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Three positive radial solutions for elliptic equations in a ball

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Abstract

In this work we deal with a class of second-order elliptic problems of the form $-\Delta u = \lambda k(|x|) f(u)$ in Ω, with non-homogeneous boundary condition $u = a$ on $\partial\Omega$ where Ω is the ball of radius R_0 centered at origin, λ, a are positive parameters, $f \in C([0, +\infty), [0, +\infty))$ is an increasing function and $k \in C([0, R_0], [0, +\infty))$ is not identically zero on any subinterval of $[0, R_0]$. We obtain via a fixed point theorem of cone expansion/compression type the existence of at least three positive radial solutions.

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1. Introduction

The objective of this work is to establish the existence and multiplicity of positive radial solutions for elliptic problems of the form

$$-\Delta u = \lambda k(|x|) f(u), \quad u > 0 \text{ in } \Omega \text{ and } u = a \text{ on } \partial\Omega, \tag{1.1}$$

where Ω is the ball of radius R_0 centered at origin, λ, a are positive parameters, $f \in C([0, +\infty), [0, +\infty))$ is a increasing function and $k \in C([0, R_0], [0, +\infty))$ is not identically zero in any subinterval of $[0, R_0]$.

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The following hypotheses will be assumed through this work.

(H_0) $f(t) > 0$, for all $t > 0$.

(H_1) $\lim_{u\to 0} \frac{f(u)}{u} = 0$.

(H_2) $\lim_{u\to +\infty} \frac{f(u)}{u} = 0$.

(H_3) There exists a function $\varphi = \varphi_a \in C([0, +\infty), [0, +\infty))$ verifying

$$\int_1^{+\infty} \varphi(\tau)\tau^{-\frac{N}{N-2}} \mathrm{d}\tau < +\infty$$

and positive constants $\bar{\alpha}(a)$, $\bar{\tau}(a)$ and $M(a)$ such that for all $\tau > \bar{\tau}$, we have

$$\frac{f(\alpha\tau + a)}{f(\alpha + a)} \le M\varphi(\tau), \quad \text{for all} \quad \alpha \ge \bar{\alpha}. \tag{1.2}$$

Theorem 1.1. *Assume that the nonlinear function $f(u)$ satisfies (H_0)–(H_3). Then Problem (1.1) has at least one solution for all $a, \lambda > 0$.*

Theorem 1.2. *Assume that the function $f(u)$ satisfies (H_0)–(H_3). Then, there exists $\tilde{\delta} > 0$ such that for all $a \in (0, \tilde{\delta})$ there exists a $\tilde{\lambda}(a) > 0$ such that for all $\lambda > \tilde{\lambda}(a)$, Problem (1.1) has at least three solutions.*

Applications. Note that the hypotheses of our main results are satisfied by nonlinear functions of the form

(a) $f(u) = u^{p_1}/(1 + u^{q_1})$ with $1 < p_1$ and $0 < p_1 - q_1 < \min\{1, 2/(N-2)\}$;

(b) $f(u) = (u^{q_2} + 1)\varphi(u^{p_2})$ with $1 < p_2$, $0 < q_2 < \min\{1, 2/(N-2)\}$ and $\varphi \in C([0, +\infty), [0, +\infty))$ is such that $\lim_{u\to 0} \varphi(u)/u \ge 0$ and $\lim_{u\to +\infty} \varphi(u) > 0$.

The study of Problem (1.1) was in part motivated by several recent results for elliptic problems on annular domains with nonhomogeneous boundary conditions. Among others we mention [1–6], with references therein. In some of these articles existence and multiplicity in annular domains were established using fixed point theorems of expansion/compression type. In their arguments the following elementary property of nonnegative concave functions in $C([0, 1], \mathbb{R})$ was crucial: for all $\alpha, \beta \in (0, 1)$ we have

$$\inf_{t\in[\alpha,\beta]} u(t) \ge \alpha(1 - \beta)\|u\|_\infty. \tag{1.3}$$

Our approach is also based on a fixed point theorem of cone expansion/compression type. There are, however, some substantial differences between the problems in annular domains and balls. Our strategy consists of, by a suitable change of variables, transforming our problem into an ODE problem such that the fixed point operator associated preserves the cone of concave and nonnegative functions. To this end it is crucial to find an alternative property to replace (1.3) which in our case is more delicate (see Lemma 2.3 below).

The rest of this work is organized as follows. Section 2 contains preliminary results and Section 3 is devoted to proving our main results, Theorems 1.1 and 1.2.

2. Preliminary results

We will establish the existence of positive radial solutions of the problem (1.1). In fact, we will obtain positive solutions $u = u(r)$ of the problem of ordinary equations

$$-(r^{N-1}u')' = r^{N-1}\lambda k(r) f(u+a) \text{ in } (0, R_0) \text{ and } u(R_o) = u'(0) = 0. \tag{2.1}$$

Consider $a : (0, R_0] \to [0, +\infty)$; $a(r) = (r^{2-N} - R_0^{(2-N)})/(N-2)$. Performing the change of variable $t = a(r)$, $z(t) = u(r(t))$, we see that (2.1) can be rewritten as

$$-z''(t) = \lambda r^{2(N-1)}(t)h(t) f(z(t)+a) \text{ in } (0, +\infty) \text{ and } z(0) = z'(+\infty) = 0, \tag{2.2}$$

where $h(t) = k(a^{-1}(t))$ is a continuous function which does not vanish identically on any subinterval of $(0, +\infty)$. Finally, integrating the equations of (2.2) twice and using the boundary conditions, (2.2) can be brought into the form of the following integral equation:

$$z(t) = \lambda \int_0^t \int_s^{+\infty} G(\tau)h(\tau) f(z(\tau)+a)\mathrm{d}\tau\mathrm{d}s \tag{2.3}$$

where $G(\tau) = (R_0^{2-N} + (N-2)\tau)^{2(1-N)/(N-2)}$. Consequently, we may solve (2.1) using fixed point techniques. For this, we make use of the following well known fixed point theorem (see [7,8]).

Lemma 2.1. *Let X be a Banach space with norm $|\cdot|$, and let $\mathcal{C} \subset X$ be a cone in X. For $R > 0$, define $\mathcal{C}_R = \mathcal{C} \cap B[0, R]$, where $B[0, R] = \{x \in X : |x| \le R\}$ denotes the closed ball of radius R centered at the origin of X, which is a completely continuous map for which there exists $0 < r < R$ such that*

$$|Fx| < |x|, x \in \partial\mathcal{C}_r \text{ and } |Fx| > |x|, x \in \partial\mathcal{C}_R, \text{ or}$$
$$|Fx| > |x|, x \in \partial\mathcal{C}_r \text{ and } |Fx| < |x|, x \in \partial\mathcal{C}_R,$$

where $\partial\mathcal{C}_R = \{x \in \mathcal{C} : |x| = R\}$. Then F has a fixed point $u \in \mathcal{C}$ with $r < |u| < R$.

X will denote the space of continuous and bounded functions $z : [0, +\infty) \to \mathbb{R}$ endowed with the norm $|z|_\infty = \sup\{|z(t)| : t \in [0, +\infty)\}$.

$\mathcal{C}_1$ will denote the cone of nonnegative and concave functions of X such that $z(0) = 0$. Notice that the elements of $\mathcal{C}_1$ are increasing functions.

In $\mathcal{C}_1$ we consider the operator $F : \mathcal{C}_1 \to X$ given by

$$F(z)(t) := \lambda \int_0^t \int_s^{+\infty} G(\tau)h(\tau) f(z(\tau)+a)\mathrm{d}\tau\mathrm{d}s.$$

Lemma 2.2. *We have that F is well defined, $F(\mathcal{C}_1) \subset \mathcal{C}_1$, and F is a completely continuous operator.*

Proof. Notice that for all $s \ge 0$,

$$\int_s^{+\infty} G(\tau)\mathrm{d}\tau = \frac{1}{N} G(s)^{N/2(N-1)} \qquad \text{and} \qquad \int_0^{+\infty} \left(\int_s^{+\infty} G(\tau)\mathrm{d}\tau \right) \mathrm{d}s < +\infty.$$

Hence, F is well defined.

Also, notice that the function $F(z)(t)$ belongs to class C^2 and its derivatives are given by

$$\frac{\mathrm{d}}{\mathrm{d}t} F(z(t)) = \int_t^{+\infty} G(\tau) f(z(\tau)+a)\mathrm{d}\tau \qquad \text{and} \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2} F(z)(t) = -G(t) f(z(t)+a).$$

Thus $F(z)(t)$ is increasing and concave. Therefore, $F(\mathcal{C}_1) \subset \mathcal{C}_1$.

It remains to prove that F is a completely continuous operator. Let $(z_n) \in \mathcal{C}_1$ such that $|z_n|_\infty \leq c_0$ and $M_1 = \max\{f(t+a) : t \in [0, c_0]\}$. Thus, it follows that

$$|F(z_n)(t)| \leq M_1 \int_0^{+\infty} \int_s^{+\infty} G(\tau)\mathrm{d}\tau\mathrm{d}s \qquad \text{and} \qquad \left|\frac{\mathrm{d}}{\mathrm{d}t} F(z_n)(t)\right| \leq M_1 \int_0^{+\infty} G(\tau)\mathrm{d}\tau.$$

By the Arzelá–Ascoli compactness criterion for uniform convergence, up to a subsequence, we can assume that $(F(z_n))$ is uniformly convergent on compact subsets of $[0, +\infty)$. To prove that there exists a uniformly convergent subsequence of $F(z_n)$ it suffices to recall that given $\epsilon > 0$, there is $T = T(\epsilon)$ such that

$$\int_T^{+\infty} \int_s^{+\infty} G(\tau)\mathrm{d}\tau\mathrm{d}s < \epsilon.$$

We now verify that F is continuous. Let (z_n) be a sequence in $\mathcal{C}_1$ such that $|z_n - z_0|_\infty \to 0$ as $n \to \infty$. Thus

$$|F(z_n)(t) - F(z_0)(t)| \leq \int_0^{+\infty} |\Gamma_n(s) - \Gamma_0(s)|\mathrm{d}s$$

where

$$\Gamma_n(s) = \int_s^{+\infty} G(\tau) f(z_n(\tau) + a)\mathrm{d}\tau \qquad \text{and} \qquad \Gamma_0(s) = \int_s^{+\infty} G(\tau) f(z_0(\tau) + a)\mathrm{d}\tau.$$

It follows from $|z_n - z_0|_\infty \to 0$ that $\Gamma_n(s) \to \Gamma_0(s)$ and that $\Gamma_n(s) \leq C/NG(s)^{N/2(N-1)}$ for all $s \in [0, +\infty)$, where C is a positive constant. By the Lebesgue dominated convergence theorem, $|F(z_n) - F(z_0)|_\infty \to 0$, which implies that F is continuous. $\square$

It is clear that given $z \in \mathcal{C}_1 \setminus \{0\}$, there exists a unique $\tau_1 = \tau_1(z)$ such that

$$2z(\tau_1) = |z|_\infty.$$

Define

$$\tau^* := \sup\{\tau_1(F(z)) : z \in \mathcal{C}_1\}$$

and

$$\mathcal{C} := \{z \in \mathcal{C}_1 \ : \ 2z(t) \geq |z|_\infty, \forall t \geq \tau^*\}.$$

Lemma 2.3. *τ^* is a positive real number and $\mathcal{C}$ is a cone invariant under F.*

Proof. Firstly we show that τ^* is a positive real number. Suppose to the contrary that $\tau^* = +\infty$. Then there must exist a sequence $z_n \subset \mathcal{C}_1 \setminus \{0\}$ such that $\tau_n = \tau_1(F(z_n))$ is a strictly increasing sequence of positive real numbers converging to $+\infty$. By definition of τ_n we have

$$\int_0^{\tau_n} \int_s^{+\infty} H_n(\tau)\mathrm{d}\tau\mathrm{d}s = \int_{\tau_n}^{+\infty} \int_s^{+\infty} H_n(\tau)\mathrm{d}\tau\mathrm{d}s \tag{2.4}$$

where $H_n(\tau) = G(\tau)h(\tau)f(z_n(\tau) + a)$. According to (2.4) and using integration by parts we have

$$2\tau_n \int_{\tau_n}^{+\infty} H_n(\tau)\mathrm{d}\tau + \int_0^{\tau_n} \tau H_n(\tau)\mathrm{d}\tau = \int_{\tau_n}^{+\infty} \tau H_n(\tau)\mathrm{d}\tau. \tag{2.5}$$

Since z_n is concave it follows from (2.5) that

$$\int_0^{\tau_n} \tau U_n(\tau)\mathrm{d}\tau \leqslant \int_{\tau_n}^{+\infty} \tau U_n(\tau)\mathrm{d}\tau \tag{2.6}$$

where $U_n(\tau) = G(\tau)h(\tau)f(\alpha_n\tau + a)$ with $\alpha_n = z_n(\tau_n)/\tau_n$. Now, since $f(\alpha_n\tau + a) \geq f(\alpha_n + a)$, for all $\tau \geq 1$, by using (2.6) we obtain that

$$\int_1^{\tau_n} \tau U_n(\tau)\mathrm{d}\tau \leqslant \int_{\tau_n}^{+\infty} \tau U_n(\tau)\mathrm{d}\tau.$$

Hence,

$$\int_1^{\tau_n} \tau h(\tau)G(\tau)\mathrm{d}\tau \leq \int_{\tau_n}^{+\infty} \tau h(\tau)G(\tau)\frac{f(\alpha_n\tau + a)}{f(\alpha_n + a)}\mathrm{d}\tau. \tag{2.7}$$

Next we consider two cases.

Case 1. There exits a subsequence (α_{n_k}) of (α_n) such that $\alpha_{n_k} < \overline{\alpha}$ for all k. In this case, from (2.7) and (H_3), we get

$$\begin{aligned}\int_1^{\tau_{n_k}} \tau h(\tau)G(\tau)\mathrm{d}\tau &\leq \int_{\tau_{n_k}}^{+\infty} \tau h(\tau)G(\tau)\frac{f(\overline{\alpha}\tau + a)}{f(\alpha_{n_k} + a)}\mathrm{d}\tau\\ &= \int_{\tau_{n_k}}^{+\infty} \tau h(\tau)G(\tau)\frac{f(\overline{\alpha}\tau + a)}{f(\overline{\alpha} + a)}\frac{f(\overline{\alpha} + a)}{f(\alpha_{n_k} + a)}\mathrm{d}\tau\\ &\leq C\int_{\tau_{n_k}}^{+\infty} \tau G(\tau)\varphi(\tau)\mathrm{d}\tau.\end{aligned}$$

Case 2. $\alpha_n \geq \overline{\alpha}$ for all n. In this case, from (2.7) and (H_3), we obtain

$$\int_1^{\tau_n} \tau h(\tau)G(\tau)\mathrm{d}\tau \leq C\int_{\tau_n}^{+\infty} \tau G(\tau)\varphi(\tau)\mathrm{d}\tau.$$

In both cases, since $G(\tau) = (R_0^{2-N} + (N-2)\tau)^{2(1-N)/(N-2)}$, it is easy to see that

$$\int_1^{\tau_n} \tau h(\tau)G(\tau)\mathrm{d}\tau \leq C\int_{\tau_n}^{+\infty} \tau^{-N/(N-2)}\varphi(\tau)\mathrm{d}\tau,$$

and from (H_3) we see that the right integral of the inequality above converges to zero when n goes to infinity. But this is impossible, since $\int_1^{+\infty} \tau h(\tau)G(\tau)\mathrm{d}\tau > 0$.

Finally, it is clear that $\mathcal{C}$ is a invariant cone under F. □

Lemma 2.4. *Assume (H_2). Given $a > 0$ and $\lambda > 0$, there exists $R_1 = R_1(a, \lambda)$ sufficiently large that*

$$|F(z)|_\infty < |z|_\infty, \text{ for each } z \in \partial\mathcal{C}_{R_1}. \tag{2.8}$$

Proof. It follows from condition (H_2) that given $\varepsilon > 0$ there exists $R > a$ such that

$$f(s) \leq \varepsilon s, \text{ for each } s \geq R.$$

Thus, for each $z \in \partial\mathcal{C}_R$, we have

$$f(z(\tau) + a) \leq f(R + a) \leq f(2R) \leq 2\varepsilon R$$

which implies that

$$|F(z)|_\infty = \max_{t\geq 0} \int_0^t \int_s^{+\infty} \lambda G(\tau)h(\tau) f(z(\tau)+a)\mathrm{d}\tau \mathrm{d}s$$
$$\leq 2\varepsilon R \int_0^{+\infty} \int_s^{+\infty} \lambda G(\tau)h(\tau)\mathrm{d}\tau \mathrm{d}s.$$

Finally, choosing $\varepsilon > 0$ sufficiently small we prove that (2.8) holds. □

Lemma 2.5. *Assume condition* (H_0). *Given* $R, a, \lambda > 0$ *there is* $R_2 = R_2(a, \lambda) \in (0, R)$ *small enough that*

$$|F(z)|_\infty > |z|_\infty, \text{ for each } z \in \partial\mathcal{C}_{R_2}. \tag{2.9}$$

Proof. Since $f(a) > 0$, given $M > 0$ there exists $r_1 \in (0, R)$ small enough that

$$f(s+a) \geq Ms, \text{ for all } s \in [0, r_1].$$

Thus, for each $z \in \partial\mathcal{C}_{R_2}$,

$$|F(z)|_\infty \geq F(z)(\tau^*) = \int_0^{\tau^*} \int_s^{+\infty} \lambda G(\tau)h(\tau) f(z(\tau)+a)\mathrm{d}\tau \mathrm{d}s$$
$$\geq \int_0^{\tau^*} \int_{\tau^*}^{+\infty} M\lambda G(\tau)h(\tau)z(\tau)\mathrm{d}\tau \mathrm{d}s$$
$$\geq |z|_\infty \frac{\tau^* M}{2} \int_{\tau^*}^{+\infty} \lambda G(\tau)h(\tau)\mathrm{d}\tau.$$

Choosing $M > 0$ sufficiently large that

$$\tau^* M \int_{\tau^*}^{+\infty} \lambda G(\tau)h(\tau)\mathrm{d}\tau > 2,$$

we prove that (2.9) holds. □

Lemma 2.6. *Assume condition* (H_0). *Then given* $R_3 > 0$ *there is a constant* $\bar{\lambda} = \bar{\lambda}(a) > 0$ *such that for all* $\lambda > \bar{\lambda}$ *we have*

$$|F(z)|_\infty > |z|_\infty, \text{ for each } z \in \partial\mathcal{C}_{R_3}. \tag{2.10}$$

Proof. For each $z \in \partial\mathcal{C}_{R_3}$, we have

$$|F(z)|_\infty \geq F(z)(\tau^*) \geq \tau^* f(z(\tau^*)) \int_{\tau^*}^{+\infty} \lambda G(\tau)h(\tau)\mathrm{d}\tau$$
$$= \lambda\tau^* f\left(\frac{R_3}{2}\right) \int_{\tau^*}^{+\infty} G(\tau)h(\tau)\mathrm{d}\tau.$$

Thus, taking $\bar{\lambda}$ such that

$$\bar{\lambda}\tau^* f\left(\frac{R_3}{2}\right) \int_{\tau^*}^{+\infty} G(\tau)h(\tau)\mathrm{d}\tau = R_3$$

we complete the proof of Lemma 2.6. □

3. Proof of the main results

Proof of Theorem 1.1. It follows from Lemmas 2.1, 2.4 and 2.5 that the operator F has a fixed point $z \in \mathcal{C}$ such that $R_2 < |z|_\infty < R_1$.

Proof of Theorem 1.2. Let R_3 and $\bar{\lambda}$ be like in Lemma 2.6. Given $\lambda > \bar{\lambda}$ take $\eta = \eta(\lambda) > 0$ such that

$$\eta \int_0^{+\infty} \int_s^{+\infty} \lambda G(\tau) h(\tau) \mathrm{d}\tau \mathrm{d}s < 1.$$

Using assumption (H_1) we can choose $R_4 \in (0, R_3)$ such that

$$f(t) \leq \eta t, \forall t \in (0, R_4).$$

Thus, for all $a \in (0, R_4/2)$ and $z \in \partial \mathcal{C}_{R_4/2}$ we have

$$f(z(\tau) + a) \leq f\left(\frac{R_4}{2} + a\right) \leq \eta \left(\frac{R_4}{2} + a\right) \leq \eta R_4.$$

Therefore,

$$\begin{aligned} |F(z)|_\infty &\leq \eta R_4 \int_0^{+\infty} \int_s^{+\infty} \lambda G(\tau) h(\tau) \mathrm{d}\tau \mathrm{d}s \\ &< R_4. \end{aligned}$$

Finally, by using a combination of Lemmas 2.4 and 2.5, and by choosing R_1 and R_2 such that $0 < R_2 < R_4 < R_3 < R_1$ we have that there exist three fixed points z_1, z_2 and z_3 of operator F in $\mathcal{C}$ such that

$$R_2 < |z_1|_\infty < R_4 < |z_2|_\infty < R_3 < |z_3|_\infty < R_1.$$

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