Multiplicity of positive solutions for a class of quasilinear nonhomogeneous Neumann problems*

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Abstract

In this paper we study the existence, nonexistence and multiplicity of positive solutions for nonhomogeneous Neumann boundary value problem of the type

$$\begin{cases} -\Delta_p u + \lambda u^{p-1} = u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \eta} = \varphi & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $I\!R^n$ with smooth boundary, $1 < p < n$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $p-1 < q \leq p^*-1$, $p^* = np/(n-p)$, $\varphi \in C^\alpha(\overline{\Omega})$, $0 < \alpha < 1$, $\varphi \not\equiv 0$, $\varphi(x) \geq 0$ and λ is a real constant. The proofs of our main results rely on different methods: lower and upper solutions and variational approach.

Keywords and phrases: Nonlinear elliptic problems, Neumann boundary value problems, positive solutions, lower and upper solutions, variational methods, p-Laplacian, critical exponent.

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1 Introduction

In this paper we deal with quasilinear elliptic problems of the form

$$\begin{cases} -\Delta_p u + \lambda u^{p-1} = u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \eta} = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1_λ}$$

where Ω is a bounded domain in $I\!R^n$ with smooth boundary, $\varphi \in C^\alpha(\overline{\Omega}),\ 0 < \alpha < 1,\ \varphi \not\equiv 0,\ \varphi(x) \geq 0,\ 1 < p < n,\ \Delta_p u = \mathrm{div}(\mid \nabla u \mid^{p-2} \nabla u)$ is the p-Laplacian operator, $p-1 < q \leq p^* - 1$, $p^* = np/(n-p)$ is the critical exponent for the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ and λ is a real constant.

When $p = 2$, (1_λ) becomes the second order semilinear elliptic problem

$$\begin{cases} -\Delta u + \lambda u = u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \eta} = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

with $1 < q \leq 2^* - 1 = (n-2)/(n+2)$.

The study of semilinear elliptic problems involving critical growth and Neumann boundary conditions has received considerable attention in recent years. First we would like to mention the progress for problem involving homogeneous boundary condition, which correspond to $\varphi \equiv 0$ in (1.1). They have been studied for instance by [1, 2, 4, 7], among others. Problem (1.1) with nonhomogeneous Neumann boundary conditions which correspond to $\varphi \not\equiv 0$ has been investigated by Deng-Peng [5]. In the present paper we will improved the main results in [5]. We prove that there exists $\lambda^* > 0$ such that problem (1_λ) has at least two positive solutions if $\lambda > \lambda^*$, has at least one positive solution if $\lambda = \lambda^*$ and has no positive solution if $\lambda < \lambda^*$. The proofs of our main results rely on different methods: lower and upper solutions method and variational approach.

The special features of this class of problems, considered in this paper, is that involve critical growth and a nonlinear operator. The arguments used in [5] to prove the existence of the second solutions can not be carried out for a quasilinear problem as (1_λ). Moreover, because we are dealing with $p-$Laplacian equations, it is technically much involved than in [5], in our case some estimates involving the minimax level become more subtle to be established.

Next we describe in a more precise way our main results.

Theorem 1.1 *For each $q \in (p-1, p^*-1]$, there exists $\lambda^* > 0$ such that:*

(*i*) *problem* (1_λ) *possesses a minimal positive solution* u_λ *if* $\lambda \in [\lambda^*, \infty)$ *and there is no positive solution if* $\lambda < \lambda^*$.

(*ii*) u_λ *is decreasing with respect to* λ *if* $\lambda \in [\lambda^*, \infty)$.

(*iii*) u_λ *is bounded uniformly in* $W^{1,p}(\Omega)$ *and* $u_\lambda \to 0$ *as* $\lambda \to \infty$.

Theorem 1.2 *For each* $\lambda \in (\lambda^*, +\infty)$ *and* $q \in (p-1, p^*-1]$, *problem* (1_λ) *possesses at least two positive solutions* v_λ *and* w_λ.

The rest of this paper is organized as follows. The existence of minimal solution u_λ for (1_λ) are obtained in section 2. The main tool is a general method of lower- and upper-solutions described in section 2, similar to that given in [12]. Section 3 is devoted to proving Theorem 1.2.

The underling idea for proving Theorem 1.2 is first to show with the help of the minimal solution u_λ that there exists a solution v_λ which is a local minimum of the associated functional J_λ to problem (1_λ) in $W^{1,p}(\Omega)$. For proving the existence of the second solution we consider the perturbed functional $I_\lambda(u) := J_\lambda(u + v_\lambda)$. We prove that this functional has the mountain pass geometry and using the Ekeland variational principle we obtain a Palais-Smale sequence at this mountain pass level $c(v_\lambda)$ of I_λ. Finally, doing an argument similar in spirit to that used in the classical result due to Brezis-Nirenberg [3], we obtain a nontrivial critical point u of I_λ. Thus, $w_\lambda = u + v_\lambda$ is a second solution of problem (1_λ).

Notation. In this paper we make use of the following notation.
If $p \in (1, \infty)$, p' denotes the number $p/(p-1)$ so that $p' \in (1, \infty)$ and $1/p + 1/p' = 1$;
$L^p(\Omega)$, denotes Lebesgue spaces with the norm $\|.\|_{L^p(\Omega)}$;
$W^{1,p}(\Omega)$, denotes Sobolev spaces with the norm $\|.\|_{1,p}$;
$C^{k,\alpha}(\Omega)$, with k a non-negative integer and $0 \le \alpha < 1$, denotes Hölder spaces;
C, C_0, C_1, C_2, ..., denote (possibly different) positive constants;
$|A|$, denotes the Lebesgue measure of the set $A \subset I\!R^n$;
ω_{n-1} is the $(n-1)$-dimensional measure of the $n-1$ unit sphere in $I\!R^n$;
We denote by R^n_+ the half-space, that is, $R^n_+ := \{(x', x_n) \in I\!R^n : x_n \ge 0\}$;
$D^{1,p}_o(\Omega)$ is the completeness of $C^\infty_o(\Omega)$ with respect to the norm $\|u\| := \left(\int_\Omega |\nabla u|^p \, dx\right)$. We are denoting by S the best constant to the Sobolev emdedding $D^{1,p}_o(\Omega) \hookrightarrow L^{p^*}(\Omega)$, that is,

$$S = \inf_{D^{1,p}_o(\Omega)} \left\{ \int_\Omega |\nabla u|^p \, dx \ ; \ \int_\Omega |u|^{p^*} \, dx = 1 \right\}.$$

We remark also that, S is independent of Ω and depends only of n. Moreover, when $\Omega = I\!\!R^n$ this infimum S is achieved by the functions u_ϵ given by

$$u_\epsilon(x) = C_n \epsilon^{(n-p)/p^2}(\epsilon + |x|^{p/(p-1)})^{(p-n)/p},$$

where the constant C_n is chosen of form that

$$-\Delta_p u_\epsilon = u_\epsilon^{p^*-1} \quad \text{in } I\!\!R^n.$$

Thus,

$$S = \frac{K_1}{K_2^{(n-p)/n}}$$

with

$$K_1 := \int_{I\!\!R^n} |\nabla u_\epsilon|^p \, dx \ \text{ and } \ K_2 := \int_{I\!\!R^n} |u_\epsilon|^{p^*} \, dx. \tag{1.2}$$

2 Proof of Theorem 1.1

Our argument to prove the existence of the first solution to problem (1_λ) rely in the lower and upper solution methods. Our first solution is a minimal solution u_λ of problem (1_λ), in sense that $u_\lambda \leq w$, for all w solution of (1_λ). The main of our next subsection is to prove the existence of such minimal solution.

2.1 The existence of minimal solution

Let us first recall some definitions. We say that $u \in W^{1,p}(\Omega)$ is a **weak solution** of problem (1_λ) if for all $v \in W^{1,p}(\Omega)$ we have

$$\int_\Omega \left[\mid \nabla u \mid^{p-2} \nabla u \nabla v + \lambda \mid u \mid^{p-2} uv \right] dx = \int_\Omega |u|^{q-1} uv \, dx + \int_{\partial\Omega} \varphi v \, d\sigma_y. \tag{2.3}$$

Hence the weak solutions of (1_λ) correspond to nontrivial critical points of the energy functional

$$J_\lambda(u) = \frac{1}{p}\int_\Omega [|\nabla u|^p + \lambda |u|^p] \, dx - \frac{1}{q+1}\int_\Omega |u|^{q+1} \, dx - \int_{\partial\Omega} \varphi u \, d\sigma_y, \ u \in W^{1,p}(\Omega).$$

A function $\underline{u} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is said to be a **lower solution** of (1_λ) if

$$\int_\Omega \left[\mid \nabla \underline{u} \mid^{p-2} \nabla \underline{u} \nabla v + \lambda \mid \underline{u} \mid^{p-2} \underline{u} v \right] dx \leq \int_\Omega |\underline{u}|^{q-1} \underline{u} v \, dx + \int_{\partial\Omega} \varphi v \, d\sigma_y,$$

for all $v \in W^{1,p}(\Omega),\ v \geq 0$. In the same way, a function $\overline{u} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is said to be a **upper solution** of (1_λ) if

$$\int_\Omega \left[\mid \nabla \overline{u} \mid^{p-2} \nabla \overline{u} \nabla v + \lambda \mid \overline{u} \mid^{p-2} \overline{u} v \right] dx \geq \int_\Omega |\overline{u}|^{q-1} \overline{u} v \ dx + \int_{\partial\Omega} \varphi v \ d\sigma_y,$$

for $v \in W^{1,p}(\Omega),\ v \geq 0$.

Lemma 2.1 (Maximum Principle) *Let $u_1, u_2 \in W^{1,p}(\Omega)$ be nonnegative functions such that for all $v \in W^{1,p}(\Omega)$, $v \geq 0$ we have*

$$\int_\Omega \left[\mid \nabla u_1 \mid^{p-2} \nabla u_1 \nabla v + \lambda u_1^{p-1} v \right] dx \leq \int_\Omega \left[\mid \nabla u_2 \mid^{p-2} \nabla u_2 \nabla v + \lambda u_2^{p-1} v \right] dx, \quad (2.4)$$

then $u_1 \leq u_2$ almost everywhere in Ω.

Proof. Taking $v = (u_1 - u_2)^+ \in W^{1,p}(\Omega)$ in (2.4) we have

$$\begin{aligned} 0 \ &\geq \int_\Omega [|\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2].\nabla (u_1 - u_2)^+ \ dx \\ &+\lambda \int_\Omega (u_1^{p-1} - u_2^{p-1})(u_1 - u_2)^+ \ dx \\ &= \int_\Omega \frac{|\nabla u_1|^{p-2} + |\nabla u_2|^{p-2}}{2} |\nabla (u_1 - u_2)^+|^2 \ dx \\ &+ \int_{u_1 \geq u_2} \frac{|\nabla u_1|^{p-2} - |\nabla u_2|^{p-2}}{2} [|\nabla u_1|^2 - |\nabla u_2|^2] \ dx \\ &+\lambda \int_\Omega (u_1^{p-1} - u_2^{p-1})(u_1 - u_2)^+ \ dx. \end{aligned}$$

Observe that every summand in this last expression is nonnegative, and hence we obtain that $(u_1 - u_2)^+ = 0$ almost everywhere in Ω or, equivalently, $u_1 \leq u_2$ almost everywhere in Ω. ■

In order to prove the existence of first solution to problem (1_λ), we consider the following auxiliary problem:

$$\begin{cases} -\Delta_p w + \lambda w^{p-1} = f(x) & \text{in } \ \Omega, \\ |\nabla w|^{p-2} \frac{\partial w}{\partial \eta} = \varphi & \text{on } \ \partial\Omega, \end{cases} \qquad (2_\lambda)$$

Our next result concerns existence of solutions for problem (2_λ) and some properties of the associated solution operator.

Lemma 2.2 *If $\varphi \in C^{\alpha}(\overline{\Omega})$, $0 < \alpha < 1$, $\varphi \not\equiv 0$ and $\varphi \geq 0$, then for each nonnegative function $f \in L^{p'}(\Omega)$, the problem (2_λ) possesses a unique weak positive solution $w_\lambda \in C^{1,\alpha}(\bar{\Omega})$ for all $\lambda > 0$. Moreover, the associated operator $T_\lambda : L^{p'}(\Omega) \to W^{1,p}(\Omega)$, $f \mapsto w_\lambda$ is continuous and nondecreasing.*

Proof. First we use variational argument to prove the existence of solution. More precisely, we use minimization argument to the associated energy functional of the problem (2_λ),

$$I_\lambda(w) = \frac{1}{p}\int_\Omega [|\nabla w|^p + \lambda |w|^p]\ dx - \int_\Omega fw\ dx - \int_{\partial\Omega} \varphi w\ d\sigma_y,$$

defined on the reflexive Banach space $W^{1,p}(\Omega)$. Note that I_λ is coercive. Indeed,

$$\begin{aligned} I_\lambda(w) &\geq C_1\|w\|_{1,p}^p - \|f\|_{L^{p'}(\Omega)}\|w\|_{L^p(\Omega)} - \|\varphi\|_{L^{p'}(\partial\Omega)}\|w\|_{L^p(\partial\Omega)} \\ &\geq C_2\|w\|_{1,p}^p - C_3, \end{aligned}$$

where above we have used Holder inequality, Sobolev embedding and trace embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$.

Now, we proceed to prove that I_λ is sequentially weakly lower semi-continuous. To this end is sufficient to show that for $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$ we have

$$\int_\Omega fu_n\ dx \to \int_\Omega fu\ dx \tag{2.5}$$

and

$$\int_{\partial\Omega} \varphi u_n\ d\sigma_y \to \int_{\partial\Omega} \varphi u\ d\sigma_y. \tag{2.6}$$

Since $f \in L^{p'}(\Omega)$, (2.5) follows from the definition of weak convergence. Finally, (2.6) follows from the trace embedding.

Let u_i be a weak solutions of (2_λ) associated to $f_i \in L^{p'}(\Omega)$, that is

$$\int_\Omega \big[\ |\ \nabla u_i\ |^{p-2}\ \nabla u_i \nabla v + \lambda |u_i|^{p-1} u_i v\big]\ dx = \int_\Omega f_i v\ dx + \int_{\partial\Omega} \varphi v\ d\sigma_y$$

for all $v \in W^{1,p}(\Omega)$ and $i = 1, 2$.

If $f_1 \leq f_2$, using Lemma 2.1, we obtain that $u_1 \leq u_2$. From this we get the uniqueness and that T_λ is nondecreasing.

Using regularity result due to Lieberman [13] we may prove that $u \in C^{1,\alpha}(\overline{\Omega})$. Finally, by the maximum principle or Hanark's inequality it is standard to prove that $u > 0$ (see [14, 16]). This completes the proof of Lemma 2.2. ■

Proposition 2.3 *Let $\underline{u}, \overline{u} \in W^{1,p}(\Omega) \cap L^\infty(\overline{\Omega})$ be, respectively, a lower solution and a upper solution of problem (1_λ), with $0 \leq \underline{u}(x) \leq \overline{u}(x)$ almost everywhere in Ω. Then there exists a minimal (and, respectively, a maximal) weak solution u_* (resp. u^*) for problem (1_λ).*

Proof. Consider the interval $[\underline{u}, \overline{u}]$ with the topology of $W^{1,p}(\Omega)$ and the operator $S : [\underline{u}, \overline{u}] \to L^{p'}(\Omega)$ defined by $Sv := v^q$. Since $\overline{u} \in L^\infty(\Omega)$, we see that S is well defined. Moreover, for $u_n, u \in [\underline{u}, \overline{u}]$ with $u_n \to u$ in $W^{1,p}(\Omega)$, we have that $\|Su_n - S_u\|_{L^{p'}(\Omega)} \to 0$, and hence S is continuous.

Considering the operators: $[\underline{u}, \overline{u}] \xrightarrow{S} L^{p'}(\Omega) \xrightarrow{T_\lambda} W^{1,p}(\Omega)$, we can define $F : [\underline{u}, \overline{u}] \mapsto W^{1,p}(\Omega)$; given by $F = T_\lambda \circ S$, where $F(v) = w$ is the unique weak positive solution of the boundary value problem

$$\begin{cases} -\Delta_p w + \lambda w^{p-1} = v^q & \text{in } \Omega, \\ |\nabla w|^{p-2} \frac{\partial w}{\partial \eta} = \varphi & \text{on } \partial\Omega. \end{cases}$$

It is clear that F is continuous and nondecreasing.

Writing $u_1 = F(\underline{u})$ and $u^1 = F(\overline{u})$, for all $v \in W^{1,p}(\Omega)$ with $v \geq 0$, we have

$$\begin{aligned} \int_\Omega \left[\mid \nabla u_1 \mid^{p-2} \nabla u_1 \nabla v + \lambda u_1^{p-1} v \right] dx &= \int_\Omega \underline{u}^q v \, dx + \int_{\partial\Omega} \varphi v \, d\sigma_y \\ &\geq \int_\Omega \left[\mid \nabla \underline{u} \mid^{p-2} \nabla \underline{u} \nabla v + \lambda \underline{u}^{p-1} v \right] dx \end{aligned}$$

and

$$\begin{aligned} \int_\Omega \left[\mid \nabla u^1 \mid^{p-2} \nabla u^1 \nabla v + \lambda (u^1)^{p-1} v \right] dx &= \int_\Omega \overline{u}^q v \, dx + \int_{\partial\Omega} \varphi v \, d\sigma_y \\ &\leq \int_\Omega \left[\mid \nabla \overline{u} \mid^{p-2} \nabla \overline{u} \nabla v + \lambda \overline{u}^{p-1} v \right] dx. \end{aligned}$$

Thus, applying Lemma 2.1 and taking into account that F is nondecreasing, we get

$$\underline{u} \leq F(\underline{u}) \leq F(u) \leq F(\overline{u}) \leq \overline{u}, \text{ a.e. in } \Omega.$$

Repeating the same reasoning, we can obtain the existence of sequences (u^n) and (u_n) in $W^{1,p}(\Omega)$ satisfying

$$\begin{aligned} &u^0 = \overline{u}, \ u^{n+1} = F(u^n), \\ &u_0 = \underline{u}, \ u_{n+1} = F(u_n), \end{aligned}$$

and for every weak solution $u \in [\underline{u}, \overline{u}]$ of problem (1_λ), we have

$$u_0 \leq u_1 \leq ... \leq u_n \leq u \leq u^n \leq ... \leq u^1 \leq u^0 \text{ a.e. in } \Omega.$$

Since

$$\begin{aligned}\int_\Omega \left[\mid \nabla u_{n+1} \mid^{p-2} \nabla u_{n+1} \nabla v + \lambda u_{n+1}^{p-1} v \right] dx &= \int_\Omega u_n^q v \, dx + \int_{\partial\Omega} \varphi v \, d\sigma_y \\ &\leq \int_\Omega \overline{u}^q v \, dx + \int_{\partial\Omega} \varphi v \, d\sigma_y\end{aligned}$$

and

$$\begin{aligned}\int_\Omega \left[\mid \nabla u^{n+1} \mid^{p-2} \nabla u^{n+1} \nabla v + \lambda (u^{n+1})^{p-1} v \right] dx &= \int_\Omega (u^n)^q v \, dx + \int_{\partial\Omega} \varphi v \, d\sigma_y \\ &\leq \int_\Omega \overline{u}^q v \, dx + \int_{\partial\Omega} \varphi v \, d\sigma_y,\end{aligned}$$

we obtain that (u^n) and (u_n) are bounded in $W^{1,p}(\Omega)$. Therefore, up to subsequences, we have $u_n \rightharpoonup u_*$, $u^n \rightharpoonup u^*$ weakly in $W^{1,p}(\Omega)$, $u_n \to u_*$, $u_n \to u^*$ in $L^r(\Omega)$ for $1 \leq r < p^*$ and $u_n \to u_*$, $u^n \to u^*$ almost everywhere in Ω. Moreover, by construction we have u_*, $u^* \in [\underline{u}, \overline{u}]$ and $u_* \leq u^*$ almost everywhere in Ω. Now, using that $S(u_n) \to S(u_*)$, $S(u^n) \to S(u^*)$ and the continuity of T_λ we conclude that $u_{n+1} = F(u_n) \to F(u_*)$ and $u^{n+1} = F(u^n) \to F(u^*)$ in $W^{1,p}(\Omega)$. Thus $u_*, u^* \in W^{1,p}(\Omega)$ with $u_* = F(u_*)$, $u^* = F(u^*)$. This completes the proof of Proposition 2.3. ■

Lemma 2.4 *There exists $\lambda^* \geq 0$, such that problem (1_λ) possesses a minimal positive solution for each $\lambda \in (\lambda^*, +\infty)$ and (1_λ) has no positive solution for $\lambda \in (-\infty, \lambda^*)$.*

Proof. Notice that $\underline{u} \equiv 0$ is a lower solution of (1_λ) for all $\lambda \geq 0$. Now, we take w_1 the positive solution of problem (2_λ) with $f \equiv 0$ and $\lambda = 1$. Thus, $\overline{u} = w_1$ is a upper solution of (1_{λ_0}) with $\lambda_0 = 1 + \max_{x \in \overline{\Omega}} w_1^{q-p+1}$. Using Proposition 2.3 we get a minimal solution u_{λ_0} of (1_{λ_0}). Finally, by Harnack's inequality (see Theorem 1.2 in [16]) we have $\underline{u} \equiv 0 < u_{\lambda_0} < \overline{u}$. Thus

$$\Lambda = \{\lambda \in I\!R \ : \ (1)_\lambda \ \text{ possesses at least one positive solution}\} \tag{2.7}$$

is a nonempty set. Notice that u_{λ_0} is a upper solution of (1_λ) for all $\lambda \geq \lambda_o$. Thus, using the same argument above we conclude that $[\lambda_o, \infty) \subset \Lambda$. Moreover, $u_{\lambda_1} \leq u_{\lambda_2}$ if $\lambda_2 \leq \lambda_1$ and $\Lambda \subset [0, +\infty)$, because for u_λ solution of (1_λ) then u_λ satisfies (2.3) and taking $v = 1$ as test function we get

$$\lambda \int_\Omega u_\lambda^{p-1}\, dx = \int_\Omega u_\lambda^q\, dx + \int_{\partial\Omega} \varphi\, d\sigma_y > 0,$$

which implies that $\lambda > 0$. Consequently, setting

$$\lambda^* = \inf \Lambda,$$

we have $\lambda^* \in [0, +\infty)$. Moreover, for all $\lambda \in (\lambda^*, \infty)$, (1_λ) possesses one minimal solution and has no solution if $\lambda \in (-\infty, \lambda^*)$. ■

Lemma 2.5 *λ^* is positive real number and the problem (1_{λ^*}) possesses a minimal positive solution.*

Proof. Our goal is prove that λ^* is attained. To this end, let us take (λ_j) a decreasing sequence in (λ^*, ∞), satisfying $\lim_{j\to\infty} \lambda_j = \lambda^*$ and (u_j) in $W^{1,p}(\Omega)$ the correspondent sequence of minimal positive solutions of problem (1_{λ_j}) given in Lemma 2.4. We claim that (u_j) is bounded in $W^{1,p}(\Omega)$. Indeed, suppose by contradiction (up to subsequences) that $\|u_j\|_{1,p} \to +\infty$, as $j \to +\infty$. From this we will prove that

$$\int_\Omega u_j^{p-1}\, dx \to \infty \text{ as } j \longrightarrow +\infty. \tag{2.8}$$

Setting $w_j = u_j/\|u_j\|_{1,p}$, we have $\|w_j\|_{1,p} = 1$ and $w_j > 0$ in Ω. Thus, (up to subsequences) there exists $w \in W^{1,p}(\Omega)$ such that $w_j \rightharpoonup w$ weakly in $W^{1,p}(\Omega)$, $w_j \to w$ in $L^r(\Omega)$ for $1 \leq r < p^*$, and $w_j \to w$ almost everywhere in Ω. Taking $v = w/\|u_j\|_{1,p}^{p-1}$ as a test function in (2.3), we obtain

$$\int_\Omega |\nabla w_j|^{p-2}\nabla w_j \nabla w\, dx + \int_\Omega \frac{(\lambda_j u_j^{p-1} - u_j^q)}{\|u_j\|_{1,p}^{p-1}} w\, dx = \frac{1}{\|u_j\|_{1,p}^{p-1}} \int_{\partial\Omega} \varphi w\, d\sigma_y. \tag{2.9}$$

Passing to the limit in (2.9) and using a convergence result due Lucio-Bocardo (see Theorem 2.1 in [17]) we concluded that

$$\int_\Omega \frac{(\lambda_j u_j^{p-1} - u_j^q)}{\|u_j\|_{1,p}^{p-1}} w\, dx \to \int_\Omega |\nabla w|^p\, dx. \tag{2.10}$$

Similarly, taking $v = w_j/\|u_j\|_{1,p}^{p-1}$ in (2.3) and passing to the limit we obtain

$$\int_\Omega |\nabla w_j|^p \, dx - \int_\Omega \frac{(u_j^q - \lambda_j u_j^{p-1})}{\|u_j\|_{1,p}^{p-1}} w_j \, dx \to 0. \tag{2.11}$$

From (2.10)-(2.11) we conclude that

$$\|\nabla w_j\|_{L^p} \to \|\nabla w\|_{L^p}. \tag{2.12}$$

Now, observe that w_j satisfies

$$\begin{cases} -\Delta_p w_j + \lambda w_j^{p-1} = f_j & \text{in } \Omega, \\ |\nabla w_j|^{p-2} \frac{\partial w_j}{\partial \eta} = \varphi_j & \text{on } \partial\Omega, \end{cases} \tag{2.13}$$

where $f_j = u_j^q/\|u_j\|_{1,p}^{p-1}$ and $\varphi_j = \varphi/\|u_j\|_{1,p}^{p-1}$. It is not difficult to see that $f_j \rightharpoonup f$ weakly in $L^p(\Omega)$, and $\varphi_j \to 0$ almost everywhere in $\partial\Omega$. By a convergence result due Lucio-Bocardo (see Theorem 2.1 in [17]) and Brézis-Lieb's Lemma (see [18]), we conclude that $\nabla w_j \to \nabla w$ strongly in $(L^p(\Omega))^n$. This fact implies that $w_j \to w$ strongly in $L^{p^*}(\Omega)$. Since Ω is a bounded domain, we conclude that $w_j \to w$ strongly in $W^{1,p}(\Omega)$. Observe that $w \geq 0$ and $w \not\equiv 0$. Therefore, there exists a subset $\mathcal{V} \subset \Omega$ of positive Lebesgue measure such that $w > 0$ almost everywhere in $\mathcal{V}$. Thus, there exists j_o such that for all $j \geq j_o$ we have $u_j \to +\infty$ almost everywhere in $\mathcal{V}$. Therefore, given $M > 0$ there exists j_o such that $u_j(x) \geq M$ for all $j \geq j_o$ and almost everywhere in $\mathcal{V}$. So, for each $1 \leq r \leq p^*$, we have

$$M^r |\mathcal{V}| \leq \int_{\mathcal{V}} u_j^r \, dx \leq \int_\Omega u_j^r \, dx.$$

Thus, making $M \to +\infty$, we obtain (2.8).

On the other hand, choosing $v = 1$ in (2.3) and using the Holder's inequality we have

$$C(\Omega, q, p) \left(\int_\Omega u_j^{p-1} \, dx \right)^{\frac{q}{p-1}} \leq \int_\Omega u_j^q \, dx = \lambda_j \int_\Omega u_j^{p-1} \, dx - \int_{\partial\Omega} \varphi(y) \, d\sigma_y. \tag{2.14}$$

where $C = C(\Omega, q, p) > 0$, which is a contradiction with (2.8). Since (u_j) is bounded in $W^{1,p}(\Omega)$, taking subsequence if necessary, we can assume that there exists a function $u \in W^{1,p}(\Omega)$ such that $u_j \rightharpoonup u$ weakly in the spaces

$W^{1,p}(\Omega)$, $L^{p+1}(\Omega))^*$, $L^p(\partial\Omega)$ and $L^q(\Omega)$ for each $q \in (1, p^*)$. Since u_j satisfies (1_{λ_j}), we have

$$\int_\Omega \left[\mid \nabla u_j \mid^{p-2} \nabla u_j \nabla v + \lambda_j \mid u_j \mid^{p-2} u_j v \right] dx = \int_\Omega u_j^q v \, dx + \int_{\partial\Omega} \varphi v \, d\sigma_y. \tag{2.15}$$

Hence, using a convergence result due to Lucio-Bocardo (see Theorem 2.1 in [17]) we have $\nabla w_n \to \nabla w$ strongly. Moreover, by Brezis-Lieb's Lemma, we have after taking the limit

$$\int_\Omega \left[\mid \nabla u \mid^{p-2} \nabla u \nabla v + \lambda^* \mid u \mid^{p-2} uv \right] dx = \int_\Omega u^q v \, dx + \int_{\partial\Omega} \varphi v \, d\sigma_y. \tag{2.16}$$

Therefore, u is a weak solution of $(1)_{\lambda^*}$. Finally, applying Proposition 2.3 and using the fact that $\underline{u} \equiv 0$ is a lower solution of $(1)_{\lambda^*}$, we conclude that there exists a minimal solution u_{λ^*} of $(1)_{\lambda^*}$. ■

We notice that until this moment we have proved the items (i) and (ii) of Theorem 1.1.

2.2 Asymptotic behavior of the minimal solution

Next we are going to prove the last item of Theorem 1.1. For this end firstly we observe that taking $v = u_\lambda$ as a test function in (2.3) we obtain

$$\|\nabla u_\lambda\|^p_{L^p(\Omega)} = \int_{\partial\Omega} \varphi(y) u_\lambda \, d\sigma_y + \int_\Omega (u_\lambda^{q+1} - \lambda u_\lambda^p) \, dx. \tag{2.17}$$

Let λ_1 be a fixed element in Λ. From (ii) in Theorem 1.1 follows that for each $\lambda \geq \lambda_1$, the respective minimal solutions u_λ satisfies $u_\lambda \leq u_{\lambda_1}$ in Ω. Thus, using this fact and the Hölder's inequality with $1/p' + 1/p = 1$,

$$\|\nabla u_\lambda\|^p_{L^p(\Omega)} \leq \|\varphi\|_{L^{p'}(\partial\Omega)} \|u_\lambda\|_{L^p(\partial\Omega)} + \int_{\{u_\lambda \leq 1\}} 1 \, dx + \int_{\{u_\lambda \geq 1\}} u_{\lambda_1}^{q+1} \, dx - \lambda \int_\Omega u_\lambda^p \, dx. \tag{2.18}$$

Thus, applying the trace embedding theorem and Young's inequality, we have

$$\begin{aligned} \|\varphi\|_{L^{p'}(\partial\Omega)} \|u_\lambda\|_{L^p(\partial\Omega)} &\leq \|\varphi\|_{L^{p'}(\partial\Omega)} \|u_\lambda\|_{1,p} \\ &\leq C_\epsilon \|\varphi\|^{p'}_{L^{p'}(\partial\Omega)} + \epsilon(\|\nabla u_\lambda\|^p_{L^p(\Omega)} + \|u_\lambda\|^p_{L^p(\Omega)}), \end{aligned} \tag{2.19}$$

which together with (2.18) and (2.19) implies that

$$\begin{aligned}(1-\epsilon)\|\nabla u_\lambda\|^p_{L^p(\Omega)} \leq \ & C_\epsilon \|\varphi\|^{p'}_{L^{p'}(\partial\Omega)} + \epsilon\Big(\int_{\{u_\lambda \leq 1\}} dx + \int_{\{u_\lambda \geq 1\}} u^p_{\lambda_1}\, dx\Big) \\ & + \int_{\{u_\lambda \leq 1\}} dx + \int_{\{u_\lambda \geq 1\}} u^{q+1}_{\lambda_1}\, dx - \lambda \int_\Omega u^p_\lambda \, dx. \end{aligned} \tag{2.20}$$

Therefore, taking $\epsilon \in (0,1)$ and using (2.20) we conclude that $u_\lambda \to 0$ as $\lambda \to \infty$ in $L^p(\Omega)$. Since $u_\lambda \in C^{1,\alpha}$, we deduce that $u_\lambda \to 0$ as $\lambda \to \infty$. This completes the proof of Theorem 1.1.

3 Proof of Theorem 1.2

In order to prove Theorem 1.2 we first show with the help of the minimal solution u_λ that there exists a solution v_λ which is a local minimum of the associated functional J_λ to problem (1_λ) in $W^{1,p}(\Omega)$. This is necessary because the minimal solution u_λ is not a variational solution so it is not clear how to get an estimate to its the energy level. For proving the existence of the second solution we consider the perturbed functional $I_\lambda(u) := J_\lambda(u + v_\lambda)$ and we prove that this functional has the mountain pass geometry. Using the Ekeland variational principle we obtain a Palais-Smale sequence at this mountain pass level $c(v_\lambda)$ of I_λ. Finally, doing an argument similar in spirit to that used in the classical result due to Brezis-Nirenberg [3], we obtain a nontrivial critical point u of I_λ. Thus, $w_\lambda = u + v_\lambda$ is a second solution of problem (1_λ).

3.1 Existence of a local minimum

Here we are going to prove the existence of a local minimum of the energy functional J_λ for all $\lambda > \lambda^*$. To do that, it is crucial in our argument the existence of the minimal solution obtained in the last section.

Proposition 3.1 *For each $\lambda \in (\lambda^*, +\infty)$, the functional J_λ has a local minimum v_λ in $W^{1,p}(\Omega)$.*

Proof. Fixed $\lambda \in (\lambda^*, +\infty)$, we can take real numbers $\lambda_1, \lambda_2 \geq \lambda^*$ such that $\lambda_2 < \lambda < \lambda_1$. Let u_{λ_i} be the positive minimal solution associated to the problem (1_{λ_i}), for $i \in \{1,2\}$ given by Theorem 1.1. Thus,

$$0 < u_{\lambda_1} \leq u_{\lambda_2}. \tag{3.21}$$

Since $\lambda_2 < \lambda < \lambda_1$, for all $v \geq 0$ we have

$$\begin{aligned}\int_\Omega \left[\mid \nabla u_{\lambda_1} \mid^{p-2} \nabla u_{\lambda_1} \nabla v \right. & \left. + \lambda u_{\lambda_1}^{p-1} v \right] \, dx \\ & < \int_\Omega \left[\mid \nabla u_{\lambda_1} \mid^{p-2} \nabla u_{\lambda_1} \nabla v + \lambda_1 u_{\lambda_1}^{p-1} v \right] \, dx \\ & = \int_\Omega u_{\lambda_1}^q v \, dx + \int_{\partial\Omega} \varphi v \, d\sigma_y,\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}\int_\Omega u_{\lambda_2}^q v \, dx + \int_{\partial\Omega} \varphi v \, d\sigma_y & = \int_\Omega \left[\mid \nabla u_{\lambda_2} \mid^{p-2} \nabla u_{\lambda_2} \nabla v + \lambda_2 u_{\lambda_2}^{p-1} v \right] \, dx \\ & < \int_\Omega \left[\mid \nabla u_{\lambda_2} \mid^{p-2} \nabla u_{\lambda_2} \nabla v + \lambda u_{\lambda_2}^{p-1} v \right] \, dx.\end{aligned} \tag{3.23}$$

Thus, using (3.21), (3.22) and (3.23), for all $v \geq 0$ we get

$$\int_\Omega \left[\mid \nabla u_{\lambda_1} \mid^{p-2} \nabla u_{\lambda_1} \nabla v + \lambda u_{\lambda_1}^{p-1} v \right] \, dx < \int_\Omega \left[\mid \nabla u_{\lambda_2} \mid^{p-2} \nabla u_{\lambda_2} \nabla v + \lambda u_{\lambda_2}^{p-1} v \right] \, dx. \tag{3.24}$$

Next, we apply the minimization methods to the Euler Lagrange functional

$$\tilde{J}_\lambda(u) = \frac{1}{p} \int_\Omega [|\nabla u|^p + \lambda |u|^p] \, dx - \int_\Omega \tilde{F}(u_+) \, dx - \int_{\partial\Omega} \varphi u_+ d\sigma_y,$$

associated to the problem

$$\begin{cases} -\Delta_p u + \lambda u^{p-1} = \tilde{f}(u) & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \eta} = \varphi & \text{on } \partial\Omega, \end{cases}$$

where $\tilde{F}(t) = \int_0^t \tilde{f}(s) \, ds$ is the primitive of function

$$\tilde{f}(u(x)) = \begin{cases} u_{\lambda_1}^q(x) & \text{if } u(x) \leq u_{\lambda_1}(x), \\ u^q(x) & \text{if } u_{\lambda_1}(x) \leq u(x) \leq u_{\lambda_2}(x), \\ u_{\lambda_2}^q(x) & \text{if } u_{\lambda_2}(x) \leq u(x). \end{cases}$$

It is not difficult to prove that the functional $\tilde{J}_\lambda$ is coercive and bounded below on $W^{1,p}(\Omega)$. Indeed, it is enough to observe that

$$\int_\Omega \tilde{F}(u_{\lambda_1}(x)) \, dx \leq \int_\Omega \tilde{F}(u(x)) \, dx \leq \int_\Omega \tilde{F}(u_{\lambda_2}(x)) \, dx.$$

Therefore, we get a minimizer v_λ to $\tilde{J}_\lambda$ in $W^{1,p}(\Omega)$, which without loss of generality we can assume that v_λ is positive. By regularity theory $v_\lambda \in C^{1,\alpha}$. Moreover,

$$-\Delta_p u_{\lambda_1} + \lambda u_{\lambda_1}^{p-1} \leq \tilde{f}(u_{\lambda_1}) \leq \tilde{f}(v_\lambda) \leq f(u_{\lambda_2}) \leq -\Delta_p u_{\lambda_2} + \lambda u_{\lambda_2}^{p-1}.$$

Thus, by weak comparison principle (see Lemma 2.1), we have

$$u_{\lambda_1} \leq v_\lambda \leq u_{\lambda_2}.$$

Set

$$\mathcal{K} := \{x \in \overline{\Omega} : v_\lambda(x) = u_{\lambda_2}(x)\}.$$

Using (3.24), we have that $\mathcal{K} \neq \overline{\Omega}$ and so by the Proposition 2.1 in Guedda-Veron[8], we obtain that $0 < v_\lambda < u_{\lambda_2}$. Therefore, there exists $\epsilon_0 > 0$ such that for each $\epsilon \in (0, \epsilon_0)$,

$$u_{\lambda_1}(x) + \epsilon\delta(x) \leq v_\lambda \leq u_{\lambda_2}(x) - \epsilon\delta(x),$$

where $\delta(x) = \inf\{|x-y|\,;\, y \in \partial\Omega\}$. Moreover, it is easy to see that the function $\hat{F}(u) := \widetilde{F}(u) - F(u)$ on the interval of functions $[u_{\lambda_1}, u_{\lambda_2}]$ is independent of u, so $\widetilde{J}_\lambda - J_\lambda$ is constant in $\mathcal{C}^1$-ball, $\{u \in C^1(\Omega) \cap W^{1,p}(\Omega) : \|u - v_\lambda\|_{1,0} \leq \epsilon\}$, which means that v_λ is a local minimum of J_λ in the $\mathcal{C}^1$-topology. Finally, using the same argument as in the proof of Theorem 1.1 in [11] (see also [6]) we obtain that v_λ is also a local minimum of functional J_λ in the space $W^{1,p}(\Omega)$. ∎

3.2 The perturbed functional

Here, we are denoting by v_λ the local minimum obtained in the Proposition 3.1. Next we are going to prove that the perturbed functional $I_\lambda(u) := J_\lambda(u + v_\lambda)$ has the mountain pass geometry.

Lemma 3.2 (mountain pass geometry) *The functional J_λ satisfies:*
(i) there exist $\alpha \in \mathbb{R}$ and $\rho > 0$ such that

$$J_\lambda(u) \geq \alpha \quad \textit{for } u \in W^{1,p}(\Omega) \textit{ with } \|u - v_\lambda\| = \rho;$$

(ii) there exists $\tilde{u}_\lambda \in W^{1,p}(\Omega)$ such that $\|\tilde{u}_\lambda\| > \rho$ and $J_\lambda(\tilde{u}_\lambda) < \alpha$.

Proof. (i) follows from the fact that v_λ is local minimum of J_λ. To prove (ii) it is enough to observe that

$$\begin{aligned} J_\lambda(v_\lambda + tv_\lambda) &= \frac{(1+t)^p}{p}\|v_\lambda\|_{1,p}^p - \frac{(1+t)^{q+1}}{q+1}\|v_\lambda\|_{L^q(\Omega)}^{q+1} - (1+t)\int_{\partial\Omega} v_\lambda \varphi \, d\sigma_y \\ &\leq \frac{(1+t)^p}{p}\|v_\lambda\|_{1,p}^p - \frac{(1+t)^{q+1}}{q+1}\|v_\lambda\|_{L^q(\Omega)}^{q+1} \end{aligned}$$

and $q+1 > p$. ∎

Therefore, we can conclude that the set

$$\Gamma = \left\{\gamma \in C([0,1], W^{1,p}(\Omega)) \ : \ \gamma(0) = v_\lambda \text{ and } J_\lambda(\gamma(1)) < J_\lambda(v_\lambda)\right\},$$

is nonempty and the *mountain pass level*

$$c(v_\lambda) := \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} J_\lambda(\gamma(t)),$$

is well defined. Moreover, following [9] we have the following characterization to the minimax level $c(v_\lambda)$,

$$c(v_\lambda) = \inf_{v\in W^{1,p}(\Omega)\setminus\{0\}} \max_{t\geq 0} I_\lambda(tv) = \inf_{v\in W^{1,p}(\Omega)\setminus\{0\}} \max_{t\geq 0} J_\lambda(v_\lambda + tv). \tag{3.25}$$

Next, using this characterization we can state.

Proposition 3.3 *If $q = p^* - 1$, then the following estimate is true*

$$c(v_\lambda) < J_\lambda(v_\lambda) + \frac{1}{2n}S^{n/p}.$$

Proof. By (3.25) we have

$$c(v_\lambda) \leq \max_{t\geq 0} J_\lambda(v_\lambda + tv), \text{ for all } v \in W^{1,p}(\Omega)\setminus\{0\}. \tag{3.26}$$

Since the equation (1_λ) is equivariant with respect to rotations and translations in $I\!R^n$, we can assume without lost of generality that $x_0 = 0 \in \partial\Omega$ and $\Omega \subset \{x_n > 0\}$. For each $x \in I\!R^n$ we write $x = (x', x_n) \in I\!R^{n-1} \times I\!R$. In the following, we assume that in some neighborhood of origin, the boundary of Ω is give by

$$x_n = h(x') = g(x') + o(|x'|^2), \ \ \forall \ \ x' = (x_1, ...x_{n-1}) \in D(0,\delta), \tag{3.27}$$

where

$$D(0,\delta) = B(0,\delta) \cap \{x_n = 0\}, \qquad g(x') := \frac{1}{2}\sum_{i=1}^{n-1} \alpha_i x_i^2$$

and $\alpha_i > 0$ are the principal curvatures of $\partial\Omega$ in $x_0 = 0$.

Next, we are going to estimate

$$\begin{aligned} J(v_\lambda + tu_\epsilon) &= \frac{1}{p}\int_\Omega [|\nabla(v_\lambda + tu_\epsilon)|^p + \lambda |v_\lambda + tu_\epsilon|^p]\ dx \\ &- \frac{1}{p^*}\int_\Omega |v_\lambda + tu_\epsilon|^{p^*}\ dx - \int_{\partial\Omega} \varphi(v_\lambda + tu_\epsilon)\ d\sigma_y. \end{aligned}$$

For the sake of clarity we estimate separability the gradient term, critical and subcritical term. We are going to use the following notations

$$K_{1,s}(\epsilon) := \int_\Omega |\nabla u_\epsilon|^s\ dx, \quad K_{2,r}(\epsilon) := \int_\Omega u_\epsilon^r\ dx.$$

(i) Estimate of the gradient term: Let $t \in [0,\infty)$, $p \in [2,3)$, $\alpha \in [0,2\pi]$ and $\gamma \in [p-1,2]$. The following elementary inequality holds

$$(1 + t^2 + 2t\cos\alpha)^{p/2} \leq 1 + t^p + pt\cos\alpha + Ct^\gamma. \tag{3.28}$$

Since

$$\int_\Omega |\nabla(v_\lambda + tu_\epsilon)|^p\ dx = \int_\Omega |\nabla v_\lambda|^p \Big(1 + 2t\frac{\nabla v_\lambda \nabla u_\epsilon}{|\nabla v_\lambda|^2} + t^2\frac{|\nabla u_\epsilon|^2}{|\nabla v_\lambda|^2}\Big)^{p/2}\ dx,$$

from (3.28) we obtain

$$\int_\Omega |\nabla(v_\lambda + tu_\epsilon)|^p\ dx \leq \int_\Omega \Big(|\nabla v_\lambda|^p + t^p|\nabla u_\epsilon|^p + pt|\nabla v_\lambda|^{p-2}\langle\nabla v_\lambda \nabla u_\epsilon\rangle + t^\gamma|\nabla u_\epsilon|^\gamma\Big)\ dx,$$

which together with L^∞ estimate due to Lierbmann [13] and Cauchy-Schwarz's inequality implies

$$\int_\Omega |\nabla(v_\lambda + tu_\epsilon)|^p\ dx \leq \int_\Omega |\nabla v_\lambda|^p\ dx + t^p\int_\Omega |\nabla u_\epsilon|^p\ dx + t^\gamma K_{1,\gamma}(\epsilon). \tag{3.29}$$

(ii) Estimate of the critical power term: In order to estimate the critical power term we consider the elementary inequality

$$(1+s)^{p^*} \geq 1 + s^{p^*} + p^*s + p^*s^{p^*-1} + Cs^\gamma, \qquad s \geq 0, \tag{3.30}$$

where $\gamma \in (1, p^*-1]$ (see [10] for more details). Thus, from (3.30),

$$\int_\Omega (v_\lambda + tu_\epsilon)^{p^*}\, dx \geq \int_\Omega v_\lambda^{p^*}\, dx + t^{p^*}\int_\Omega u_\epsilon^{p^*}\, dx + p^* t^{p^*-1}\int_\Omega u_\epsilon^{p^*-1} v_\lambda\, dx. \tag{3.31}$$

(iii) Estimate of the subcritical power term: Firstly, we notice that for each $a, b \geq 0$ and $1 < p < n$ we have

$$(a+b)^p \leq a^p +^p + C\max\{ab^{p-1}, ba^{p-1}\},$$

which implies that

$$\int_\Omega |v_\lambda + tu_\epsilon|^p\, dx \leq \int_\Omega v_\lambda^p\, dx + t^p\int_\Omega u_\epsilon^p\, dx + C_1 t^{p-1}\int_\Omega v_\lambda u_\epsilon^{p-1}\, dx + C_2 t\int_\Omega v_\lambda^{p-1} u_\epsilon\, dx.$$

Since $v_\lambda \in L^\infty(\overline{\Omega})$, we get

$$\int_\Omega |v_\lambda + tu_\epsilon|^p\, dx \leq \int_\Omega v_\lambda^p\, dx + t^p\int_\Omega u_\epsilon^p\, dx + C_3 t^{p-1}\int_\Omega u_\epsilon^{p-1}\, dx + C_4 t\int_\Omega u_\epsilon\, dx. \tag{3.32}$$

Using the estimates (3.29), (3.31) and (3.32) we obtain

$$J_\lambda(v_\lambda + tu_\epsilon) \leq J_\lambda(v_\lambda) + F_\lambda(t,\epsilon) + G_\lambda(t,\epsilon), \tag{3.33}$$

where

$$F_\lambda(t,\epsilon) = \frac{t^p}{p}(K_{1,p} + \lambda K_{2,p}) - \frac{t^{p^*}}{p^*}K_{2,p^*},$$

and

$$G_\lambda(t,\epsilon) = C_1 t^\gamma K_{1,\gamma}(\epsilon) + C_2 t^{p-1}K_{2,p-1}(\epsilon) + C_3 t K_{2,1}(\epsilon) - t^{p^*-1}\int_\Omega u_\epsilon^{p^*-1} v_\lambda\, dx.$$

To finish the proof of Proposition 3.3, we need the following result.

Lemma 3.4 *For each $\lambda > 0$ and $\epsilon > 0$ sufficiently small we have*

$$\max_{t>0} F_\lambda(t,\epsilon) < \frac{1}{2n}S^{n/p} \tag{3.34}$$

and

$$G(t,\epsilon) = t^\gamma O(\epsilon^\alpha) + t^{p-1}O(\epsilon^\beta) + tO(\epsilon^\delta) - t^{p^*-1}O(\epsilon^\eta), \tag{3.35}$$

where

$$\begin{aligned}
\alpha &= \frac{n-p}{p^2}\gamma + \frac{\gamma}{p} - \frac{n\gamma}{p} + \frac{p-1}{p}n, \\
\beta &= \frac{n-p}{p^2}(p-1) - \frac{(n-p)}{p}(p-1) + \frac{p-1}{p}n, \\
\delta &= \frac{n-p}{p^2} - \frac{(n-p)}{p} + \frac{p-1}{p}n, \\
\eta &= \frac{n-p}{p^2}(p^*-1) - \frac{(n-p)}{p}(p^*-1) + \frac{p-1}{p}n.
\end{aligned}$$

Proof. We begin by proving estimate (3.34). For this purpose, we consider two cases: $p^2 \le n$ and $p^2 > n$.

Case: $p^2 \le n$**.** Notice that

$$K_{1,p}(\epsilon) = \int_{I\!R^n_+} |\nabla u_\epsilon|^p dx - \int_{D(0,\delta)} dx' \int_0^{h(x')} |\nabla u_\epsilon|^p dx_n + O(\epsilon^{(n-p)/p}), \tag{3.36}$$

because

$$-\int_{I\!R^n_+} |\nabla u_\epsilon|^p \, dx + \int_\Omega |\nabla u_\epsilon|^p \, dx + \int_{D(0,\delta)} dx' \int_0^{h(x')} |\nabla u_\epsilon|^p dx_n = O(\epsilon^{(n-p)/p}).$$

Since

$$\begin{aligned}
&\Big| \int_\Omega |\nabla u_\epsilon|^p \, dx - \int_{I\!R^n_+} |\nabla u_\epsilon|^p \, dx + \int_{D(0,\delta)} dx' \int_0^{h(x')} |\nabla u_\epsilon|^p dx_n \Big| \\
= \;& \Big| -\int_{I\!R^n_+\setminus\Omega} |\nabla u_\epsilon|^p \, dx + \int_{D(0,\delta)} dx' \int_0^{h(x')} |\nabla u_\epsilon|^p dx_n \Big| \\
\le \;& \int_{I\!R^n_+\setminus B_+(0,\delta)} |\nabla u_\epsilon|^p \, dx = C(n,p)\epsilon^{(n-p)/p} \int_{I\!R^n_+\setminus B_+(0,\delta)} \frac{|x|^{p/(p-1)}}{(\epsilon + |x|^{p/(p-1)})^n} \, dx \\
\le \;& C(n,p)\epsilon^{(n-p)/p} \int_\delta^\infty \frac{r^{p/(p-1)+n-1}}{r^{p(n-1)/(p-1)}} dr = C(n,p)\epsilon^{(n-p)/p} \int_\delta^\infty \frac{1}{r^{(n-1)/(p-1)}} dr < \infty,
\end{aligned}$$

because $1 < p^2 \le n$ implies $2p-1 < p^2 \le n$ and consequently $(n-p)/(p-1) > 1$. Now, notice that

$$K_1 = 2\int_{I\!R^n_+} |\nabla u_\epsilon|^p dx = \int_{I\!R^n} |\nabla u_\epsilon|^p dx = \Big(\frac{n-p}{p-1}\Big)^n \int_{I\!R^n} \frac{|x|^{p/(p-1)}}{(1+|x|^{p/(p-1)})^n} \, dx. \tag{3.37}$$

Thus, K_1 does not depend of ϵ.

From (3.36)-(3.37) it follows that

$$\begin{aligned} K_{1,p}(\epsilon) &= \frac{1}{2}K_1 - \int_{D(0,\delta)} dx' \int_0^{g(x')} |\nabla u_\epsilon|^p dx_n \\ &- \int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')} |\nabla u_\epsilon|^p dx_n + O(\epsilon^{(n-p)/p}) \\ &= \frac{1}{2}K_1 - \int_{\mathbb{R}^{n-1}} dx' \int_0^{g(x')} |\nabla u_\epsilon|^p dx_n - \int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')} |\nabla u_\epsilon|^p dx_n \\ &+ \int_{\mathbb{R}^{n-1}} dx' \int_0^{g(x')} |\nabla u_\epsilon|^p dx_n - \int_{D(0,\delta)} dx' \int_0^{g(x')} |\nabla u_\epsilon|^p dx_n + O(\epsilon^{(n-p)/p}). \end{aligned}$$

Thus

$$\begin{aligned} K_{1,p}(\epsilon) &= \frac{1}{2}K_1 - \int_{\mathbb{R}^{n-1}} dx' \int_0^{g(x')} |\nabla u_\epsilon|^p \, dx_n \\ &- \int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')} |\nabla u_\epsilon|^p \, dx_n + O(\epsilon^{(n-p)/p}), \end{aligned} \tag{3.38}$$

where in the last inequality we have used the following estimate

$$\begin{aligned} &\int_{\mathbb{R}^{n-1}} dx' \int_0^{g(x')} |\nabla u_\epsilon|^p dx_n - \int_{D(0,\delta)} dx' \int_0^{g(x')} |\nabla u_\epsilon|^p \, dx_n \\ = &\int_{\mathbb{R}^{n-1}\setminus D(0,\delta)} dx' \int_0^{g(x')} |\nabla u_\epsilon|^p dx_n \\ = &\; C(n,p)\epsilon^{(n-p)/p} \int_{\mathbb{R}^{n-1}\setminus D(0,\delta)} dx' \int_0^{g(x')} \frac{|x|^{p/(p-1)}}{(\epsilon + |x|^{p/(p-1)})^n} dx_n \\ \leq &\; C(n,p)\epsilon^{(n-p)/p} \int_{\mathbb{R}^{n-1}\setminus D(0,\delta)} dx' \int_0^{g(x')} \frac{1}{(\epsilon + |x'|^{p/(p-1)})^{n-1}} dx_n. \end{aligned}$$

Using radial variable we deduce

$$\begin{aligned}\int_{I\!R^{n-1}} dx' \int_0^{g(x')} |\nabla u_\epsilon|^p dx_n \quad &- \quad \int_{D(0,\delta)} dx' \int_0^{g(x')} |\nabla u_\epsilon|^p dx_n \\ &\leq \quad C_1(n,p)\epsilon^{(n-p)/p} \int_\delta^\infty \frac{r^2 r^{n-2}}{r^{p(n-1)/(p-1)}} dr \\ &\leq \quad C_2(n,p)\epsilon^{(n-p)/p} \int_\delta^\infty \frac{1}{r^{(n-p)/(p-1)}} dr < \infty.\end{aligned}$$

Now, notice that

$$\begin{aligned} I(\epsilon) \quad &:= \int_{I\!R^{n-1}} dx' \int_0^{g(x')} |\nabla u_\epsilon|^p dx_n \\ &= \Big(\frac{n-p}{p-1}\Big)^p \epsilon^{(n-p)/p} \int_{I\!R^{n-1}} dx' \int_0^{g(x')} \frac{|x|^{p/(p-1)}}{(\epsilon + |x|^{p/(p-1)})^n} dx_n \\ &= \Big(\frac{n-p}{p-1}\Big)^p \int_{I\!R^{n-1}} dx' \int_0^{\epsilon^{(p-1)/p} g(x')} \frac{|x|^{p/(p-1)}}{(1 + |x|^{p/(p-1)})^n} dx_n. \end{aligned} \tag{3.39}$$

Thus,

$$\lim_{\epsilon\to 0} \frac{I(\epsilon)}{\epsilon^{(p-1)/p}} = \Big(\frac{n-p}{p-1}\Big)^p \int_{I\!R^{n-1}} \frac{|x'|^{p/(p-1)} g(x')}{(1+|x'|^{p/(p-1)})^n} dx'$$

which implies that

$$I(\epsilon) = O(\epsilon^{(p-1)/p}).$$

Moreover,

$$\begin{aligned} |I_1(\epsilon) : | \quad &= \Big| \int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')} |\nabla u_\epsilon|^p dx_n \Big| \\ &= C(n,p)\epsilon^{n-p/p} \Big| \int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')} \frac{|x|^{p/(p-1)}}{(\epsilon + |x|^{p/(p-1)})^n} dx_n \Big| \\ &= C(n,p)\epsilon^{(n-p)/p} \Big| \int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')} \frac{|x|^{p/(p-1)}}{(\epsilon + |x|^{p/(p-1)})(\epsilon + |x|^{p/(p-1)})^{n-1}} dx_n \Big| \\ &\leq C(n,p)\epsilon^{(n-p)/p} \Big| \int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')} \frac{1}{(\epsilon + |x'|^{p/(p-1)})^{n-1}} dx_n \Big| \\ &\leq C(n,p)\epsilon^{(n-p)/p} \int_{D(0,\delta)} \frac{|h(x') - g(x')|}{(\epsilon + |x'|^{p/((p-1)})^{n-1}} dx'. \end{aligned}$$

Since $h(x') = g(x') + o(|x'|^2)$, it follows that for all $\sigma > 0$, there exists $C(\epsilon) > 0$ such that $|h(x') - g(x')| \leq \sigma|x'|^2 + C(\sigma)|x'|^\alpha$ for all $x' \in D(0,\delta)$, where $2 < \alpha < (n-1)/(p-1)$. Thus,

$$I_1(\epsilon) \leq C(n,p)\epsilon^{(n-p)/p} \int_{D(0,\delta)} \frac{\sigma|x'|^2 + C(\sigma)|x'|^\alpha}{(\epsilon + |x'|^{p/(p-1)})^{n-1}} dx'.$$

Now, observing that

$$\epsilon^{(n-p)/p}/\epsilon^{(p-1)/p} \int_{D(0,\delta)} \frac{|x'|^2}{(\epsilon + |x'|^{p/(p-1)})^{n-1}} dx' \leq C,$$

and

$$\epsilon^{(n-p)/p}/\epsilon^{(p-1)/p} \int_{D(0,\delta)} \frac{|x'|^\alpha}{(\epsilon + |x'|^{p/(p-1)})^{n-1}} dx' \leq C(n,p)\epsilon^{(p-1)(\alpha-2)/p},$$

we obtain

$$I_1(\epsilon) \leq C(n,p)\epsilon^{(p-1)/p}(\sigma + C(\sigma)\epsilon^{(p-1)(\alpha-2)/p}).$$

Since σ is arbitrary and $\alpha > 2$ we conclude that $I_1(\epsilon) = o(\epsilon^{(p-1)/p})$. Therefore,

$$\begin{aligned} K_{1,p}(\epsilon) &= \tfrac{1}{2}K_1 - I(\epsilon) - I_1(\epsilon) + O(\epsilon^{(n-p)/p}) \\ &= \tfrac{1}{2}K_1 - I(\epsilon) + o(\epsilon^{(p-1)/p}). \end{aligned} \tag{3.40}$$

Now, let us obtain a more refined estimate of $K_{2,p^*}(\epsilon)$. To this end, firstly notice that

$$\begin{aligned} K_{2,p^*}(\epsilon) &= \int_{I\!R^n_+} |u_\epsilon|^{p^*} dx - \int_{D(0,\delta)} dx' \int_0^{h(x')} |u_\epsilon|^{p^*} dx_n + O(\epsilon^{n/p}) \\ &= \frac{1}{2}K_2 - \int_{D(0,\delta)} dx' \int_0^{g(x')} |u_\epsilon|^{p^*} dx_n - \int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')} |u_\epsilon|^{p*} dx_n + O(\epsilon^{n/p}) \\ &= \frac{1}{2}K_2 - \int_{I\!R^{n-1}} dx' \int_0^{g(x')} |u_\epsilon|^{p^*} dx_n - \int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')} |u_\epsilon|^{p*} dx_n + O(\epsilon^{n/p}) \\ &= \frac{1}{2}K_2 - II(\epsilon) - III(\epsilon) + O(\epsilon^{n/p}). \end{aligned}$$

Since

$$II(\epsilon) := \int_{I\!R^{n-1}} dx' \int_0^{g(x')} u_\epsilon^{p^*} dx_n = \epsilon^{n/p} \int_{I\!R^{n-1}} dx' \int_0^{g(x')} \frac{1}{(\epsilon + |x|^{p/(p-1)})^n} dx_n$$

$$= \int_{\mathbb{R}^{n-1}} dy' \int_0^{\epsilon^{(p-1)/p} g(y')} \frac{1}{(1+|y|^{p/(p-1)})^n} dy_n, \tag{3.41}$$

we have $II(\epsilon) = O(\epsilon^{(p-1)/p})$. Using the same estimate as in $I_1(\epsilon)$ we have $III(\epsilon) = o(\epsilon^{(p-1)/p})$. Thus, for $1 < p^2 \leq n$ we have

$$K_{2,p^*}(\epsilon) = \frac{1}{2} K_2 - II(\epsilon) + o(\epsilon^{(p-1)/p}). \tag{3.42}$$

We can now proceed analogously to obtain a refined estimate for $K_{2,p}(\epsilon)$. To this end, we consider two cases $p^2 < n$ and $p^2 = n$ separably.

Case 1: $p^2 < n$. In this case we have

$$\begin{aligned} K_{2,p}(\epsilon) &= \int_\Omega u_\epsilon^p \, dx \leq \int_{\mathbb{R}^n} u_\epsilon^p \, dx = \epsilon^{(n-p)/p} \int_{\mathbb{R}^n} \frac{1}{(\epsilon + |x|^{p/(p-1)})^{n-p}} dx \\ &= w_n \epsilon^{(n-p)/p} \Big(\int_0^1 \frac{r^{n-1}}{(\epsilon + |r|^{p/(p-1)})^{n-p}} dr + \int_1^\infty \frac{r^{n-1}}{(\epsilon + |r|^{p/(p-1)})^{n-p}} dr \Big) \\ &= O(\epsilon^{(n-p)/p}) \\ &= o(\epsilon^{(p-1)/p}). \end{aligned}$$

Case 2: $p^2 = n$. Let $R > 0$ such that $\Omega \subset B(0,R)$. Notice that

$$\begin{aligned} K_{2,p}(\epsilon) &= \int_\Omega u_\epsilon^p \, dx \leq \int_{B(0,R)} u_\epsilon^p \, dx = \epsilon^{(n-p)/p} \int_{B(0,R)} \frac{1}{(\epsilon + |x|^{p/(p-1)})^{n-p}} dx \\ &= w_n \epsilon^{(n-p)/p} \int_0^R \frac{r^{n-1}}{(\epsilon + |r|^{p(p-1)})^{n-p}} dr \\ &= w_n \epsilon^{p-1} \int_0^{R/\epsilon^{(p-1)/p}} \frac{s^{n-1}}{(1 + |s|^{p/(p-1)})^{n-p}} ds \\ &= C\epsilon^{p-1} \Big(1 - \log(\epsilon^{(p-1)/p}) \Big) \\ &= \epsilon^{(p-1)/p} \Big(\epsilon^{(p-1)^2/p} - \epsilon^{(p-1)^2/p} \log(\epsilon^{(p-1)/p}) \Big) \\ &= o(\epsilon^{(p-1)/p}). \end{aligned}$$

Hence, for $1 < p^2 \leq n$ we have

$$K_{2,p}(\epsilon) = o(\epsilon^{(p-1)/p}). \tag{3.43}$$

Since $p^* > p$, there exist $t_\epsilon > 0$ such that

$$J_\lambda(t_\epsilon u_\epsilon) = \max_{t>0} \Big\{ \frac{1}{p}(K_{1,p}(\epsilon) + \lambda K_{2,p}(\epsilon))t^p - \frac{K_{2,p}(\epsilon)}{p^*} t^{p^*} \Big\}. \quad (3.44)$$

Follows from estimates (3.40), (3.42) and (3.43), that there exists $\epsilon_0 > 0$, $K' > 0$ and $K'' > 0$ such that

$$K_{2,p^*}(\epsilon) \geq K' \;\text{ and }\; K_{1,p}(\epsilon) + K_{2,p}(\epsilon) \leq K'',\; \forall\, \epsilon \in (0, \epsilon_0). \quad (3.45)$$

Consequently, t_ϵ is uniformly bounded in $(0, \epsilon_0)$. Since, $K_3(\epsilon) = o(\epsilon^{(p-1)/p})$ for $p^2 \leq n$ we get

$$\begin{aligned} J_\lambda(t_\epsilon) &= \sup_{t>0} \Big\{ \frac{1}{p} K_1(\epsilon) t^p - \frac{K_2(\epsilon)}{p^*} t^{p^*} \Big\} + o(\epsilon^{(p-1)/p}) \\ &= \frac{1}{p} K_1(\epsilon) \frac{K_2(\epsilon)}{K_1(\epsilon)} t_1^{p^*} - \frac{1}{p^*} K_2(\epsilon) t_1^{p^*} + o(\epsilon^{(p-1)/p}) \\ &= \frac{1}{n} K_2(\epsilon) t_1^{p^*} + o(\epsilon^{(p-1)/p}) \\ &= \frac{1}{n} K_2(\epsilon) \Big(\frac{K_1(\epsilon)}{K_2(\epsilon)} \Big)^{n/p} + o(\epsilon^{(p-1)/p}) \\ &= \frac{1}{n} \Big(\frac{K_1(\epsilon)}{K_2(\epsilon)^{(n-p)/n}} \Big)^{n/p} + o(\epsilon^{(p-1)/p}). \end{aligned}$$

Finally, we observe that the statement (3.34) will be proved once we have proved the following claim

Claim 3.1 *The following estimate is holds*

$$\frac{K_1(\epsilon)}{K_2(\epsilon)^{(n-p)/n}} < 2^{-p/n} S + o(\epsilon^{(p-1)/p}). \quad (3.46)$$

From (1.2), inequality (3.46) is equivalent to

$$\begin{aligned} \frac{K_1(\epsilon)}{K_2(\epsilon)^{(n-p)/n}} &< 2^{-p/n} \frac{K_1}{K_2^{(n-p)/n}} + o(\epsilon^{(p-1)/p}) \\ &= \frac{K_1}{2} \frac{1}{(\frac{K_2}{2})^{(n-p)/n}} + o(\epsilon^{(p-1)/p}), \end{aligned}$$

that is,

$$K_1(\epsilon)\Big(\frac{K_2}{2}\Big)^{(n-p)/n} < \frac{K_1}{2}K_2(\epsilon)^{(n-p)/n} + o(\epsilon^{(p-1)/p}).$$

From (3.40)-(3.42) we have

$$\Big(\frac{K_1}{2}-I(\epsilon)\Big)\,\Big(\frac{K_2}{2}\Big)^{(n-p)/n} < \frac{K_1}{2}\Big(\frac{K_2}{2}-II(\epsilon)+o(\epsilon^{(p-1)/p})\Big)^{(n-p)/n}+o(\epsilon^{(p-1)/p}). \quad (3.47)$$

Now, notice that for $a\alpha > 0$, we have

$$(1-t)^{\alpha} = 1-\alpha t + o(t), \text{ as } t \to 0.$$

In particular, taking

$$t = \frac{II(\epsilon)+o(\epsilon^{(p-1)/p})}{\frac{K_2}{2}},$$

we obtain

$$\Big(\frac{K_2}{2}-II(\epsilon)+o(\epsilon^{(p-1)/p})\Big)^{(n-p)/n} = (\frac{K_2}{2})^{(n-p)/n}-(\frac{n-p}{n})(\frac{K_2}{2})^{-p/n}II(\epsilon)+o(\epsilon^{(p-1)/p}).$$

Thus, (3.47) is equivalent to

$$-I(\epsilon)(\frac{K_2}{2})^{(n-p)/n} < -\frac{K_1}{2}\Big(\frac{K_2}{2}\Big)^{-p/n}\Big(\frac{n-p}{n}\Big)II(\epsilon)+o(\epsilon^{(p-1)/p}).$$

Since, $II(\epsilon) = O(\epsilon^{(p-1)/p})$ we get

$$\frac{I(\epsilon)}{II(\epsilon)} > \Big(\frac{n-p}{n}\Big)\frac{K_1}{K_2}+o(1),$$

which implies that (3.46) is equivalent to

$$\lim_{\epsilon\to 0}\frac{I(\epsilon)}{II(\epsilon)} > \Big(\frac{(n-p)}{n}\Big)\frac{K_1}{K_2}. \quad (3.48)$$

From (3.39) and (3.41) we get

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{I(\epsilon)}{II(\epsilon)} &= \Big((n-p)/(p-1)\Big)^p \lim_{\epsilon\to 0}\frac{\displaystyle\int_{I\!R^{n-1}} dy' \int_0^{\epsilon^{(p-1)/p}g(y')} \frac{|y|^{p/(p-1)}}{(1+|y|^{p/(p-1)})^n}dy_n}{\displaystyle\int_{I\!R^{n-1}} dy' \int_0^{\epsilon^{(p-1)/p}g(y')} \frac{1}{(1+|y|^{p/(p-1)})^n}dy_n} \\
&= \left(\tfrac{n-p}{p-1}\right)^p \frac{\displaystyle\int_{I\!R^{n-1}} \frac{|y'|^{p/(p-1)}}{(1+|y'|^{p/(p-1)})^n}dy'}{\displaystyle\int_{I\!R^{n-1}} \frac{1}{(1+|y'|^{p/(p-1)})^n}dy'} \\
&= \left(\tfrac{n-p}{p-1}\right)^p \frac{\displaystyle\int_0^\infty \frac{r^{n+p/(p-1)}}{(1+r^{p/(p-1)})^n}dr}{\displaystyle\int_0^\infty \frac{r^n}{(1+r^{p/(p-1)})^n}dr}.\end{aligned} \tag{3.49}$$

Now we calculate the last term in 3.49). If $p/(p-1)\le\beta\le p(n-1)+1/(p-1)$, integrating on by parts we have

$$\int_0^\infty \frac{r^{\beta-p/(p-1)}}{(1+r^{p/(p-1)})^{n-1}}dr = \frac{p(n-1)}{(p-1)\beta-1}\int_0^\infty \frac{r^\beta}{(1+r^{p/(p-1)})^n}dr. \tag{3.50}$$

Observing that

$$\frac{r^\beta}{(1+r^{p/(p-1)})^n} = \frac{r^{\beta-p/(p-1)}}{(1+r^{p/(p-1)})^{n-1}}(1-\frac{1}{1+r^{p/(p-1)}}),$$

we obtain

$$\int_0^\infty \frac{r^\beta}{(1+r^{p/(p-1)})^n}dr = \int_0^\infty \frac{r^{\beta-p/(p-1)}}{(1+r^{p/(p-1)})^{n-1}}dr - \int_0^\infty \frac{r^{\beta-p/(p-1)}}{(1+r^{p/(p-1)})^n}dr. \tag{3.51}$$

From (3.50) and (3.51) we get

$$\Big(1-\frac{(n-1)p}{(p-1)\beta-1}\Big)\int_0^\infty \frac{r^\beta}{(1+r^{p/(p-1)})^n}dr = -\int_0^\infty \frac{r^{\beta-p/(p-1)}}{(1+r^{p/(p-1)})^n}dr,$$

that is

$$\int_0^\infty \frac{r^\beta}{(1+r^{p/(p-1)})^n}dr = \frac{(p-1)\beta-1}{(n-1)p-(p-1)\beta+1}\int_0^\infty \frac{r^{\beta-p/(p-1)}}{(1+r^{p/(p-1)})^n}dr. \tag{3.52}$$

From (3.49) and (3.52) with $\beta = n + p/(p-1)$ we obtain

$$\lim_{\epsilon\to 0}\frac{I(\epsilon)}{II(\epsilon)} = \Big(\frac{n-p}{p-1}\Big)^p\frac{(p-1)(n+1)}{n-2p+1} = \frac{(n-p)^p}{(p-1)^{p-1}}\frac{n+1}{n-2p+1}. \tag{3.53}$$

By (1.2), we have

$$\Big(\frac{n-p}{n}\Big)\frac{K_1}{K_2} = \Big(\frac{n-p}{n}\Big)\frac{\displaystyle\int_0^\infty \frac{r^{n+p/(p-1)-1}}{(1+r^{p/(p-1)})^n}dr}{\displaystyle\int_0^\infty \frac{r^{n-1}}{(1+r^{p/(p-1)})^n}dr}\Big(\frac{n-p}{p-1}\Big)^p.$$

Taking $\beta = n + p/(p-1) - 1$ in (3.52) we have

$$\begin{aligned}\frac{n-p}{n}\Big(\frac{K_1}{K_2}\Big) &= \frac{n-p}{n}\frac{(p-1)((n-1)+p/(p-1))}{(n-1)p-(p-1)((n-1)+p/(p-1))}\Big(\frac{n-p}{p-1}\Big)^p\\ &= \frac{(n-p)^p}{(p-1)^{p-1}}.\end{aligned} \tag{3.54}$$

Since $n+1 > n-2p+1$, (3.53)-(3.54) yields that (3.48) is true. Therefore the claim was proved in the case $1 < p^2 \le n$.

Case 2: $p^2 > n$. Let $R > 0$ such that $\Omega \subset B(0,R)$. Notice that

$$K_3(\epsilon) = \int_\Omega u_\epsilon^p\ dx \le c\epsilon^{(n-p)/p}\int_0^R \frac{r^{n-1}}{(\epsilon + r^{p/(p-1)})^{n-p}}dr.$$

Consequently,

$$K_3(\epsilon) = O(\epsilon^{(n-p)/p}). \tag{3.55}$$

Choosing $0 < a \le A < \infty$ such that $a|x'|^2 \le h(x') \le A|x'|^2$ for $x' \in D(0,\delta)$, we have

$$\begin{aligned}K_1(\epsilon) = \int_\Omega |\nabla u_\epsilon|^p dx &= \int_{I\!R^n_+} |\nabla u_\epsilon|^p\ dx - \int_{D(0,\delta)} dx' \int_0^{h(x')} |\nabla u_\epsilon|^p dx_n + O(\epsilon^{(n-p)/p})\\ &= \frac{K_1}{2} - \int_{D(0,\delta)} dx' \int_0^{h|x'|^2} |\nabla u_\epsilon|^p dx_n + O(\epsilon^{(n-p)/p})\\ &\le \frac{K_1}{2} - \int_{D(0,\delta)} dx' \int_0^{a|x'|^2} |\nabla u_\epsilon|^p dx_n + O(\epsilon^{(n-p)/p})\end{aligned}$$

Using that $|x|^{p/(p-1)} \geq |x'|^{p/(p-1)}$, we have

$$\int_{D(0,\delta)} dx' \int_0^{a|x'|^2} |\nabla u_\epsilon|^p dx_n \geq \epsilon^{(n-p)/p} \int_{D(0,\delta)} dx' \int_0^{a|x'|^2} \frac{|x'|^{p/(p-1)}}{(\epsilon + |x|^{p/(p-1)})^n} dx_n. \quad (3.56)$$

For $\delta \in (0,1)$, we have $\epsilon + |x|^{p/(p-1)} \leq C(\epsilon + |x'|^{p/(p-1)})$. Consequently

$$\epsilon^{(n-p)/p} \int_{D(0,\delta)} dx' \int_0^{a|x'|^2} \frac{|x'|^{p/(p-1)}}{(\epsilon + |x|^{p/(p-1)})^n} dx_n \geq c_1 \epsilon^{(n-p)/p} \int_{D(0,\delta)} \frac{a|x'|^2 |x'|^{p/(p-1)}}{(\epsilon + |x'|^{p/(p-1)})^n} dx' \quad (3.57)$$

Now, observe that

$$\begin{aligned}
\epsilon^{(n-p)/p} \int_{D(0,\delta)} \frac{a|x'|^2|x'|^{p/(p-1)}}{(\epsilon + |x'|^{p/(p-1)})^n} dx' &= \epsilon^{(n-p)/p} \int_0^\delta \frac{r^2 r^{p/(p-1)} r^{n-1}}{(\epsilon + r^{p/(p-1)})^n} dr \\
&= \epsilon^{(n-p)/p} \epsilon^{(2p-n-1)/p} \int_0^{\delta/\epsilon^{(p-1)/p}} \frac{s^{p/(p-1)+n}}{(1 + s^{p/(p-1)})^n} ds \\
&\geq \epsilon^{(n-p)/p} \epsilon^{(2p-n-1)/p} \int_1^{\delta/\epsilon^{\frac{p-1}{p}}} \frac{s^{p/(p-1)+n}}{(1 + s^{p/(p-1)})^n} ds \\
&\geq \epsilon^{(n-p)/p} \epsilon^{(2p-n-1)/p} \int_1^{\delta/\epsilon^{(p-1)/p}} \frac{1}{(1 + s^{p/(p-1)})^n} ds \\
&\geq c_2 \epsilon^{(n-p)/p} \epsilon^{(2p-n-1)/p} \int_1^{\delta/\epsilon^{(p-1)/p}} \frac{1}{s^{pn/p-1}} ds,
\end{aligned}$$

where in the last inequality above we have used the fact that $1 + s^{p/(p-1)} \leq s^{p/(p-1)} + s^{p/(p-1)}$. Setting

$$f(\epsilon) := \epsilon^{(2p-n-1)/p} \int_1^{\delta/\epsilon^{(p-1)/p}} \frac{1}{s^{pn/p-1}} ds,$$

we have

$$K_1(\epsilon) \leq \frac{1}{2} K_1 - c_2 \epsilon^{(n-p)/p} f(\epsilon) + O(\epsilon^{(n-p)/p}). \quad (3.58)$$

To estimate $K_2(\epsilon)$, notice that

$$\begin{aligned}
K_2(\epsilon) &= \frac{1}{2} K_2 - \int_{D(0,\delta)} dx' \int_0^{h(x')} u_\epsilon^{p^*} dx_n + O(\epsilon^{n/p}) \\
&\geq \frac{1}{2} K_2 - \int_{D(0,\delta)} dx' \int_0^{A|x'|^2} u_\epsilon^{p^*} dx_n + O(\epsilon^{n/p})
\end{aligned}$$

and

$$\begin{aligned}\int_{D(0,\delta)} dx' \int_0^{A|x'|^2} u_\epsilon^{p^*} dx &\leq A\epsilon^{n/p} \int_{D(0,\delta)} \frac{|x'|^2}{(\epsilon + |x'|^{p/(p-1)})^n} dx' \\ &= A\epsilon^{n/p} \int_0^\delta \\ &= O(\epsilon^{n/p}).\end{aligned}$$

Thus

$$K_2(\epsilon) \geq \frac{1}{2}K_2 - O(\epsilon^{(n-p)/p}). \tag{3.59}$$

Let t_ϵ be such that

$$\max_{t>0} J_\lambda(tu_\epsilon) = J_\lambda(t_\epsilon u_\epsilon).$$

From (3.55) – (3.59) we conclude that t_ϵ is uniformly bounded for $\epsilon \in (0, \epsilon_o)$. Thus,

$$\begin{aligned}J_\lambda(t_\epsilon u_\epsilon) &\leq \sup_{t>0}\{\frac{1}{p}t^p K_1(\epsilon)t^p - \frac{1}{p^*}t^{p^*}K_2(\epsilon)\} + O(\epsilon^{p/(p-1)}) \\ &= \frac{1}{n}\Big(\frac{K_1(\epsilon)}{K_2(\epsilon)^{p/(p-1)}}\Big)^{n/p} + O(\epsilon^{p/(p-1)}).\end{aligned}$$

Now we claim that

$$\frac{K_1(\epsilon)}{K_2(\epsilon)^{p/(p-1)}} < 2^{-p/n}S - O(\epsilon^{p/(p-1)}), \tag{3.60}$$

for ϵ small(that is sufficiently to show (3.46)). Indeed by (3.58)-(3.59), we see that (3.60) is equivalent to

$$\begin{aligned}\tfrac{K_1}{2} - c_o\epsilon^{p/(p-1)}f(\epsilon) &< 2^{-p/n}S\Big(\frac{1}{2}K_2 - O(\epsilon^{p/(p-1)})\Big)^{n/p} + O(\epsilon^{p/(p-1)}) \\ &= \tfrac{1}{2}SK_2^{(n-p)/n} + O(\epsilon^{p/(p-1)}).\end{aligned}$$

Since $S = K_1/K_2^{n/p}$ we have that

$$\frac{K_1}{2} - c_o\epsilon^{p/(p-1)}f(\epsilon) < \frac{1}{2}K_1 + O(\epsilon^{p/(p-1)}), \tag{3.61}$$

because $\lim_{\epsilon\to 0} f(\epsilon) = \infty$. Therefore, (3.60) is true. Thus (3.46) is holds in the case $p^2 > n$.

Finally we are going to prove (3.35). To this, notice that

$$\begin{aligned} K_{1,\gamma}(\epsilon) &= \int_\Omega |\nabla u_\epsilon|^\gamma \, dx \\ &= (\frac{n-p}{p-1})^\gamma \epsilon^{(n-p)\gamma/p^2} \int_\Omega \frac{|x|^{\gamma/p-1}}{(\epsilon + |x|^{p/(p-1)})^{n\gamma/p}} \, dx \\ &= C\epsilon^\alpha \int_\Omega \frac{|x|^{\gamma/p-1}}{(1+|x|^{p/(p-1)})^{n\gamma/p}} \, dx \\ &= O(\epsilon^\alpha), \end{aligned}$$

where $\alpha = (n-p)\gamma/p^2 + \gamma p - n\gamma/p + (p-1)n/p$. On the outer hand, if $r > 1$ we have

$$\begin{aligned} K_{2,r}(\epsilon) : &= \int_\Omega u_\epsilon^r \, dx \\ &= \epsilon^{(n-p)r/p^2 - r(n-p)/p + (p-1)n/p} \int_\Omega \frac{1}{(1+|x|^{p/(p-1)})^{r(n-p)/p}} \, dx \\ &= O(\epsilon^{(n-p)r/p^2 - r(n-p)/p + (p-1)n/p}). \end{aligned}$$

Taking $r = p-1$ and $r = 1$, we obtain respectively, β and δ. Since $v_\lambda \in L^\infty(\overline{\Omega})$, we have

$$\Big| \int_\Omega u_\epsilon^{p^*-1} v_\lambda \Big| \, dx \le \int_\Omega u_\epsilon^{p^*-1} \, dx = O(\epsilon^{(n-p)r/p^2 - r(n-p)/p + (p-1)n/p}),$$

with $r = p^* - 1$. Thus, we obtain η. ∎

Finishing the proof of Proposition 3.3: If $p \in [2,3)$, fixe $\epsilon_0 > 0$ and consider the function $h : [0,+\infty) \times [0,\epsilon_0) \to I\!R$ defined by $h_\lambda(t,\epsilon) = F_\lambda(t,\epsilon) + G_\lambda(t,\epsilon)$.

From (3.45) and (3.34), there exists $C_1 > 0$ and $C_2 > 0$ such that

$$h_\lambda(t,\epsilon) \le C_1(t^p + t^\gamma + t^{p-1} + t) - C_2 t^{p^*-1}.$$

Since $\gamma < p^*$, there exists $t_0 > 0$ such that $t_\epsilon \le t_0$ for all $0 < \epsilon \le \epsilon_0$, where $h(t_\epsilon, \epsilon) = \max_{t\ge 0} h(t,\epsilon)$. Thus,

$$h_\lambda(t,\epsilon) \le h_\lambda(t_0,\epsilon) = F_\lambda(t_0,\epsilon) + G_\lambda(t_0,\epsilon) \le \max_{t\ge 0} F_\lambda(t,\epsilon) + G_\lambda(t_0,\epsilon).$$

From (3.34) we obtain $G(t_0,\epsilon) = O(\epsilon^\theta)$ for some $\theta > 0$. Thus, we obtain from (3.34) that

$$h_\lambda(t,\epsilon) < \frac{1}{2n} S^{n/p}.$$

Noting that $u \in C^{1,\alpha}(\overline{\Omega})$ (see Lieberman [13]) we obtain $u, \nabla u \in L^\infty$. Thus, the cases $1 < p < 2$ and $3 \le p$ follows using the same argument as in Azorero-Peral [10]. This completes the proof of Proposition 3.3. ∎

3.3 Proof of Theorem 1.2

Until this moment we have proved the existence of a local minimum v_λ of energy functional J_λ and we are ready to prove the existence of a second critical point of J_λ, which is of mountain pass type. Indeed, in view of Lemma 3.2 we can apply the Mountain-Pass Theorem to obtain a sequence (w_n) in $W^{1,p}(\Omega)$ such that $J_\lambda(w_n) \to c(v_\lambda)$ and $J'_\lambda(w_n) \to 0$ in $W^{-1,p'}(\Omega)$. Now, we consider two cases.

Subcritical case: $p - 1 < q < p^* - 1$. In this case, since the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact the result follows easily.

Critical case: $q = p^* - 1$. Here we going to proof that J_λ satisfies the $(PS)_{c(v_\lambda)}$ condition, or exists one solution w_λ such that

$$J_\lambda(w_\lambda) < J_\lambda(v_\lambda).$$

Since

$$\frac{1}{p}\int_\Omega |\nabla w_n|^p + \lambda |w_n|^p - \frac{1}{p^*}\int_\Omega w_n^{p^*} - \int_{\partial\Omega} w_n \varphi = o_n(1) + c(v_\lambda)$$

and

$$\int_\Omega |\nabla w_n|^p + \lambda |w_n|^p - \int_\Omega w_n^{p^*} - \int_{\partial\Omega} w_n \varphi = o_n(1)\|w_n\|_{1,p},$$

by Sobolev embedding and Holder's inequality, we obtain

$$(\frac{1}{p} - \frac{1}{p^*})\|w_n\|_{1,p}^p \le c(v_\lambda) + (o_n(1) + C_1\|\varphi\|_{L^{p'}(\partial\Omega)})\|w_n\|_{1,p},$$

consequently, (w_n) is bounded in $W^{1,p}(\Omega)$. Thus, we may extract a subsequence still denoted by (w_n) such that

$$\begin{aligned} w_n &\rightharpoonup w, \quad \text{weakly in } W^{1,p}(\Omega);\\ w_n &\to w, \quad \text{strongly in } L^p(\Omega);\\ w_n &\to w, \quad \text{a.e. on } \Omega. \end{aligned}$$

By a convergence result due to Lucio-Bocardo (see Theorem 2.1 in [17]) we have $\nabla w_n \to \nabla w$ almost everywhere in Ω. Using this and standard argument yield that w must be a critical point of J_λ. We observe that $w \neq 0$. In fact, by the definition of the weakly solution, we obtain

$$\lambda \int_\Omega w_n^{p-1}\, dx = \int_\Omega w_n^q\, dx + \int_{\partial\Omega} \varphi\, d\sigma_y.$$

Making $n \to +\infty$, we get a contradiction.

We shall have established the Theorem 1.2 if we prove the following:

Claim 3.2 *$J_\lambda(w) = c(v_\lambda)$, or $J_\lambda(w) < J_\lambda(v_\lambda)$.*

Applying the Brezis-Lieb Lemma, we obtain

$$\|\nabla w_n\|^p = \|\nabla w\|^p + \|\nabla(w_n - w)\|^p + o_n(1), \tag{3.62}$$

and

$$\|w_n\|^{p^*}_{L^{p^*}} = \|w\|^{p^*}_{L^{p^*}} + \|w_n - w\|^{p^*}_{L^{p^*}} + o_n(1). \tag{3.63}$$

From (3.62) and (3.63) we have

$$\frac{1}{p}\|w_n - w\|^p - \frac{1}{p^*}\|w_n - w\|^{p^*} + J_\lambda(w_n) = c(v_\lambda) + o_n(1). \tag{3.64}$$

$$\|w_n - w\|^p - \|w_n - w\|^{q+1}_{L^{p^*}} + J'_\lambda(w)w = J'_\lambda(w_n)w_n + o_n(1). \tag{3.65}$$

Substituting (3.65) in (3.64) results that

$$o_n(1) + c(v_\lambda) = J_\lambda(w) + (\frac{1}{p} - \frac{1}{p^*})\|w_n - w\|^{p^*}_{L^{p*}}, \tag{3.66}$$

or let, $\|w_n - w\|^{p^*}_{L^{p^*}} \to l \geq 0$. If $l = 0$ the proof is finished. If not, $l > 0$. By Sobolev inequality, we get

$$\|w_n - w\|_{L^{p^*}} \leq S\|w_n - w\|_{1,p}.$$

Thus,

$$l \geq S^{n/p}.$$

Returned to (3.66) we obtain

$$\begin{aligned} c(v_\lambda) &= J_\lambda(w) + (\frac{1}{p} - \frac{1}{p^*})\|w_n - w\|^{p^*}_{L^{p*}} - o_n(1) \\ &= J_\lambda(w) + (\frac{1}{p} - \frac{1}{p^*})l \\ &\geq J_\lambda(w) + \frac{1}{n}S^{n/p} \\ &> J_\lambda(w) + \frac{1}{2n}S^{n/p}. \end{aligned} \tag{3.67}$$

Since

$$c(v_\lambda) < J_\lambda(v_\lambda) + \frac{1}{2n}S^{n/p},$$

we concludes the Claim 3.2. This finishes the proof of Theorem 1.2.

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