & \text{in } \mathbb{R}^N, \end{cases} \qquad (P_\varepsilon)$$

where $\varepsilon > 0$ is a small real parameter; $\Delta_p u := div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $p^* = pN/(N-p)$, $1 < p < N$, is the critical Sobolev exponent, and $V : \mathbb{R}^N \to \mathbb{R}$ is a C^1 function satisfying

(V_0) there is a positive constant α such that

$$V(z) \geqslant \alpha \quad \forall z \in \mathbb{R}^N;$$

(V_1) there is an open bounded subset Ω of $\mathbb{R}^N$ such that

$$\inf_{\partial\Omega} V > \inf_{\Omega} V =: V_0.$$

We also assume that $f : \mathbb{R}_+ \to \mathbb{R}$ is a C^1 function satisfying the following conditions

(f_1) $f(s) = o(s^{p-1})$ as $s \to 0$;
(f_2) there are $q_1, q_2 \in (p-1, p^*-1)$, $\lambda > 0$ such that

$$f(s) \geqslant \lambda s^{q_1}, \text{ for all } s > 0 \text{ and } \lim_{s\to\infty} \frac{f(s)}{s^{q_2}} = 0;$$

(f_3) for some $\theta \in (p, q_2+1)$ we have

$$0 < \theta F(s) \equiv \theta \int_0^s f(t)\,\mathrm{d}t \leqslant f(s)s \quad \text{for all } s > 0;$$

(f_4) the function $s^{1-p} f(s)$ is nondecreasing for $s > 0$.

The main result of this paper is stated as follows:

Theorem 1.1. *Suppose that the potential V satisfies* (V_0)–(V_1) *and f satisfies* (f_1)–(f_4). *Then there is* $\varepsilon_o > 0$ *such that problem* (P_ε) *possesses a positive bound state solution* u_ε, *for all* $0 < \varepsilon < \varepsilon_o$, *provided that one of the following conditions holds*:

(a) $N \geqslant p^2$;
(b) $p < N < p^2$, $p^* - \frac{p}{p+1} - 1 < q_1 < p^* - 1$;
(c) $p < N < p^2$, $p^* - \frac{p}{p+1} - 1 \geqslant q_1$ *and large* λ.

Moreover, when $1 < p \leqslant 2$, u_ε *possesses at most one local* (*hence global*) *maximum* z_ε *in* $\mathbb{R}^N$, *which is inside* Ω, *such that*

$$\lim_{\varepsilon\to 0^+} V(z_\varepsilon) = V_o = \inf_{\Omega} V$$

and there are C and α *positive constants such that*

$$u_\varepsilon(x) \leqslant C \exp\left(-\alpha\left|\frac{x-z_\varepsilon}{\varepsilon}\right|\right) \text{ for all } \quad x \in \mathbb{R}^N.$$

The study of such a class of problems in the semilinear case, which corresponds to $p=2$, has been motivated in part by the search for standing waves for the nonlinear Schrödinger equation

$$\mathrm{i}\varepsilon\frac{\partial\psi}{\partial t} = -\frac{\varepsilon^2}{2m}\Delta\psi + V(z)\psi - \gamma|\psi|^{r-1}\psi, \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

namely, solutions of the form $\psi(z,t) = \exp(-\mathrm{i}Et/\varepsilon)v(z)$, where ε, m, γ, and p are positive constants, $p > 1$, $E \in \mathbb{R}$ and v is real. In fact, it is readily checked that ψ satisfies (1.1) if, and only if, the function $v(z)$ solves the semilinear elliptic equation

$$-\frac{\varepsilon^2}{2m}\Delta u + (V(z) - E)v = \gamma|v|^{r-1}v \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

Floer and Weinstein [12], using Lyapunov–Schmidt methods have proved the existence of standing wave solutions concentrating at each given nondegenerate critical point of the potential V, provided that V is bounded, $N=1$ and $r=3$. Their method was extended by Oh [20,21] to higher dimensions, in the case $2 < r < (N+2)/(N-2)$. Rabinowitz [22] among others results obtained existence results under the assumption that $\inf V < \liminf_{|z|\to\infty} V(z)$ and $1 < r < (N+2)/(N-2)$. He used variational methods based on variants of the mountain-pass theorem and considered the case of degenerated critical point of the potential V. For this class of problems, Wang [28] complemented the work of Rabinowitz obtaining the concentration behavior of the solutions. Recently, in [9], del Pino and Felmer have proved the existence and concentration behavior of bounded state solutions under the potential conditions $(V_0)-(V_1)$ for subcritical nonlinearities. This result was complemented in [2] to elliptic problems involving critical growth. In this paper we extend these results, since we are considering a more general class of operator. To prove the existence of solutions, we adapt some ideas from [2,9] and to prove the decay of the solutions, we make use of the local estimates of Serrin [23]. To obtain the concentration behavior of solutions we use a recent result contained in [24], about symmetry of ground states solutions of quasilinear equations.

Several papers have appeared recently about the p-Laplacian problems involving critical growth. For the case of bounded domains, we mention the works of Azorero and Alonso [3], Egnell [11] and Guedda and Veron [15], and references therein. As to unbounded domains, we recall the results of Alves et al. [1], Ben-Naoum et al. [4], Gonçalves and Alves [14] and Jianfu and Xi Ping [16]. We referred to their references for other related results.

The underlying idea for proving Theorem 1.1 has two basic steps. First, in Section 2, we modify the function $f(u)$ outside the domain Ω such that the associated energy functional to the modified problem satisfies the Palais–Smale condition and to which we may apply the mountain-pass theorem. In the last section, with the aid of some local estimates, we prove that the solution of the modified problem is in fact a solution for the original problem and we study the concentration behavior of the solution.

Notation. In this paper we make use of the following notation:

Let U be a domain in $\mathbb{R}^N$. $C^{k,\alpha}(U)$, with k being a nonnegative integer and $0 \leqslant \alpha < 1$, denotes Hölder spaces; the norm in $C^{k,\alpha}(U)$ is denoted by $\|u\|_{k,\alpha,\Omega}$;

$L^p(U)$, $1 \leqslant p \leqslant \infty$, denotes Lebesgue spaces; the norm in L^p is denoted by $|u|_p$;

$W^{1,p}(U)$, denotes Sobolev spaces; the norm in $W^{1,p}(U)$ is denoted by $\|u\|_{W^{1,p}}$;

$C, C_0, C_1, C_2, \ldots$ denote (possibly different) positive constants;

$u_+ = \max\{u, 0\}$ and $u_- = \min\{u, 0\}$;

χ_A denotes the characteristic function of subset A of $\mathbb{R}^N$;

We denote by S the best Sobolev constant of the Sobolev embedding, $D^{1,m}(\mathbb{R}^N) \hookrightarrow L^{m^*}(\mathbb{R}^N)$, that is,

$$S = \inf\{|\nabla u|^p_{L^p}/|u|^p_{L^{p*}} \ : \ u \in D^{1,m}(\mathbb{R}^N)\backslash\{0\}\}. \tag{1.3}$$

According to Lemma 2 in [25], S is attained by the functions w_ε given by

$$w_\varepsilon(z) = \frac{C(N,p)\varepsilon^{(N-p)/p^2}}{[\varepsilon + |z|^{p/(p-1)}]^{(N-p)/p}} \text{ with } C(N,p) = \left[N\left(\frac{N-p}{p-1}\right)^{p-1}\right]^{(N-p)/p^2} \tag{1.4}$$

for any $z \in \mathbb{R}^N$ and any $\varepsilon > 0$.

2. The modified functional

Since we seek positive solutions, it is convenient to define $f(s) = 0$, for $s \leqslant 0$. Also, we modify the nonlinearity f into a more appropriate one to obtain a existence result as an application of the mountain-pass theorem. Namely, we consider the following Carathéodory function:

$$g(z,s) = \begin{cases} \chi_\Omega(z)(f(s) + s^{p^*-1}) + \chi_D(z)\widetilde{f}(s) & \text{if } s \geqslant 0, \\ 0 & \text{if } s < 0, \end{cases}$$

where

$$D = \mathbb{R}^N \backslash \Omega, \quad \widetilde{f}(s) = \begin{cases} f(s) + s^{p^*-1} & \text{if } s \leqslant a, \\ k^{-1}s^{p-1}\alpha & \text{if } s > a, \end{cases}$$

$k > \theta(\theta - p)^{-1} > 1$ and $a > 0$ is such that $f(a) + a^{p^*-1} = k^{-1}a^{p-1}\alpha$. We set $\widetilde{F}(s) = \int_o^s \widetilde{f}(t)\,dt$ and $G(z,s) = \chi_\Omega(F(s) + \frac{1}{2^*}s^{2^*}) + \chi_D \widetilde{F}(s)$.

Notice that, using (f_1)–(f_4) it is easy to check that the nonlinearity $g(x,u)$ satisfies the following properties:

(g_1) $g(z,s) = f(s) + s^{p^*-1} = o(s^{p-1})$, near the origin, uniformly in $z \in \mathbb{R}^N$;

(g_2) $g(z,s) \leqslant f(s) + s^{p^*-1}$ for all $s > 0$, $z \in \mathbb{R}^N$;

(g_3) $0 < \theta G(z, s) \leqslant g(z, s)s$ for all $z \in \Omega$, $s > 0$ or $z \in D$ and $s \leqslant a$ and $0 \leqslant pG(z, s) \leqslant g(z, s)s \leqslant \frac{1}{k}(z)s^p$ for all $z \in D$, s > 0.

(g_4) The function $g(z, s)/s^{p-1}$ is increasing in $s > 0$ for each z fixed.

Now, we consider the functional

$$J(u) = \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u|^p + V(z)|u|^p)\,\mathrm{d}z - \int_{\mathbb{R}^N} G(z, u)\,\mathrm{d}z,$$

defined on the reflexive Banach space

$$E = \left\{ u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(z)|u|^p\,\mathrm{d}z < \infty \right\},$$

endowed with the norm $\|u\| := \{\int_{\mathbb{R}^N} (|\nabla u|^p + V(z)|u|^p)\,\mathrm{d}z\}^{1/p}$.

It is well known that J is in $C^1(E, \mathbb{R})$ (see [7]) with Fréchet derivative given by

$$J'(u)v = \int_{\mathbb{R}^N} [|\nabla u|^{p-2}\nabla u \nabla v + V(z)|u|^{p-2}uv]\,\mathrm{d}z - \int_{\mathbb{R}^N} g(z, u)v\,\mathrm{d}z.$$

It is standard to prove that J verifies the mountain-pass geometrical conditions. We include a proof for completeness.

Lemma 2.1 (*mountain-pass geometry*). *The functional J satisfies the following conditions*:

(i) *there exist* $\alpha, \beta > 0$, *such that* $J(u) \geqslant \beta$ *if* $\|u\| = \alpha$,
(ii) *for any* $u \in C_0^\infty(\Omega, [0, +\infty))$, *we have* $J_\infty(tu) \to -\infty$ *as* $t \to +\infty$.

Proof. As usual, from our assumptions we have

$$F(u) \leqslant \frac{1}{2p}|s|^p + C|s|^{p^*}.$$

Thus, using (g_2) and Sobolev inequality, we find

$$J(u) \geqslant \frac{1}{2p}\|u\|^p - C\|u\|^{p^*}.$$

Hence, there exist constants $\alpha, \beta > 0$, such that $J(u) \geqslant \beta$ if $|u| = \alpha$.

Let u be a nontrivial function in $C_0^\infty(\Omega, [0, +\infty))$. Thus, using (g_2), for all $t > 0$

$$J(tu) \leqslant \frac{t^p}{p}\|u\|^p - \frac{t^{p^*}}{p^*}\int_{\mathbb{R}^N} u^{p^*}.$$

Of course, $J(t\eta) \to -\infty$ as $t \to \infty$. This completes the proof. □

Proposition 2.2. *There exists a bounded sequence* $(u_n) \subset E$ *such that*

$$J(u_n) \to c, \quad 0 < c < \frac{1}{N}S^{N/p}, \quad \text{and} \quad J'(u_n) \to 0, \quad \text{as} \quad n \to \infty, \tag{2.1}$$

provided that one of conditions (1), (2) *or* (3) *in Theorem* 1.1 *holds*.

Proof. In view of Lemma 2.1, we may apply a version of mountain-pass theorem without a Palais–Smale condition (cf. [7,18]), to obtain a Palais–Smale sequence associated to the functional J, more precisely, $(u_n) \subset E$ such that

$$J(u_n) \to c > 0 \quad \text{and} \quad J'(u_n) \to 0, \tag{2.2}$$

where

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t))$$

and

$$\Gamma = \{\gamma \in C([0, 1], E) : \gamma(0) = 0 \text{ and } J(\gamma(1)) < 0\}.$$

Next, as a consequence of assumptions on g we prove that the mountain-pass minimax level c can be characterized in a simpler way, as has been established in [9,10] and [22] to the semilinear case.

Assertion 2.1.

$$c = \inf_{\substack{v \in E \\ v_+ \neq 0}} \max_{t \geqslant 0} J(tv). \tag{2.3}$$

Proof. From our hypothesis and Lemma 2.1, it is easy to check that for each fixed $u \in E$, such that $u_+ \neq 0$, the function $t \longmapsto J(tu)$ has at most one critical point $t_u \in (0, +\infty)$ and it satisfies

$$\|u\|^p = \frac{1}{t_u^{p-1}} \int_{\mathbb{R}^N} g(z, t_u u) u \, \mathrm{d}z.$$

Furthermore,

$$\max_{t \geqslant 0} J(tu) = J(t_u u).$$

Thus,

$$\inf_{\substack{v \in E \\ v_+ \neq 0}} \max_{t \geqslant 0} J(tv) \leqslant \inf_{v \in M} J(v),$$

where $M = \{v = t_u u : u \in E - \{0\} \text{ and } t_u \in (0, +\infty)\}$. It is obvious that $c \leqslant \inf_{v \in M} J(v)$. In order to prove the other inequality, given $\gamma \in \Gamma$, it is enough to obtain $t_\gamma \in (0, 1)$ such that $u_\gamma = \gamma(t_\gamma) \in M$. Let us assume the contrary, that is, there is no $t \in (0, 1)$ such that

$$\|\gamma(t)\|^p = \int_{\mathbb{R}^N} g(z, \gamma(t))\gamma(t) \, \mathrm{d}z.$$

Thus, since $\gamma(0) = 0$, from (g_1) we must have

$$\|\gamma(t)\|^p > \int_{\mathbb{R}^N} g(z, \gamma(t))\gamma(t) \, \mathrm{d}z \quad \forall t \in (0, 1).$$

This estimate together with condition (g_3) implies

$$J(\gamma(t)) = \frac{1}{p}\|\gamma(t)\|^p - \int_{\mathbb{R}^N} G(z,\gamma(t))\,dz > \frac{1}{p}\int_{\mathbb{R}^N} [g(z,\gamma(t))\gamma(t) - pG(z,\gamma(t))]\,dz$$
$$\geqslant \frac{\theta - p}{p}\int_{\Omega} G(z,\gamma(t))\,dz \geqslant 0$$

for all $t \in (0, 1)$. But this is contrary to $\gamma \in \Gamma$. Thus, Assertion 2.1 holds. □

Assertion 2.2. *Every sequence* $(u_n) \subset E$ *satisfying* (2.2) *is bounded in* E.

Proof. Using assumption (g_3) we find

$$\begin{aligned} J(u_n) - \frac{1}{\theta}J'(u_n)u_n &= \left(\frac{1}{p} - \frac{1}{\theta}\right)\|u_n\|^p + \frac{1}{\theta}\int_{\mathbb{R}^N} [u_n g(z,u_n) - \theta G(z,u_n)]\,dz \\ &\geqslant \left(\frac{1}{p} - \frac{1}{\theta}\right)\|u_n\|^p + \frac{1}{\theta}\int_{\mathbb{R}^N-\Omega} [u_n g(z,u_n) - \theta G(z,u_n)]\,dz \\ &\geqslant \left(\frac{1}{p} - \frac{1}{\theta}\right)\|u_n\|^p + \frac{(p-\theta)}{\theta}\int_{\mathbb{R}^N-\Omega} G(z,u_n)\,dz \\ &\geqslant \left(\frac{1}{p} - \frac{1}{\theta}\right)\|u_n\|^p + \frac{(p-\theta)}{pk\theta}\int_{\mathbb{R}^N-\Omega} V(z)|u_n|^p\,dz \\ &= \left(\frac{\theta - p}{p\theta}\right)\int_{\mathbb{R}^N} \left[|\nabla u_n|^p + \left(1 - \frac{1}{k}\right)V(z)|u_n|^p\right]dz, \end{aligned} \quad (2.4)$$

which together with (2.2) implies that (u_n) is bounded in E. □

Now we closely follow the approach from [3], to prove the next result.

Assertion 2.3. *There exists* $v \in E - \{0\}$ *such that*

$$\max_{t \geqslant 0} J(tv) < \frac{S^{N/p}}{N}. \quad (2.5)$$

Proof. For each $\zeta > 0$, consider the functions

$$\beta_\zeta(z) = \phi(z)w_\zeta(z) \text{ and } v_\zeta(z) = \frac{\beta_\zeta(z)}{\left(\int_{|z|\leqslant 2} \beta_\zeta^{p^*}\,dz\right)^{1/p^*}},$$

where $\phi \in C_0^\infty(\mathbb{R}^N, [0, 1])$, $\phi(z) = 1$ if $|z| \leqslant 1$ and $\phi(z) = 0$ if $|z| \geqslant 2$. By a similar argument to that used in [3], we show that the functions w_ζ, β_ζ and v_ζ satisfy the following estimates:

(A) $\int_{|z|\geqslant 1} |\nabla \beta_\zeta(z)|^p \, dz = O(\zeta^{(N-p)/p})$,
(B) $k_1 < \int_{|z|\leqslant 2} \beta_\zeta^{p^*}(z) \, dz < k_2$ for ζ sufficiently small,
(C) $\int_{|z|\leqslant 1} |z| w_\zeta^{p^*}(z) \, dz = O(\zeta^{(N-p)/p})$,
(D) $\int_{\mathbb{R}^N} |\nabla v_\zeta(z)|^p \, dz \leqslant S + O(\zeta^{(N-p)/p})$.

In view of Lemma 2.1, for each $\zeta > 0$ small, there exists $t_\zeta > 0$ such that

$$J(t_\zeta v_\zeta) = \max\{J(t v_\zeta) : t \geqslant 0\}.$$

Notice that, at $t = t_\zeta$, we have $d/dt \, J(t v_\zeta) = 0$. Thus, using (g_1) and (f_2),

$$t_\zeta^{p-1} \int_{\mathbb{R}^N} [|\nabla v_\zeta|^p + V(z)|v_\zeta|^p] \, dz = \int_{\mathbb{R}^N} g(z, t_\zeta v_\zeta) v_\zeta \, dz \leqslant \lambda \int_{\mathbb{R}^N} (t_\zeta v_\zeta)^{q_1} \, dz + t_\zeta^{p^*-1}.$$

From this, condition (f_2) and estimates (A)–(D), it follows that there is $\zeta_0 > 0$ such that $t_\zeta > \alpha_0 > 0$ for all $0 < \zeta < \zeta_0$, where α_0 is a positive constant independent of ζ. Furthermore, by straightforward calculations we find

$$\begin{aligned} J(t_\zeta v_\zeta) &\leqslant \frac{S^{N/p}}{N} + O(\zeta^{(N-p)/p}) + \int_{|z|\leqslant 2} [C_1 V(z) v_\zeta^p - C_2 \lambda v_\zeta^{q_1+1}] \, dz \\ &\leqslant \frac{S^{N/p}}{N} + \zeta^{(N-p)/p} \left[C_0 + \zeta^{(p-N)/p} \int_{|z|\leqslant 2} [C_1 V(z) v_\zeta^p - C_2 \lambda v_\zeta^{q_1+1}] \, dz \right], \end{aligned}$$

where C_0, C_1 and C_2 are positive constants independent of ζ.

Assertion 2.4. *There is $\zeta > 0$ sufficiently small such that*

$$C_0 + \zeta^{(p-N)/p} \int_{|z|\leqslant 2} [C_1 V(z) v_\zeta^p - C_2 \lambda v_\zeta^{q_1+1}] \, dz < 0.$$

From the above estimate we easily see that

$$\max_{t\geqslant 0} J(t v_\zeta) = J(t_\zeta v_\zeta) < \frac{S^{N/p}}{N},$$

and taking $u = t_\zeta v_\zeta$ we obtain (2.5) and the proof of Assertion 2.3 is complete. □

Proof of Assertion 2.4. Now we proceed to prove Assertion 2.4. Using estimates (A)–(D), and the expression of v_ζ we have

$$\zeta^{(p-N)/p} \int_{|z|\leqslant 2} [C_1 V(z) v_\zeta^p - C_2 \lambda v_\zeta^{q_1+1}] \, dz \leqslant \Phi(\zeta) + \Psi(\zeta),$$

where

$$\Phi(\zeta) = \zeta^{(p-N)/p} \int_{|z|\leqslant 1} [C_3 V(z) v_\zeta^p - C_4 \lambda v_\zeta^{q_1+1}]\, dz$$
$$\leqslant C_5 \zeta^{(p^2-N)/p} K_\zeta - C_6 \lambda \zeta^{(q_1+1)[(N-p)/p^2-(N-p)/p]+(p-1)N/p+(p-N)/p}$$
$$K_\zeta = \int_0^{\zeta^{(1-p)/p}} \frac{s^{N-1}}{(1+s^{p/(p-1)})^{N-p}}\, ds$$
$$\Psi(\zeta) = \zeta^{(p-N)/p} \int_{1\leqslant |z|\leqslant 2} [C_3 V(z) v_\zeta^p - C_4 \lambda v_\zeta^{q_1+1}]\, dz$$
$$\leqslant C_5 \zeta^{(p-N)/p} \int_{1\leqslant |z|\leqslant 2} w_\zeta^p \leqslant C_6 \int_{1\leqslant |z|\leqslant 2} |z|^{p(p-N)/(p-1)}\, dz \leqslant C_7$$

for some positive constants C_3–C_7 independent of ζ. Finally, using these estimates and studying separately the conditions (a), (b) or (c) in Theorem 1.1, we prove that there exists $\zeta > 0$ such that

$$\Phi(\zeta) \leqslant C_7 - C_0.$$

Thus, Assertion 2.4 holds. □

Finally, to complete the proof of Proposition 2.2 it is enough to use Assertions 2.1 and 2.2 □

Lemma 2.3. *Let* $(u_n) \subset E$ *be a sequence satisfying* (2.1). *Then there is a sequence* $(y_n) \subset \mathbb{R}^N$, *and* $\rho, \eta > 0$ *such that*

$$\limsup_{n\to+\infty} \int_{B_\rho(y_n)} |u_n|^p\, dz \geqslant \eta.$$

Furthermore, the sequence (y_n) *is bounded in* $\mathbb{R}^N$.

Proof. Suppose that the first part of the Lemma is not satisfied. Since (u_n) is bounded in E, using Lemma 1.1 in [17], it follows that

$$\int_{\mathbb{R}^N} |u_n|^{q+1}\, dz \to 0 \text{ as } n \to +\infty$$

for all $p < q+1 < p^*$. Thus, from our assumption on f we find

$$\theta \int_{\mathbb{R}^N} F(u_n)\, dz = \int_{\mathbb{R}^N} u_n f(u_n)\, dz = o_n(1).$$

Now, the expression of $g(z,u)$ and (g_3) yield

$$\int_{\mathbb{R}^N} G(z,u_n)\, dz \leqslant \frac{1}{p^*} \int_{\Omega\cup\{u_n\leqslant a\}} (u_n)_+^{p^*} dz + \frac{\alpha}{pk} \int_{(\mathbb{R}^N-\Omega)\cap\{u_n>a\}} |u_n|^p\, dz + o_n(1) \tag{2.6}$$

and

$$\int_{\mathbb{R}^N} u_n g(z, u_n)\,\mathrm{d}z = \int_{\Omega\cup\{u_n \leqslant a\}} (u_n)_+^{p^*}\,\mathrm{d}z + \frac{\alpha}{k}\int_{(\mathbb{R}^N-\Omega)\cap\{u_n>a\}} |u_n|^p\,\mathrm{d}z + o_n(1). \quad (2.7)$$

From (2.7) and $J'(u_n).u_n = o_n(1)$, we conclude that

$$\|u_n\|^p - \frac{\alpha}{k}\int_{(\mathbb{R}^N-\Omega)\cap\{u_n>a\}} |u_n|^p\,\mathrm{d}z + o_n(1) = \int_{\Omega\cup\{u_n \leqslant a\}} (u_n)_+^{p^*}\,\mathrm{d}z. \quad (2.8)$$

Let $\ell \geqslant 0$ be such that

$$\|u_n\|^p - \frac{\alpha}{k}\int_{(\mathbb{R}^N-\Omega)\cap\{u_n>a\}} |u_n|^p\,\mathrm{d}z \to \ell.$$

Notice that $\ell > 0$, otherwise we have

$$\|u_n\|^p \leqslant C\left(\|u_n\|^p - \frac{\alpha}{k}\int_{(\mathbb{R}^N-\Omega)\cap\{u_n>a\}} |u_n|^p\,\mathrm{d}z\right) \to 0,$$

that is, $u_n \to 0$ in E, which implies that $c = 0$ and we get a contradiction with $c \geqslant \beta > 0$. Thus, from (2.8) we have

$$\int_{\Omega\cup\{u_n \leqslant a\}} (u_n)_+^{p^*}\,\mathrm{d}z \to \ell.$$

From $J(u_n) = c + o_n(1)$ and (2.6) it follows that

$$\begin{aligned} c + o_n(1) &= \frac{1}{p}\|u_n\|^p - \int_{\mathbb{R}^N} G(z, u_n)\,\mathrm{d}z \\ &\geqslant \frac{1}{p}\left(\|u_n\|^p - \frac{\alpha}{k}\int_{(\mathbb{R}^N-\Omega)\cap\{u_n>a\}} |u_n|^p\,\mathrm{d}z\right) - \frac{1}{p^*}\int_{\Omega\cup\{u_n \leqslant a\}} (u_n)_+^{p^*}\,\mathrm{d}z \end{aligned}$$

and letting $n \to \infty$ we get

$$\ell \leqslant Nc. \quad (2.9)$$

Now, using (1.3) we have

$$\|u_n\|^p - \frac{\alpha}{k}\int_{(\mathbb{R}^N-\Omega)\cap\{u_n>a\}} |u_n|^p\,\mathrm{d}z \geqslant S\left(\int_{\Omega\cup\{u_n \leqslant a\}} u_n^{p^*}\,\mathrm{d}z\right)^{p/p^*}.$$

Thus, passing to the limit and using (2.9) we achieve

$$c \geqslant \frac{1}{N} S^{N/p},$$

which is a contradiction with (2.1).

It remains to prove that (y_n) is bounded in $\mathbb{R}^N$. For this end we consider as test function $u_n\psi_\rho$, where $\psi_\rho \in C_0^\infty(\mathbb{R}^N, [0, 1])$, $\psi_\rho(z) = 0$ if $|z| \leqslant \rho$, $\psi_\rho(z) = 1$ if $|z| \geqslant 2\rho$ and $|\nabla\psi_\rho(z)| \leqslant C\rho^{-1}$ for all $z \in \mathbb{R}^N$.

Since, $J'(u_n)(\psi_\rho u_n) = o_n(1)$, we obtain

$$\begin{aligned}\alpha\left(1-\frac{1}{k}\right)\int_{\mathbb{R}^N}|u_n|^p\psi_\rho\,\mathrm{d}z &\leqslant \int_{\mathbb{R}^N}\left[|\nabla u_n|^p+\left(V(z)-\frac{\alpha}{k}\right)|u_n|^p\right]\psi_\rho\,\mathrm{d}z\\ &= -\int_{\mathbb{R}^N}u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_\rho\,\mathrm{d}z\\ &\quad+\int_{\mathbb{R}^N}\left[g(z,u_n)u_n-\frac{\alpha}{k}|u_n|^p\right]\psi_\rho\,\mathrm{d}z+o_n(1).\end{aligned}$$

If ρ is large enough, $\Omega \subset B_\rho(0)$, and from (g_3) we have

$$\alpha\left(1-\frac{1}{k}\right)\int_{\mathbb{R}^N}|u_n|^p\psi_\rho\,\mathrm{d}z \leqslant \frac{C}{\rho}\|u_n\|^p_{W^{1,p}}+o_n(1). \tag{2.10}$$

From (2.10), we conclude that (y_n) is bounded in $\mathbb{R}^N$. □

Now we are ready to state the following existence result:

Theorem 2.4. *For all* $\varepsilon > 0$, *the functional*

$$J_\varepsilon(u) \doteq \frac{1}{p}\int_{\mathbb{R}^N}[\varepsilon^p|\nabla u|^p+V(z)|u|^p]\,\mathrm{d}z-\int_{\mathbb{R}^N}G(z,u)\,\mathrm{d}z$$

possesses a positive critical point $u_\varepsilon \in E$ *at the level*

$$c_\varepsilon = \inf_{v\in E-\{0\}}\max_{t\geqslant 0} J_\varepsilon(tv). \tag{2.11}$$

Proof. We know that there exists a bounded sequence $(u_n) \subset E$ such that

$$J_\varepsilon(u_n) \to c_\varepsilon, \quad 0 < c_\varepsilon < \frac{1}{N}S^{N/p}, \quad \text{and} \quad J'_\varepsilon(u_n) \to 0, \quad \text{as} \quad n \to \infty.$$

Then, up to a subsequence, $u_n \rightharpoonup u_\varepsilon$ weakly in E. Now, using the same kind of ideas contained in [1,3,16] we can prove that, for all $\phi \in E$, we have

$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}\nabla u_n\nabla\phi\,\mathrm{d}z &\to \int_{\mathbb{R}^N}|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\nabla\phi\,\mathrm{d}z,\\ \int_{\mathbb{R}^N}V(z)|u_n|^{p-2}u_n\phi\,\mathrm{d}z &\to \int_{\mathbb{R}^N}V(z)|u_\varepsilon|^{p-2}u_\varepsilon\phi\,\mathrm{d}z,\\ \int_{\mathbb{R}^N}g(z,u_n)\phi\,\mathrm{d}z &\to \int_{\mathbb{R}^N}g(z,u_\varepsilon)\phi\,\mathrm{d}z.\end{aligned}$$

From these facts, together with $J'_\varepsilon(u_n) \to 0$ and passing to the limit, we easily obtain

$$\int_{\mathbb{R}^N}[|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\nabla\phi+V(z)|u_\varepsilon|^{p-2}u_\varepsilon\phi]\,\mathrm{d}z=\int_{\mathbb{R}^N}g(z,u_\varepsilon)\phi\,\mathrm{d}z \quad \forall\phi\in E, \tag{2.12}$$

that is, u_ε is a critical point of J_ε.

Assertion 2.5. $u_\varepsilon > 0$ *on* $\mathbb{R}^N$.

Proof. First, using Lemma 2.3, we are going to prove that u_ε is nontrivial. We know that $B_\rho(y_n) \subset B_R(0)$ for all n, with suitable $R > 0$. Thus, up to a subsequence,

$$0 < \sqrt[p]{\eta} \leqslant |u_n|_{L^p(B_\rho(y_n))} \leqslant |u_n|_{L^p(B_R(0))} \leqslant |u_\varepsilon|_{L^p(B_R(0))} + |u_n - u_\varepsilon|_{L^p(B_R(0))},$$

which together with the Sobolev's compact embedding theorem implies that u_ε is nontrivial.

Let $u_\varepsilon = (u_\varepsilon)_+ + (u_\varepsilon)_-$ and take $\phi = (u_\varepsilon)_-$ as test the function in (2.12); then

$$\int_{\mathbb{R}^N} \{|\nabla (u_\varepsilon)_-|^p + V(z)\|(u_\varepsilon)_-|^p\}\,\mathrm{d}z = \int_{\mathbb{R}^N} g(z, (u_\varepsilon)_-)(u_\varepsilon)_-\,\mathrm{d}z = 0.$$

Hence $(u_\varepsilon)_- = 0$ almost everywhere in $\mathbb{R}^N$. Therefore $u_\varepsilon \geqslant 0$ almost everywhere in $\mathbb{R}^N$. Now, we claim that $u_\varepsilon > 0$. Indeed, by contradiction, suppose that there exists $x_0 \in \mathbb{R}^N$ such that $u_\varepsilon(x_0) = 0$. Notice that u_ε is a weak supersolution of the problem

$$\begin{cases} -\Delta_p u + V(x)u^{p-1} = g(x, u)), & x \in B_r(x_0), \\ u(x) = 0 & x \in \partial B_r(x_0). \end{cases}$$

Now, using a standard bootstrap argument we may show that $u_\varepsilon \in L^\infty(\mathbb{R}^N)$; see Proposition 3.6. Hence, by Harnack's inequality, see Theorem 1.2 in [27], we have $u_\varepsilon \equiv 0$ in $B_r(x_0)$, which is a contradiction. □

Finally, it only remains to prove that the critical point u_ε is in the level given in (2.11). For that matter, we use assumption (g_3) and Fatou's lemma, to obtain

$$\begin{aligned} c_\varepsilon &\leqslant \max_{t \geqslant 0} J_\varepsilon(tu_\varepsilon) = J_\varepsilon(u_\varepsilon) = J_\varepsilon(u_\varepsilon) - \frac{1}{p} J'_\varepsilon(u_\varepsilon)u_\varepsilon \\ &= \frac{1}{p} \int_{\mathbb{R}^N} [u_\varepsilon g(z, u_\varepsilon) - pG(z, u_\varepsilon)]\,\mathrm{d}z \\ &\leqslant \liminf_{n\to\infty} \left\{ \frac{1}{p} \int_{\mathbb{R}^N} [u_n g(z, u_n) - pG(z, u_n)] \right\} \mathrm{d}z \\ &= \liminf_{n\to\infty} \left\{ J_\varepsilon(u_n) - \frac{1}{p} J'_\varepsilon(u_n)u_n \right\} = c_\varepsilon. \end{aligned}$$

Thus, u_ε is a solution with minimal energy $J_\varepsilon(u_\varepsilon) = c_\varepsilon$ and the proof of Theorem 2.4 is complete. □

3. Proof of Theorem 1.1

3.1. *Existence of solution*

Let I_ε denote the energy functional

$$I_\varepsilon(u) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u|^p + V(\varepsilon x)|u|^p)\,\mathrm{d}x - \int_{\mathbb{R}^N} G(\varepsilon x, u)\,\mathrm{d}x$$

defined in

$$E_\varepsilon = \{u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\varepsilon x)|u|^p \,\mathrm{d}x < \infty\},$$

associated with the problem

$$\begin{cases} -\Delta_p u + V(\varepsilon x)u^{p-1} = g(\varepsilon x, u) & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N. \end{cases}$$

From Theorem 2.4, the family of positive functions

$$v_\varepsilon(x) = u_\varepsilon(z) = u_\varepsilon(\varepsilon x), \;\; z = \varepsilon x$$

is such that each v_ε is a critical point of I_ε at the level

$$b_\varepsilon = I_\varepsilon(v_\varepsilon) = \inf_{v \in E_\varepsilon \setminus \{0\}} \max_{t \geqslant 0} I_\varepsilon(tv).$$

It is easy to check that $b_\varepsilon = \varepsilon^{-N} c_\varepsilon$. Furthermore, using Assertion 2.3 we also conclude that $b_\varepsilon < \frac{1}{N} S^{N/p}$.

In order to derive a useful estimate of the mountain-pass minimax level b_ε we consider a test function related to the solution of the autonomous problem

$$\begin{cases} -\Delta_p u + V_0 u^{p-1} = f(u) + u^{p^*-1} & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{3.1}$$

It is known that under assumptions (f_1)–(f_4), problem (3.1) possesses a ground state solution ω at the level

$$c_0 = I_0(\omega) = \inf_{v \in E - \{0\}} \max_{t \geqslant 0} I_0(tv) < \frac{1}{N} S^{\frac{N}{2}}, \tag{3.2}$$

where I_0 is defined as

$$I_0(u) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u|^p + V_0|u|^p)\,\mathrm{d}x - \int_{\mathbb{R}^N} \left[F(u) + \frac{u_+^{p^*}}{p^*} \right] \mathrm{d}x$$

(cf. [8]). Furthermore, in the case $1 < p \leqslant 2$, we have that ω must be radially symmetric about some origin O in $\mathbb{R}^N$ and the corresponding function $\omega(r)$ obeys $\omega'(r) < 0$ for all $r > 0$ (see details in [5,6,24]).

Lemma 3.1. $\limsup_{\varepsilon \to 0} b_\varepsilon \leqslant c_0$.

Proof. Let ω be a ground state solution of problem (3.1), which without loss of generality we may assume maximizes at zero. Now consider the test function $\varpi_\varepsilon(x) = \phi(\varepsilon x)\omega(x)$, where $\phi \in C_0^\infty(\mathbb{R}^N, [0, 1])$, $\phi(x) = 1$ if $|x| \leqslant 1$ and $\phi(x) = 0$ if $|x| \geqslant 2$. It is easy to check that $\varpi_\varepsilon \to \omega$ in $W^{1,p}(\mathbb{R}^N)$, $I_0(\varpi_\varepsilon) \to I_0(\omega)$, as $\varepsilon \to 0$, and the support of ϖ_ε is contained in $\Omega_\varepsilon = \{x \in \mathbb{R}^N : \varepsilon x \in \Omega\}$. For each $\varepsilon > 0$ consider $t_\varepsilon \in (0, +\infty)$ such that $\max_{t \geqslant 0} I_\varepsilon(t\varpi_\varepsilon) = I_\varepsilon(t_\varepsilon \varpi_\varepsilon)$, thus

$$\begin{aligned} b_\varepsilon &= \inf_{v \in E_\varepsilon \setminus \{0\}} \max_{t \geqslant 0} I_\varepsilon(tv) \leqslant \max_{t \geqslant 0} I_\varepsilon(t\varpi_\varepsilon) = I_\varepsilon(t_\varepsilon \varpi_\varepsilon) \\ &= \frac{t_\varepsilon^p}{p} \int_{\mathbb{R}^N} [|\nabla \varpi_\varepsilon|^p + V(\varepsilon x)|\varpi_\varepsilon|^p]\, \mathrm{d}x - \int_{\mathbb{R}^N} \left[F(t_\varepsilon \varpi_\varepsilon) + \frac{(t_\varepsilon \varpi_\varepsilon)^{p^*}}{p^*} \right] \mathrm{d}x. \end{aligned}$$

Assertion 3.1. $t_\varepsilon \to 1$ *as* $\varepsilon \to 0$.

Proof. Since $I'_\varepsilon(t_\varepsilon \varpi_\varepsilon)(t_\varepsilon \varpi_\varepsilon) = 0$, using assumption (f_2) we have

$$\begin{aligned} t_\varepsilon^p \int_{\mathbb{R}^N} (|\nabla \varpi_\varepsilon|^p + V(\varepsilon x)|\varpi_\varepsilon|^p)\, \mathrm{d}x &= \int_{\mathbb{R}^N} [f(t_\varepsilon \varpi_\varepsilon) t_\varepsilon \varpi_\varepsilon + (t_\varepsilon \varpi_\varepsilon)^{p^*}]\, \mathrm{d}x \\ &\geqslant \int_{\mathbb{R}^N} [\lambda (t_\varepsilon \varpi_\varepsilon)^{q_1+1} + (t_\varepsilon \varpi_\varepsilon)^{p^*}]\, \mathrm{d}x. \end{aligned} \tag{3.3}$$

Since (ϖ_ε) is bounded, from this estimate we derive easily that (t_ε) is bounded from above and below. Thus, up to a subsequence, we have $t_\varepsilon \to t_1 > 0$. Passing to the limit in the first expression of (3.3), we have

$$\int_{\mathbb{R}^N} (|\nabla \omega|^p + V_0|\omega|^p)\, \mathrm{d}x = t_1^{-p} \int_{\mathbb{R}^N} [f(t_1\omega) t_1 \omega + (t_1 \omega)^{p^*}]\, \mathrm{d}x. \tag{3.4}$$

Now, subtracting (3.4) from

$$\int_{\mathbb{R}^N} (|\nabla \omega|^p + V_0|\omega|^p)\, \mathrm{d}x = \int_{\mathbb{R}^N} [f(\omega)\omega + \omega^{p^*}]\, \mathrm{d}x,$$

we achieve

$$0 = \int_{\mathbb{R}^N} \left[\frac{f(t_1\omega)}{(t_1\omega)^{p-1}} - \frac{f(\omega)}{\omega^{p-1}} \right] \omega^p\, \mathrm{d}x + (t_1^{p^*-p} - 1) \int_{\mathbb{R}^N} \omega^{p^*}\, \mathrm{d}x.$$

Thus, from (f_4), we have $t_1 = 1$. □

Finally, we notice that we also have

$$I_\varepsilon(t_\varepsilon \varpi_\varepsilon) = I_0(t_\varepsilon \varpi_\varepsilon) + \frac{t_\varepsilon^p}{p} \int_{\mathbb{R}^N} (V(\varepsilon x) - V_0)|\varpi_\varepsilon|^p)\, \mathrm{d}x.$$

Thus, taking the limit as $\varepsilon \to 0$, using the fact that $V(\varepsilon x)$ in bounded on the support ofϖ_ε and the Lebesgue dominated convergence theorem, we conclude the proof of Lemma 3.1. □

Now we have $I_\varepsilon(v_\varepsilon) \leqslant c_0 + o_\varepsilon(1)$, where $o_\varepsilon(1)$ goes to zero as $\varepsilon \to 0$. Since for $u \in W^{1,p}(\mathbb{R}^N)$, from (V_0) and (g_2),

$$I_\varepsilon(u) \geqslant \bar{I}(u) \doteq \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u|^p + \alpha |u|^p)\, \mathrm{d}x - \int_{\mathbb{R}^N} \left[F(u) + \frac{1}{p^*}(u_+)^{p^*} \right] \mathrm{d}x,$$

then each b_ε is bounded from below by $\bar{c} > 0$, the mountain-pass minimax level of the functional $\bar{I}$.

Notice that

$$|v_\varepsilon|_{E_\varepsilon}^p = \int_{\mathbb{R}^N} g(\varepsilon x, v_\varepsilon) v_\varepsilon \, \mathrm{d}x$$

and for ε suitable small,

$$\frac{\theta}{p} |v_\varepsilon|_{E_\varepsilon}^p \leqslant \int_{\mathbb{R}^N} \theta G(\varepsilon x, v_\varepsilon) \, \mathrm{d}x + \theta c_0 + 1,$$

which together with assumption (g_3) implies that

$$\begin{aligned}
\left(\frac{\theta}{p} - 1\right) |v_\varepsilon|_{E_\varepsilon}^p &\leqslant \int_{\mathbb{R}^N} [\theta G(\varepsilon x, v_\varepsilon) - g(\varepsilon x, v_\varepsilon) v_\varepsilon] \, \mathrm{d}x + \theta c_0 + 1 \\
&\leqslant \int_{\mathbb{R}^N - \Omega_\varepsilon} [\theta G(\varepsilon x, v_\varepsilon) - g(\varepsilon x, v_\varepsilon) v_\varepsilon] \, \mathrm{d}x + \theta c_0 + 1 \\
&\leqslant \int_{\mathbb{R}^N - \Omega_\varepsilon} (\theta - p) G(\varepsilon x, v_\varepsilon) \, \mathrm{d}x + \theta c_0 + 1 \\
&\leqslant \int_{\mathbb{R}^N - \Omega_\varepsilon} \frac{(\theta - p)}{kp} V(\varepsilon x) v_\varepsilon^p \, \mathrm{d}x + \theta c_0 + 1 \leqslant \frac{(\theta - p)}{kp} |v_\varepsilon|_{E_\varepsilon}^p + \theta c_0 + 1.
\end{aligned}$$

Thus $|v_\varepsilon|_{E_\varepsilon} \leqslant C$, where C is a positive constant independent of ε. Of course, we have that $(v_\varepsilon)_{\{0<\varepsilon\leqslant\varepsilon_0\}}$ is bounded in $W^{1,p}(\mathbb{R}^N)$.

Lemma 3.2. *There are $\varepsilon_0 > 0$, a family $(y_\varepsilon)_{\{0<\varepsilon\leqslant\varepsilon_0\}} \subset \mathbb{R}^N$ and positive constants R, β such that*

$$\int_{B_R(y_\varepsilon)} v_\varepsilon^p \, \mathrm{d}x \geqslant \beta \text{ for all } 0 < \varepsilon \leqslant \varepsilon_0.$$

Proof. Assume, for the sake of contradiction, that there is a sequence $\varepsilon_n \searrow 0$ such that for all $R > 0$,

$$\lim_{n\to\infty} \sup_{x \in \mathbb{R}^N} \int_{B_R(x)} v_{\varepsilon_n}^p \, \mathrm{d}x = 0.$$

Using Lemma 1.1 in [17], it follows that

$$\int_{\mathbb{R}^N} F(v_{\varepsilon_n}) \, \mathrm{d}x = \int_{\mathbb{R}^N} v_{\varepsilon_n} f(v_{\varepsilon_n}) \, \mathrm{d}x = o_n(1).$$

This implies the estimates

$$\int_{\mathbb{R}^N} G(z, v_{\varepsilon_n}) \, \mathrm{d}x \leqslant \frac{1}{p^*} \int_{\Omega_{\varepsilon_n} \cup \{v_{\varepsilon_n} \leqslant a\}} v_{\varepsilon_n}^{p^*} \, \mathrm{d}x + \frac{\alpha}{pk} \int_{D_{\varepsilon_n} \cap \{v_{\varepsilon_n} > a\}} v_{\varepsilon_n}^p \, \mathrm{d}x + o_n(1) \tag{3.5}$$

and

$$\int_{\mathbb{R}^N} v_{\varepsilon_n} g(z, v_{\varepsilon_n})\,\mathrm{d}x = \int_{\Omega_{\varepsilon_n} \cup \{v_{\varepsilon_n} \leqslant a\}} v_{\varepsilon_n}^{p^*}\,\mathrm{d}x + \frac{\alpha}{k} \int_{D_{\varepsilon_n} \cap \{v_{\varepsilon_n} > a\}} v_{\varepsilon_n}^{p}\,\mathrm{d}x + o_n(1), \quad (3.6)$$

where $D_\varepsilon = \mathbb{R}^N \backslash \Omega_\varepsilon$. From equality (3.6) and $I'_{\varepsilon_n}(v_{\varepsilon_n}).v_{\varepsilon_n} = 0$ we have

$$\|v_{\varepsilon_n}\|^p - \frac{\alpha}{k} \int_{D_{\varepsilon_n} \cap \{v_{\varepsilon_n} > a\}} v_{\varepsilon_n}^{p}\,\mathrm{d}x + o_n(1) = \int_{\Omega_{\varepsilon_n} \cup \{v_{\varepsilon_n} \leqslant a\}} v_{\varepsilon_n}^{p^*}\,\mathrm{d}x. \quad (3.7)$$

Since $\bar{c} > 0$, we have

$$\|v_{\varepsilon_n}\|^p - \frac{\alpha}{k} \int_{D_{\varepsilon_n} \cap \{v_{\varepsilon_n} > a\}} v_{\varepsilon_n}^{p}\,\mathrm{d}x \to \ell > 0. \quad (3.8)$$

Thus, from (3.7) and (3.8),

$$\int_{\Omega_{\varepsilon_n} \cup \{v_{\varepsilon_n} \leqslant a\}} v_{\varepsilon_n}^{p^*}\,\mathrm{d}x \to \ell.$$

Using estimate (3.5) and the fact that $I_{\varepsilon_n}(v_{\varepsilon_n}) \leqslant c_0 + o_n(1)$, we conclude that

$$\ell \leqslant N c_0. \quad (3.9)$$

Now, using (1.3) we have

$$\|v_{\varepsilon_n}\|^p - \frac{\alpha}{k} \int_{(\mathbb{R}^N - \Omega) \cap \{v_{\varepsilon_n} > a\}} |v_{\varepsilon_n}|^p\,\mathrm{d}x \geqslant S\left(\int_{\Omega \cup \{v_{\varepsilon_n} \leqslant a\}} v_{\varepsilon_n}^{p^*}\,\mathrm{d}x\right)^{p/p^*}.$$

Thus, passing to the limit and using (3.9) we achieve

$$c_0 \geqslant \frac{1}{N} S^{N/p},$$

which is a contradiction for (3.2), and the proof of Lemma 3.2 is complete. □

Lemma 3.3. *The family* $(\varepsilon y_\varepsilon)_{\{0<\varepsilon \leqslant \varepsilon_0\}}$ *is bounded in* $\mathbb{R}^N$ *and moreover*, $\mathit{dist}(\varepsilon y_\varepsilon, \Omega) \leqslant \varepsilon R$.

Proof. For each $\delta > 0$, we set $K_\delta = \{x \in \mathbb{R}^N : \mathit{dist}(x, \Omega) \leqslant \delta\}$ and $\psi_\varepsilon(x) = \psi(\varepsilon x)$, where $\psi \in C^\infty(\mathbb{R}^N, [0, 1])$ is such that $\psi(x) = 1$ if $x \notin K_\delta$, $\psi(x) = 0$ if $x \in \Omega$ and $|\nabla \psi| \leqslant C\delta^{-1}$. Notice that $|\nabla \psi_\varepsilon| \leqslant C\delta^{-1}\varepsilon$. Using assumption (V_0), we see that

$$\alpha\left(1 - \frac{1}{k}\right) \int_{\mathbb{R}^N} |v_\varepsilon|^p \psi_\varepsilon\,\mathrm{d}x \leqslant \int_{\mathbb{R}^N} \left[|\nabla v_\varepsilon|^p + \left(V(\varepsilon x) - \frac{\alpha}{k}\right) v_\varepsilon^p\right] \psi_\varepsilon\,\mathrm{d}x.$$

On the other hand, using $I'_\varepsilon(v_\varepsilon)v_\varepsilon\psi_\varepsilon = 0$, condition (g_3) and the fact that the support of ψ_ε does not intercept Ω_ε, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} [|\nabla v_\varepsilon|^p + \left(V(\varepsilon x) - \frac{\alpha}{k}\right) v_\varepsilon^p]\psi_\varepsilon \, dx \\
&\leqslant - \int_{\mathbb{R}^N} v_\varepsilon |\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon \nabla \psi_\varepsilon \, dx + \int_{\mathbb{R}^N} \left[g(z, v_\varepsilon)v_\varepsilon - \frac{\alpha}{k} v_\varepsilon^p\right]\psi_\varepsilon \, dx \\
&= - \int_{\mathbb{R}^N} v_\varepsilon |\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon \nabla \psi_\varepsilon \, dx + \int_{\mathbb{R}^N} \left[\tilde{f}(v_\varepsilon)v_\varepsilon - \frac{\alpha}{k} v_\varepsilon^p\right]\psi_\varepsilon \, dx \\
&\leqslant - \int_{\mathbb{R}^N} v_\varepsilon |\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon \nabla \psi_\varepsilon \, dx \\
&\leqslant C\delta^{-1}\left(\int_{\mathbb{R}^N} \varepsilon |\nabla v_\varepsilon|^p \, dx\right)^{(p-1)/p}\left(\int_{\mathbb{R}^N} |v_\varepsilon|^p \, dx\right)^{1/p} \leqslant C\delta^{-1}\varepsilon|v_\varepsilon|^p.
\end{aligned}$$

Thus, from these estimates, we have

$$\alpha\left(1 - \frac{1}{k}\right)\int_{\mathbb{R}^N} |v_\varepsilon|^p \psi_\varepsilon \, dx \leqslant C\delta^{-1}\varepsilon|v_\varepsilon|^p.$$

Notice that, if there is a sequence $\varepsilon_n \searrow 0$ such that

$$B_R(y_{\varepsilon_n}) \cap \{x \in \mathbb{R}^N : \varepsilon_n x \in K_\delta\} = \emptyset,$$

then

$$\alpha\left(1 - \frac{1}{k}\right)\int_{B_R(y_{\varepsilon_n})} |v_{\varepsilon_n}|^p \psi_{\varepsilon_n} \, dx \leqslant C\delta^{-1}\varepsilon_n|v_{\varepsilon_n}|^p.$$

But this is contrary to Lemma 3.2. Thus, for all $\varepsilon > 0$ there is an x such that $\varepsilon x \in K_\delta$ and $|x - y_\varepsilon| \leqslant R$, which implies that $dist(\varepsilon y_\varepsilon, \Omega) \leqslant \varepsilon R + \delta$. From this we conclude the proof. □

Remark 1. From Lemma 3.3, we can see that the family $(\varepsilon y_\varepsilon)_{\{0<\varepsilon\leqslant\varepsilon_0\}}$ given in Lemma 3.2, can be taken such that $\varepsilon y_\varepsilon \in \Omega$ for all $0 < \varepsilon < \varepsilon_0$. Indeed, since $dist(\varepsilon y_\varepsilon, \Omega) \leqslant \varepsilon R$, if necessary, we can replace y_ε by $\varepsilon^{-1}x_\varepsilon$ where $x_\varepsilon \in \Omega$ and $|y_\varepsilon - \varepsilon^{-1}x_\varepsilon| < R$. Thus,

$$0 < \beta \leqslant \int_{B_R(y_\varepsilon)} |v_\varepsilon|^p \, dx \leqslant \int_{B_{2R}(\varepsilon^{-1}x_\varepsilon)} |v_\varepsilon|^p \, dx$$

and if we replace R by $2R$ in the Lemma 3.2, we have our claim.

Next, we are going to prove that there is $\varepsilon_0 > 0$ such that the set

$$\mathscr{E}_\varepsilon = \{x \in \mathbb{R}^N : v_\varepsilon(x) \geqslant a \text{ and } \varepsilon x \notin \Omega\}$$

is empty, for all $0 < \varepsilon < \varepsilon_0$. For that matter we have the following basic result:

Lemma 3.4. *The following limits hold*:

(i) $\lim_{\varepsilon\to 0} b_\varepsilon = c_0$;

(ii) $\lim_{\varepsilon\to 0} V(\varepsilon y_\varepsilon) = V_0$;
(iii) $\lim_{\varepsilon\to 0}|\mathscr{E}_\varepsilon| = 0$, *where* $|\mathscr{E}_\varepsilon|$ *denotes the Lebesgue measure of* $\mathscr{E}_\varepsilon$.

Proof. (i) Let $\varepsilon_n \searrow 0$ and $y_n = y_{\varepsilon_n}$. Since $\varepsilon_n y_{\varepsilon_n} \in \overline{\Omega}$, up to a subsequence, we have $\varepsilon_n y_n \to x_0 \in \overline{\Omega}$. Set

$$v_n(x) = v_{\varepsilon_n}(x),\ \omega_n(x) = v_{\varepsilon_n}(x + y_n),\ \ \mathscr{E}_n = \mathscr{E}_{\varepsilon_n} \text{ and } \mathscr{F}_n = \mathscr{F}_{\varepsilon_n},$$

where

$$\mathscr{F}_\varepsilon = \{x \in \mathbb{R}^N : v_\varepsilon(x + y_\varepsilon) \geqslant a \text{ and } \varepsilon x + \varepsilon y_\varepsilon \notin \Omega\}. \tag{3.10}$$

It is clear that $|\mathscr{E}_\varepsilon| = |\mathscr{F}_\varepsilon|$, since $\mathscr{F}_\varepsilon$ is a translation of $\mathscr{E}_\varepsilon$. From the definition of ω_n, we have for all $\phi \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \{|\nabla\omega_n|^{p-2}\nabla\omega_n\nabla\phi + V(\varepsilon_n x + \varepsilon_n y_n)\omega_n^{p-1}\phi\}\,\mathrm{d}x = \int_{\mathbb{R}^N} g(\varepsilon_n x + \varepsilon_n y, \omega_n)\phi\,\mathrm{d}x \tag{3.11}$$

and $\|\omega_n\|_{W^{1,p}} = \|v_n\|_{W^{1,p}}$ is bounded. Thus, we may assume that there is $\omega_0 \in W^{1,p}(\mathbb{R}^N)$ such that $\omega_n \rightharpoonup \omega_0$ in $W^{1,p}(\mathbb{R}^N)$ and $\omega_n(x) \to \omega_0(x)$ a.e. in $\mathbb{R}^N$. Using Lemma 3.2, and taking $R_0 > 0$ such that $B_R(y_n) \subset B_{R_0}(0)$ for all n, we have

$$\sqrt[p]{\beta} \leqslant |\omega_n|_{L^p(B_R(y_n))} \leqslant |\omega_n|_{L^p(B_R(0))} \leqslant |\omega_n - \omega_0|_{L^p(B_{R_0}(y))} + |\omega_0|_{L^p(B_{R_0}(y))}.$$

From this estimate, using the Sobolev's compact embedding theorem, we conclude that ω_0 is nontrivial and so nonnegative.

Now, taking the limit in (3.11) and proceeding as in the proof of Theorem 2.4, we achieve that ω_0 is a critical point of the energy functional

$$\tilde{I}(\omega) = \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla\omega|^p + V(x_0)|\omega|^p)\,\mathrm{d}x - \int_{\mathbb{R}^N} \tilde{G}(x,\omega)\,\mathrm{d}x,$$

that is,

$$\int_{\mathbb{R}^N} [|\nabla\omega_0|^{p-2}\nabla\omega_0\nabla\phi + V(x_0)\omega_0^{p-1}\phi]\,\mathrm{d}x = \int_{\mathbb{R}^N} \tilde{g}(x,\omega_0)\phi\,\mathrm{d}x \quad \forall\phi \in C_0^\infty(\mathbb{R}^N), \tag{3.12}$$

where $\tilde{G}$ is the primitive of

$$\widetilde{g}(x,\omega_0) = \chi(x)[f(\omega_0) + \omega_0^{p^*-1}] + (1 - \chi(x))\widetilde{f}(\omega_0)$$

and

$$\chi(x) = \lim_{n\to\infty} \chi_\Omega(\varepsilon_n x + \varepsilon_n y_n) \quad \text{a.e.} \quad \text{in } \mathbb{R}^N.$$

Notice that, if $x_0 \in \Omega$, we have $\chi(x) = 1$ for all $x \in \mathbb{R}^N$, and so ω_0 is a critical point of the energy functional

$$\tilde{I}_{x_0}(\omega) = \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla\omega|^p + V(x_0)|\omega|^p)\,\mathrm{d}x - \int_{\mathbb{R}^N} \left[F(\omega) + \frac{\omega^{p^*}}{p^*}\right]\mathrm{d}x.$$

On the other hand, if $x_0 \in \partial\Omega$, without loss of generality we suppose that the outer normal vector v in x_0 is $(1, 0, \dots, 0)$. Let $P = \{x \in \mathbb{R}^N : x_1 < 0\}$. Notice that $\chi \equiv 1$ on P, since for each $x \in P$, we have that $\varepsilon_n x + \varepsilon_n y_n \in \Omega$, for n large, because $\varepsilon_n y_n \in \Omega$. Thus, in both cases $\widetilde{g}(x, s) = f(s) + s^{2^*-1}$, for all $x \in P$. This implies that the mountain-pass minimax level $\tilde{c}$ associated to the functional $\tilde{I}$ is equal to the mountain-pass minimax level $\tilde{c}_{x_0}$ associated to the functional $\tilde{I}_{x_0}$. Indeed, from (g_2), we have $\tilde{I}_{x_0}(u) \leqslant \tilde{I}(u)$, for all $u \in W^{1,p}(\mathbb{R}^N)$ and then $\tilde{c} \leqslant \tilde{c}_{x_0}$. On the other hand, $\tilde{I}_{x_0}(u) = \tilde{I}(u)$ for all u with support contained in P.

From (3.2), using Fatou's Lemma and Lemma 3.1, we get

$$
\begin{aligned}
pc_0 \leqslant p\widetilde{I}(\omega_0) &= \int_{\mathbb{R}^N} [\omega_0 \widetilde{g}(x, \omega_0) - p\widetilde{G}(x, \omega_0)]\,\mathrm{d}x \\
&\times \liminf_{n\to\infty} \int_{\mathbb{R}^N} [\omega_n g(\varepsilon_n x + \varepsilon_n y_n, \omega_n) - pG(\varepsilon_n x + \varepsilon_n y_n, \omega_n)]\,\mathrm{d}x \\
&\leqslant \liminf_{n\to\infty} \int_{\mathbb{R}^N\setminus\mathscr{F}_n} [\omega_n g(\varepsilon_n x + \varepsilon_n y_n, \omega_n) - pG(\varepsilon_n x + \varepsilon_n y_n, \omega_n)]\,\mathrm{d}x \\
&\leqslant \liminf_{n\to\infty} \int_{\mathbb{R}^N\setminus\mathscr{E}_n} [v_n g(\varepsilon_n x, v_n) - pG(\varepsilon_n x, v_n)]\,\mathrm{d}x \\
&= \liminf_{n\to\infty} [pI_{\varepsilon_n}(v_{\varepsilon_n}) - I'_{\varepsilon_n}(v_{\varepsilon_n})v_{\varepsilon_n}] \leqslant pc_0.
\end{aligned} \tag{3.13}
$$

Thus (i) holds.

Notice that if (ii) does not hold (that is, $V(x_0) > V_0$) we have

$$c_0 < \widetilde{c} \leqslant \widetilde{I}(\omega_0) = c_0,$$

which is a contradiction; then $V(x_0) = V_0$.

To show (iii), we observe that from (3.13), we have

$$\lim_{n\to\infty} \frac{1}{p} \int_{\mathbb{R}^N\setminus\mathscr{E}_n} [v_n g(\varepsilon_n x, v_n) - pG(\varepsilon_n x, v_n)]\,\mathrm{d}x = c_0 \tag{3.14}$$

and

$$\lim_{n\to\infty} \frac{1}{p} \int_{\mathbb{R}^N} [v_n g(\varepsilon_n x, v_n) - pG(\varepsilon_n x, v_n)]\,\mathrm{d}x = c_0.$$

Thus

$$\lim_{n\to\infty} \frac{1}{p} \int_{\mathscr{E}_n} [v_n g(\varepsilon_n x, v_n) - pG(\varepsilon_n x, v_n)]\,\mathrm{d}x = 0. \tag{3.15}$$

Now, using the definition of $g(z, s)$ and assumption (f_3) we can prove that

$$\int_{\mathscr{E}_n} [v_n g(\varepsilon_n x, v_n) - pG(\varepsilon_n x, v_n)]\,\mathrm{d}x = \left[\left(1 - \frac{p}{p^*}\right) a^{p^*} + af(a) - pF(a)\right] |\mathscr{E}_n| > 0,$$

which together with (3.15) implies that $\lim_{n\to\infty} |\mathscr{E}_n| = 0$, and the proof is complete. □

Lemma 3.5. *$v_\varepsilon(x + y_\varepsilon)\chi_{(\mathbb{R}^N\setminus\mathscr{F}_\varepsilon)}(x)$ converges to ω_0 in $L^{p^*}(\mathbb{R}^N)$.*

Proof. Following the notation of Lemma 3.4, we set $\omega_\varepsilon(x)=v_\varepsilon(x+y_\varepsilon)$. We have proved in last lemma that $b_\varepsilon \to c_0$ and that ω_n converges in the weak sense to ω_0, a ground state solution of the autonomous problem (3.1). From the proof of Lemma 3.4 and the regularity result below, we also have that $g(\varepsilon x+\varepsilon y_\varepsilon,\omega_\varepsilon)$ converges uniformly over compacts to $f(\omega_0)+\omega_0^{p^*}$. Moreover, it follows from the definition of g and (3.10), using the Hölder's inequality, that

$$\int_{\mathscr{F}_\varepsilon} g(\varepsilon x+\varepsilon y_\varepsilon,\omega_\varepsilon)\omega_\varepsilon\,\mathrm{d}x=\frac{\alpha}{k}\int_{\mathscr{F}_\varepsilon}\omega_\varepsilon^p \leqslant \frac{\alpha}{k}|\omega_\varepsilon|^p_{L^{p^*}}|\mathscr{F}_\varepsilon|^{\frac{p^*-p}{p^*}}=o_\varepsilon(1),$$

which together with assumption (g_3) implies that

$$\int_{\mathscr{F}_\varepsilon} G(\varepsilon x+\varepsilon y_\varepsilon,\omega_\varepsilon)\,\mathrm{d}x=o_\varepsilon(1).$$

Now from (3.14) and definition of g, we have

$$\begin{aligned} pc_0+o_n(1) &= \int_{\mathbb{R}^N\setminus\mathscr{F}_n}[\omega_n g(\varepsilon_n x+\varepsilon_n y_n,\omega_n)-pG(\varepsilon_n x+\varepsilon_n y_n,\omega_n)]\,\mathrm{d}x\\ &= \int_{\mathbb{R}^N\setminus\mathscr{F}_n}[\omega_n f(\omega_n)-pF(\omega_n)]\,\mathrm{d}x+\left(1-\frac{p^*}{p}\right)\int_{\mathbb{R}^N\setminus\mathscr{F}_n}\omega_n^{p^*}\,\mathrm{d}x. \end{aligned}$$

Thus

$$\int_{\mathbb{R}^N\setminus\mathscr{F}_n}\omega_n^{p^*}\,\mathrm{d}x\to\int_{\mathbb{R}^N}\omega_0^{p^*}\,\mathrm{d}x$$

and the proof is complete. □

We use the classical interactions method due to Moser to prove the regularity of the weak solutions, more precisely, we shall prove the following result:

Proposition 3.6. *v_ε belongs to $L^s(\mathbb{R}^N)$ for all $s\in[p^*,+\infty]$. Moreover, $|v_\varepsilon|_\infty\leqslant C$, for all $0<\varepsilon<\varepsilon_0$ and the solutions v_ε decay uniformly to zero as $|x|\to+\infty$.*

Proof. In order to use the same kind of argument in [23], we are going to prove by induction that $v_\varepsilon\in L^{\sigma_n}(\mathbb{R}^N)$ for all $\sigma_n=p\gamma^n$, where $\gamma=N/(N-p)$. By Gagliardo–Nirenberg–Sobolev inequality we have that $v_\varepsilon\in L^{\sigma_1}$. Assume that $v_\varepsilon\in L^{\sigma_n}(\mathbb{R}^N)$; then, we shall prove that $v_\varepsilon\in L^{\sigma_{n+1}}(\mathbb{R}^N)$. For this end we consider the test function $\phi=\psi^p v_\varepsilon[T_k(v_\varepsilon)]^{s_n}$, where $T_k(v_\varepsilon)=\min\{k,v_\varepsilon\}$, $s_n=p(\gamma^n-1)$ and $\psi\in C_0^\infty(\mathbb{R}^N,[0,1])$.

Using the fact that v_ε is a critical point of I_ε and assumptions (g_2), (f_1) and (f_2) we find

$$\int_{\mathbb{R}^N}[|\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon\nabla\phi+V(\varepsilon x)v_\varepsilon^{p-1}\phi]\,\mathrm{d}x\leqslant\int_{\mathbb{R}^N}\left[\frac{\alpha}{2}v_\varepsilon^{p-1}+C(\alpha)v_\varepsilon^{p^*-1}\right]\phi\,\mathrm{d}x,$$

which implies that

$$\int_{\mathbb{R}^N}|\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon\nabla\phi\,\mathrm{d}x\leqslant C(\alpha)\int_{\mathbb{R}^N}v_\varepsilon^{p^*-1}\phi\,\mathrm{d}x. \tag{3.16}$$

From (3.16), it is easy to see that

$$\int_{\mathbb{R}^N} \psi^p [T_k(v_\varepsilon)]^{s_n} |\nabla v_\varepsilon|^p \, dx + s_n \int_{\mathbb{R}^N} \psi^p v_\varepsilon [T_k(v_\varepsilon)]^{s_n - 1} |\nabla v_\varepsilon|^{p-2} \nabla v_\varepsilon \nabla [T_k(v_\varepsilon)] \, dx$$
$$\leqslant - p \int_{\mathbb{R}^N} \psi^{p-1} v_\varepsilon [T_k(v_\varepsilon)]^{s_n} |\nabla v_\varepsilon|^{p-2} \nabla v_\varepsilon \nabla \psi + \int_{\mathbb{R}^N} v_\varepsilon^{p^*} \psi^p [T_k(v_\varepsilon)]^{s_n} \, dx. \tag{3.17}$$

By Young's inequality it follows that

$$\left| \int_{\mathbb{R}^N} \psi^{p-1} v_\varepsilon [T_k(v_\varepsilon)]^{s_n} |\nabla v_\varepsilon|^{p-2} \nabla v_\varepsilon \nabla \psi \, dx \right|$$
$$\leqslant \frac{(p-1)\delta^{\frac{p}{p-1}}}{p} \int_{\mathbb{R}^N} \psi^p [T_k(v_\varepsilon)]^{s_n} |\nabla v_\varepsilon|^p \, dx + \frac{1}{p\delta^p} \int_{\mathbb{R}^N} v_\varepsilon^p [T_k(v_\varepsilon)]^{s_n} |\nabla \psi|^p \, dx. \tag{3.18}$$

Using Gagliardo–Nirenberg–Sobolev inequality, we obtain

$$|\psi v_\varepsilon [T_k(v_\varepsilon)]^{\frac{s_n}{p}}|^p_{L^{p^*}} \leqslant C \left\{ \int_{\mathbb{R}^N} |\nabla \psi|^p v_\varepsilon^p [T_k(v_\varepsilon)] u^{s_n} \, dx + \int_{\mathbb{R}^N} \psi^p [T_k(v_\varepsilon)]^{s_n} |\nabla v_\varepsilon|^p \, dx \right.$$
$$\left. + \left(\frac{s_n}{p}\right)^p \int_{\mathbb{R}^N} \psi^p v_\varepsilon^p [T_k(v_\varepsilon)]^{s_n - p} |\nabla [T_k(v_\varepsilon)]|^p \, dx \right\},$$

where $C = C(N, p, \alpha)$. This estimate together with (3.17) and (3.18) implies

$$|\psi v_\varepsilon [T_k(v_\varepsilon)]^{\frac{s_n}{p}}|^p_{L^{p^*}} \leqslant C \gamma^{p(n-1)} \left\{ \int_{\mathbb{R}^N} |\nabla \psi|^p v_\varepsilon^p [T_k(v_\varepsilon)]^{s_n} \, dx \right.$$
$$\left. + \int_{\mathbb{R}^N} v_\varepsilon^{p^*} \psi^p [T_k(v_\varepsilon)]^{s_n} \, dx \right\}.$$

Now, in order to prove that $u \in L^{\sigma_{n+1}}(|x| \geqslant \rho)$ for some large $\rho > 0$, we consider the function $\psi \in C_0^\infty(\mathbb{R}^N, [0, 1])$ such that $\psi \equiv 1$ if $|x| \geqslant \rho > 4$, $\psi \equiv 0$ if $|x| \leqslant \rho - 2$ and $|\nabla \psi| \leqslant 1$. Hence, by Holder's inequality,

$$\int_{\mathbb{R}^N} v_\varepsilon^{p^*} \psi^p [T_k(v_\varepsilon)]^{s_n} \, dx \leqslant |\psi v_\varepsilon [T_k(v_\varepsilon)]^{\frac{s_n}{p}}|^p_{L^{p^*}} |v_\varepsilon|^{p^*-p}_{L^{p^*}(|x| \geqslant \rho/2)}.$$

Thus

$$|\psi v_\varepsilon [T_k(v_\varepsilon)]^{\frac{s_n}{p}}|^p_{L^{p^*}} \leqslant C \gamma^{p(n-1)} \{ ||\nabla \psi| v_\varepsilon [T_k(v_\varepsilon)]^{\frac{s_n}{p}}|^p_{L^p}$$
$$+ |\psi v_\varepsilon [T_k(v_\varepsilon)]^{\frac{s_n}{p}}|^p_{L^{p^*}} |v_\varepsilon|^{p^*-p}_{L^{p^*}(|x| \geqslant \rho/2)} \}.$$

Since $|v_\varepsilon|_{L^{p^*}} \leqslant C$, for all $\varepsilon \in (0, \varepsilon_0)$, we can take ρ suitably large such that $[2C\gamma^{p(n-1)}]$ $|v_\varepsilon|^{p^*-p}_{L^{p^*}(|x|\geqslant \rho/2)} \leqslant 1$, for all $\varepsilon \in (0, \varepsilon_0)$. Thus, we get the estimate

$$\begin{aligned}|\psi v_\varepsilon [T_k(v_\varepsilon)]^{\frac{s_n}{p}}|^p_{L^{p^*}(|x|\geqslant\rho)} &\leqslant |\psi v_\varepsilon [T_k(v_\varepsilon)]^{\frac{s_n}{p}}|^p_{L^{p^*}} \\ &\leqslant C\gamma^{p(n-1)} \int_{\mathbb{R}^N} |\nabla\psi|^p v_\varepsilon^p [T_k(v_\varepsilon)]^{s_n}\,\mathrm{d}x \\ &\leqslant C\gamma^{p(n-1)} \int_{|x|\geqslant\rho/2} v_\varepsilon^{\sigma_n}\,\mathrm{d}x\end{aligned}$$

for all $\varepsilon \in (0, \varepsilon_0)$, where $C = C(N, p, \alpha, \rho)$. Therefore, letting $k \to +\infty$, by the dominated convergence theorem,

$$|v_\varepsilon|_{L^{\sigma_{n+1}}(|x|\geqslant\rho)} \leqslant C^{\frac{1}{\sigma_n}} \gamma^{\frac{n-1}{\sigma_n}} |v_\varepsilon|_{L^{\sigma_n}(|x|\geqslant\rho/2)} \quad \forall\varepsilon \in (0, \varepsilon_0). \tag{3.19}$$

We can use the same argument taking $\psi \in C_0^\infty(\mathbb{R}^N, [0, 1])$ with $\psi \equiv 1$ if $|x_0 - x| \leqslant \rho'$, $\psi \equiv 0$ if $|x_0 - x| \geqslant 2\rho'$ and $|\nabla\psi| \leqslant 2/\rho'$, to prove that

$$|v_\varepsilon|_{L^{\sigma_{n+1}}(|x|\leqslant\rho')} \leqslant C^{\frac{1}{\sigma_n}} \gamma^{\frac{n-1}{\sigma_n}} |v_\varepsilon|_{L^{\sigma_n}(|x|\leqslant 2\rho')} \quad \forall\varepsilon \in (0, \varepsilon_0), \tag{3.20}$$

where ρ' is a suitable small positive constant independent of x_0 and $C = C(N, p, \alpha, \rho')$. Therefore, from (3.19) and (3.20), by a standard covering argument we can show that

$$|v_\varepsilon|_{L^{\sigma_{n+1}}} \leqslant C^{\frac{1}{\sigma_n}} \gamma^{\frac{n-1}{\sigma_n}} |v_\varepsilon|_{L^{\sigma_n}} \quad \forall\varepsilon \in (0, \varepsilon_0).$$

Interaction yields

$$|v_\varepsilon|_{L^{\sigma_{n+1}}} \leqslant C^{\sum \frac{1}{\sigma_n}} \gamma^{\sum \frac{n-1}{\sigma_n}} |v_\varepsilon|_{L^{\sigma_1}} \quad \forall\varepsilon \in (0, \varepsilon_0),$$

where C is independent of n, since both series are convergent. Finally, letting $n \to \infty$, and observing that $|u|_\infty \leqslant \lim_{n\to\infty} |u|_{L^{\sigma_n}}$ we deduce easily that $v_\varepsilon \in L^\infty(\mathbb{R}^N)$ and, besides,

$$|v_\varepsilon|_\infty \leqslant C \quad \text{for all } 0 < \varepsilon < \varepsilon_0.$$

Since $v_\varepsilon \in L^{\sigma_1}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, using the interpolation inequality, we prove that $v_\varepsilon \in L^\sigma(\mathbb{R}^N)$ for all $\sigma \in [\sigma_1, \infty]$.

By a similar argument used to prove Theorem 1 in [23] (see also [13, Theorem 8.17]), for any open ball $B_r(x)$ of radius r centered at any $x \in \mathbb{R}^N$ and some constant $C(N, \sigma_2)$, the nonnegative function $u \in W^{1,p}(\mathbb{R}^N)$ such that

$$-\Delta_p u \leqslant h(x)$$

in the weak sense, satisfies the estimate

$$\sup_{B_r(x)} u(y) \leqslant C\{|u|_{L^p(B_{2r}(x))} + |h|_{L^{\sigma_2}(B_{2r}(x))}\}.$$

Thus, for the family $\{v_\varepsilon\}$ we have

$$\sup_{B_r(x)} v_\varepsilon \leqslant C\{|v_\varepsilon|_{L^p(B_{2r}(x))} + |v_\varepsilon^{p^*-1}|_{L^{\sigma_2}(B_{2r}(x))}\} \quad \forall\varepsilon \in (0, \varepsilon_0).$$

By the preceding results we know that $v_\varepsilon^{p^*-1} \in L^{\sigma_2}(\mathbb{R}^N)$ and, moreover, $|v_\varepsilon^{p^*-1}|_{L^{\sigma_2}(\mathbb{R}^N)} \leqslant C$ where C is independent of ε. Therefore, the uniform vanishing property of the family $\{v_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ is implied. □

The next regularity result is a direct consequence of the previous proposition and a result due to Tolksdorf (cf. [26]).

Corollary 3.7. *The functions v_ε belongs to $C^{1,\alpha}_{\mathrm{loc}}(B_r)$, where $\alpha = \alpha(r) \in (0, 1)$.*

Finally, since the solutions v_ε decay uniformly to zero as $|x| \to +\infty$, we can take $\rho > 0$ such that $\omega_\varepsilon(x) = v_\varepsilon(x + y_\varepsilon) \leqslant a$ for all $|x| \geqslant \rho$ and for all $\varepsilon \in (0, \varepsilon_0)$. On the other hand, taking ε_0 to be suitably small we see that $B_\rho(0) \subset \Omega_\varepsilon$. Therefore, in both cases we see that $g(\varepsilon x + \varepsilon y, \omega_\varepsilon) = f(\omega_\varepsilon) + \omega_\varepsilon^{p^*-1}$ in $\mathbb{R}^N$ for all $\varepsilon \in (0, \varepsilon_0)$. Therefore the existence of a positive bounded state solution of problem (P_ε) for all $\varepsilon \in (0, \varepsilon_0)$ is proved.

When $1 < p \leqslant 2$, elliptic regularity theory implies that ω_ε belongs to class C^2 and ω_ε converges in C^2 to ω_0. Using Lemma 3.6, we have that ω_ε possesses a global maximum point x_ε and after translation we may assume that $\omega_\varepsilon(0) = \max_{|x|\leqslant R} \omega_\varepsilon = \max_{\mathbb{R}^N} \omega_\varepsilon$, for some $R > 0$. Now, using that ω_0 is radially symmetric and a similar result to the Lemma 4.2 in [19], we can prove that for ε sufficiently small, ω_ε possesses no critical points other than the origin.

Finally, we are going to prove the exponential decay.

Lemma 3.8. *The family $\{\omega_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ satisfies*

$$\omega_\varepsilon(x) \leqslant C \exp(-\beta|x|) \quad \forall x \in \mathbb{R}^N,$$

where C and β are positive constants independents of ε.

Proof. Using assumption (f_1) and the fact that the solutions ω_ε decay uniformly to zero as $|x| \to +\infty$, we can take $\rho_0 > 0$ such that

$$2(f(\omega_\varepsilon(x))\omega_\varepsilon^{1-p} + \omega_\varepsilon(x)^{p^*-p}) \leqslant V_0 = \inf_{\Omega} V(x) \quad \text{for all } |x| \geqslant \rho_0.$$

Consequently,

$$-\Delta_p \omega_\varepsilon + \frac{V_0}{2}\,\omega_\varepsilon^{p-1} \leqslant f(\omega_\varepsilon(x)) + \omega_\varepsilon(x)^{p^*-1} - \frac{V_0}{2}\,\omega_\varepsilon^{p-1} \leqslant 0 \quad \text{for all } |x| \geqslant \rho_0.$$

Let α and M be positive constants such that $(p-1)\alpha^p < V_0/2$ and $\omega_\varepsilon(x) \leqslant M \exp(-\alpha\rho_0)$ for all $|x| = \rho_0$. Hence, the function $\psi(x) = M \exp(-\alpha|x|)$ satisfies

$$-\Delta_p \psi + \frac{V_0}{2}\psi^{p-1} \geqslant \left(\frac{V_0}{2} - (p-1)\alpha^p\right)\psi^{p-1} > 0 \quad \text{for all } x \neq 0.$$

Since $p > 1$, we have that the function $\zeta : \mathbb{R}^N \to \mathbb{R}$, $\zeta(x) = |x|^p$ is convex, thus

$$(|x|^{p-2}x - |y|^{p-2}y)(x - y) \geqslant 0 \quad \text{for all } x, y \in \mathbb{R}^N.$$

We now take as a test function $\eta = \max\{\omega_\varepsilon - \psi, 0\} \in W_0^{1,p}(|x| > \rho_0)$. Hence, combining these estimates,

$$0 \geqslant \int_{\mathbb{R}^N} \left[(|\nabla\omega_\varepsilon|^{p-2}\nabla\omega_\varepsilon - |\nabla\psi|^{p-2}\nabla\psi)\eta + \frac{V_0}{2}(\omega_\varepsilon^{p-1} - \psi^{p-1})\eta \right] \mathrm{d}x$$
$$\geqslant \frac{V_0}{2}\int_{\{x\in\mathbb{R}^N:\omega_\varepsilon \geqslant \psi\}} (\omega_\varepsilon^{p-1} - \psi^{p-1})(\omega_\varepsilon - \psi)\,\mathrm{d}x \geqslant 0 \quad \text{for all } |x| \geqslant \rho_0.$$

Therefore, the set $\{x \in \mathbb{R}^N : |x| \geqslant \rho_0$ and $\omega_\varepsilon(x) \geqslant \psi(x)\}$ is empty. From this we can easily conclude the proof. □

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