NoDEA
Nonlinear differ. equ. appl. 13 (2006) 167–192
1021–9722/06/020167–26
DOI 10.1007/s00030-005-0034-3

Nonlinear Differential Equations
and Applications NoDEA

On a Schrödinger equation with periodic potential and critical growth in $\mathbb{R}^2$

João Marcos do Ó*
Dep. de Matemática
Univ. Fed. Paraíba
58059.900 João Pessoa, Pb, Brazil
jmbo@mat.ufpb.br

and

Bernhard RUF
Dip. di Matematica
Università degli Studi
20133 Milano, Italy
ruf@mat.unimi.it

Abstract. The main purpose of this paper is to establish the existence of a solution of the semilinear Schrödinger equation

$$-\Delta u + V(x)u = f(u), \text{ in } \mathbb{R}^2$$

where V is a 1-periodic function with respect to x, 0 lies in a gap of the spectrum of $-\Delta + V$, and $f(s)$ behaves like $\pm\exp(\alpha s^2)$ when $s \to \pm\infty$.

2000 *Mathematics Subject Classification*: 35J60, 35J20, 35Q55.
Key words: Indefinite elliptic operator, Schrödinger equation, elliptic equations, critical growth, linking theorem, unbounded domain.

1 Introduction

This paper has been motivated by some recent works concerning the existence of a solution of the semilinear Schrödinger equation

$$-\Delta u + V(x)u = f(u), \text{ in } \mathbb{R}^N \tag{1.1}$$

*Partially supported by CNPq, PRONEX-MCT/Brazil, and Millennium Institute for the Global Advancement of Brazilian Mathematics - IM-AGIMB

where V is periodic with respect to x. First we would like to mention the progress involving subcritical nonlinearities and the so-called definite case, i.e., when $V(x)$ is a positive potential bounded away from zero. Using the Nehari variational principle, Pankov in [18] has proved an existence theorem for ground states, i.e., solutions having lowest energy among all nontrivial solutions. In [21], Rabinowitz has obtained the existence of a nontrivial solution under less restrictive assumptions on $f(s)$, based on an approximation technique with periodic functions. Coti Zelati and Rabinowitz in [6] have proved the existence of infinitely many solutions.

In the indefinite case, the operator $-\Delta + V$ on $L^2(\mathbb{R}^N)$ has a purely continuous spectrum consisting of closed disjoint intervals. Supposing that 0 lies in a gap of the spectrum of $-\Delta + V$, Troestler and Willem [23], and Kryszewski and Szulkin [15], have recently proved the existence of a nontrivial solution under the assumption that the nonlinearity $f(s)$ is superlinear and subcritical. Their proofs are based on variational methods; in particular, after decomposing the space $H^1(\mathbb{R}^N)$ into two infinite-dimensional subspaces, a generalized linking theorem is applied to the functional corresponding to equation (1.1). This approach has been simplified by Pankov and Pflüger in [19] by using the approximation technique with periodic functions. Using this approach Chabrowski and Yang in [5] have proved the existence of a nontrivial solution of the semilinear Schrödinger equation

$$\begin{aligned} -\Delta u + V(x)u &= |u|^{2^*-2}u + g(u), \text{ for } \quad x \in \mathbb{R}^N; \\ u(x) &\to 0 \qquad\qquad\qquad\quad \text{as} \quad |x| \to \infty, \end{aligned}$$

where $N \geq 4$, $2^* = 2N/(N-2)$ is the critical Sobolev exponent and the nonlinearity $g(s)$ is superlinear and subcritical.

In this paper we consider the two dimensional case. To be more precise, we deal with a semilinear elliptic problem of the form

$$-\Delta u + V(x)u = f(u), \text{ in } \mathbb{R}^2 \tag{1.2}$$

where $f(s)$ has the maximal growth in s which allows to treat the problem variationally in $H^1(\mathbb{R}^2)$, that is, the so-called Trudinger-Moser case. There are some technical difficulties in proving existence results for such kind of problems. The associated functional for such problems on $H^1(\mathbb{R}^2)$ is in general strongly indefinite near the origin. Furthermore, it is not clear whether the Palais-Smale condition holds, because of the unboundedness of the domain and the fact that the embedding of the Sobolev space $H^1(\mathbb{R}^2)$ into spaces $L^p(\mathbb{R}^2)$ $(2 \leq p < \infty)$ as well as into the Orlicz space associated to the function $\phi(s) = \exp(4\pi s^2) - 1$ is not compact. Problems involving this notion of criticality have been investigated recently, among others, in [4, 8, 9, 10], for semilinear elliptic equations, and in [1, 11, 12, 13] for quasilinear equations. In this paper we show that the approximation technique with periodic functions in combination with some of the ideas contained in [9] and [12, 13] can be used to overcome the difficulties arising from lack of compactness of the energy functional corresponding to equation (1.2).

For easy reference we state now the assumptions that will be assumed in our main result.

(A_1) $V \in C(\mathbb{R}^2, \mathbb{R})$ is a 1−periodic function in x_1 and x_2;

(A_2) 0 is in a spectral gap of the operator $-\Delta + V$;

(A_3) $f \in C(\mathbb{R})$, and there exists $\mu > 2$ such that

$$0 \le \mu F(s) := \mu \int_0^s f(t)dt \le s\, f(s), \quad \forall\, s \in \mathbb{R},$$

and there exists $\mu_1 > 0$ and $s_1 \ge 0$ such that

$$0 < F(s) \le \mu_1 |f(s)|, \ \forall\, |s| > s_1;$$

(A_4) f has critical growth, namely there exists $\alpha_0 > 0$ such that

$$\lim_{|s|\to\infty} \frac{|f(s)|}{\exp(\alpha s^2)} = +\infty, \ \forall\, \alpha < \alpha_0, \ \text{and} \ \lim_{|s|\to\infty} \frac{f(s)}{\exp(\alpha s^2)} = 0, \ \forall\, \alpha > \alpha_0$$

(A_5) for every $M > 0$ there exists $s_M > 0$ such that

$$s\, f(s) \ge M\ \exp(\alpha_0 s^2) \ \text{ for } |s| \ge s_M.$$

(A_6) $f(s) = o(|s|)$, for s near 0.

Example 1.1 The function $f(s) = \text{sign}(s)(e^{s^2} - 1)$ satisfies assumptions (A_3)–(A_5), with $\alpha_0 = 1$; for the proof, see Propostion 3.4 below.

The main result of this paper is the following

Theorem 1.2 *Under assumptions* $(A_1) - (A_6)$ *there exists a nontrivial solution* $u \in H^1(\mathbb{R}^2)$ *of* (1.2).

The underlying idea for proving Theorem 1.2 is to show that for each $k \in \mathbb{N}$ sufficiently large, there is a nontrivial solution u_k of (1.2) which is k−periodic in x_1 and x_2. The existence of u_k and some estimates will be proved in Section 2 and will follow from a version of the so-called generalized mountain-pass theorem without the Palais-Smale condition. This abstract minimax result is a consequence of the Ekeland variational principle. In Section 3, using further estimates we show that as $k \to \infty$, a subsequence of u_k converges to a solution u of (1.2). Finally, additional arguments prove that u is nontrivial.

2 Approximation by a periodic problem

2.1 The periodic problem

We shall start by recalling an abstract minimax result which is a consequence of the Ekeland variational principle (see [7]).

Theorem 2.1 *Let $X = Y \oplus Z$ be a Banach space with $\dim Y < \infty$. Let $e \in \partial B_1(0) \cap Z$ be fixed and let $\rho < R$ be given positive real numbers. Let*

$$D = \{u = y + re \ : \ \|y\| \leq R, \ 0 < r < R\}.$$

Let $I : X \to \mathbb{R}$ be a C^1 functional such that

$$b = \inf_{Z \cap \partial B_\rho} I > \max_{\partial D} I = a.$$

Then there exists a Palais-Smale sequence, that is, (u_n) in X such that

$$I(u_n) \to c > -\infty \quad \text{and} \quad I'(u_n) \to 0,$$

with

$$c = \inf_{\gamma \in \Gamma} \max_{u \in D} I(\gamma(u))$$

where

$$\Gamma = \{\gamma \in C(D, X) : \gamma(u) = u, \ u \in \partial D\}.$$

With the help of the above theorem we prove the existence of solutions of the problem:

$$-\Delta u + V(x)u = f(u), \text{ in } Q_k, \ u \in E_k := H^1_{\text{per}}(Q_k), \tag{P_k}$$

where $Q_k \subset \mathbb{R}^2$ is a cube with edge length $k \in \mathbb{N}$ and $H^1_{\text{per}}(Q_k)$ denotes the space of $H^1(Q_k)-$functions which are $k-$periodic in x_1 and x_2.

It is known that the operator $-\Delta + V$ on $L^2_{\text{per}}(Q_k)$ has a discrete spectrum with eigenvalues $\lambda_{k,1} \leq \lambda_{k,2} \leq \cdots \leq \lambda_{k,i} \leq \ldots$ diverging to $+\infty$ as $i \to \infty$. Moreover, for each k, every eigenvalue $\lambda_{k,i}$ is contained in the spectrum of $-\Delta + V$ in the whole space $L^2(\mathbb{R}^2)$ and the following minima

$$\gamma(k) = \min\{i : \lambda_{k,i} > 0\} \tag{2.3}$$

are finite. This follows from spectral decomposition, for details see [22]. In particular, it follows that if $(-a, b)$, $a, b > 0$, denotes the spectral gap around 0, assumed in (A_2), then $\lambda_{k,i} \notin (-a, b)$ for every $k, i \in \mathbb{N}$. We denote by $\phi_{k,i}$ the corresponding eigenfunctions. Notice that $E_k \subset E_{mk}$ for all $m \in \mathbb{N}$, because of the periodicity. Consequently, every eigenvalue of $L = -\Delta + V$ on $L^2_{\text{per}}(Q_k)$ is also an eigenvalue of this operator in $L^2_{\text{per}}(Q_{mk})$ for all $m \in \mathbb{N}$. We define an orthogonal decomposition of E_k by

$$E_k = Y_k \oplus Z_k, \quad \text{where} \quad Y_k = \text{span}\{\phi_{k,1}, \dots, \phi_{k,\gamma(k)-1}\}.$$

The solutions of problem (P_k) will be found as critical points of the energy functional given by

$$J_k(u) = \frac{1}{2}\int_{Q_k} (|\nabla u|^2 + V(x)u^2)dx - \int_{Q_k} F(u)dx, \quad u \in E_k.$$

By $\ell_k : E_k \to \mathbb{R}$ we denote the quadratic part of the energy functional J_k, that is,

$$\ell_k(u) = \frac{1}{2}\int_{Q_k} (|\nabla u|^2 + V(x)u^2)dx.$$

Notice that the quadratic part ℓ_k is positive on Z_k and negative on Y_k. Also we define a new scalar product $(\cdot,\cdot)_k$ on E_k with corresponding norm $\|\cdot\|_k$ such that

$$\int_{Q_k} (|\nabla u|^2 + V(x)u^2)dx = \begin{cases} -\|u\|_k^2 \text{ if } u \in Y_k, \\ \ \ \|u\|_k^2 \text{ if } u \in Z_k. \end{cases}$$

Denoting by $S_k : E_k \to Y_k$ and $T_k : E_k \to Z_k$ the orthogonal projections, the energy functional $J_k(u)$ becomes

$$J_k(u) = \frac{1}{2}(\| T_k u \|_k^2 - \| S_k u \|_k^2) - \int_{Q_k} F(u)dx, \quad u \in E_k.$$

By the assumptions (A_1) and (A_4), the functional J_k is a well defined $C^1(E_k)$ functional with Fréchet derivative given by

$$\langle J_k'(u), v\rangle = (T_k u, v)_k - (S_k u, v)_k - \int_{Q_k} f(u)v dx, \quad u \in E_k.$$

These statements are standard (see e.g. [6], [20]), taking into account that for any strongly convergent sequence $(u_n) \subset E_k$ there is a subsequence (u_{n_j}) and $h \in E_k$ such that $|u_{n_j}(x)| \le h(x)$ almost every where in $\mathbb{R}^2$, and the following Trudinger-Moser type inequality, see [4, 13].

Lemma 2.2 *If $u \in H^1(\mathbb{R}^2)$ and $\alpha > 0$, then*

$$\int_{\mathbb{R}^2} [\exp(\alpha u^2) - 1]dx < \infty.$$

Moreover, if $\|\nabla u\|_{L^2} \le 1$, $\|u\|_{L^2} \le M$ and $\alpha < 4\pi$, then there exists a constant $C = C(\alpha, M)$, which depends only on α and M, such that

$$\int_{\mathbb{R}^2} [\exp(\alpha u^2) - 1]dx \le C.$$

For the proof of the next result we refer to Lemma 2 in [19].

Lemma 2.3 *The norm* $\|\cdot\|_k$ *is equivalent to the standard norm* $\|\cdot\|_{H^1}$ *in* $H^1(Q_k)$*,*

$$a\|u\|_k \leq \|u\|_{H^1} \leq b\|u\|_k, \quad \forall u \in E_k,$$

where a *and* b *are positive constants independent of* k*.*

2.2 Behavior of J_k near the origin

We first study the behavior of the functional J_k near the origin in Z_k.

Lemma 2.4 *There exist constants* $\rho > 0$ *and* $\sigma > 0$ *independent of* k *such that* $\inf_{u \in N_k} J_k(u) \geq \sigma$*, where* $N_k = \{z \in Z_k : \|z\|_k = \rho\}$*.*

Proof. Let $z \in Z_k$, then

$$J_k(z) = \frac{1}{2}\|z\|_k^2 - \int_{Q_k} F(u)dx.$$

Assumptions (A_6) implies that $F(s) = o(|s|^2)$ for s near 0, and using (A_4), we have that for every $\epsilon > 0$, $\beta > \alpha_0$ and $q > 2$ there exists a constant $C_1 = C_1(\epsilon, \beta, q) > 0$ such that

$$F(s) \leq \epsilon s^2 + C_1|s|^q[\exp(\beta s^2) - 1], \quad \forall\ s \in \mathbb{R}. \tag{2.4}$$

To proceed further we make use of the following inequality (to be proved later).

Claim 2.5 *There exist constants* $\rho_0 > 0$ *and* $C_2 > 0$ *independent of* k *such that*

$$\int_{Q_k} |u|^q[\exp(\beta u^2) - 1]dx \leq C_2\|u\|_{H^1(Q_k)}^q, \tag{2.5}$$

for all $u \in H^1(Q_k)$ *with* $\|u\|_{H^1(Q_k)} \leq \rho_0$*.*

Thus, applying Lemma 2.3, (2.4)–(2.5) and the Sobolev embedding theorem,

$$\int_{Q_k} F(z)dx \leq C_3(\epsilon\|z\|_k^2 + C_4\|z\|_k^q),$$

for some positive constants C_3 and C_4 independent of k. Consequently

$$J_k(z) \geq \frac{1}{2}\|z\|_k^2 - C_3(\epsilon\|z\|_k^2 + C_4\|z\|_k^q).$$

Choosing $\epsilon > 0$ and $\rho > 0$ sufficiently small, the result follows.

Verification of Claim 2.5: we may assume that $u \geq 0$, since we can replace u by $|u|$ without causing any increase in the integral of the gradient. We shall use Schwarz symmetrization method (cf. [14]). Let u^* be the symmetrization of u, then it is well known that u^* depends only on $|x|$ and u^* is a decreasing function of $|x|$. Furthermore, for all $u \in H^1(Q_k)$, we have $R_k > 0$ such that $|Q_k| = |B_{R_k}|$, and

$$\int_{Q_k} |u|^r dx = \int_{B_{R_k}} |u^*|^r dx, \quad \text{for } 1 < r < \infty,$$

$$\int_{Q_k} |\nabla u|^2 dx \geq \int_{B_{R_k}} |\nabla u^*|^2 dx,$$

$$\int_{Q_k} |u|^q [\exp(\beta u^2) - 1] dx = \int_{B_{R_k}} |u^*|^q [\exp(\beta u^{*2}) - 1] dx.$$

Next, we use a continuous radial extension $P : H^1_{\text{rad}}(B_{R_k}) \to H^1_{\text{rad}}(\mathbb{R}^2)$, such that for all $v \in H^1_{\text{rad}}(B_{R_k})$,

1. $Pv|_{B_{R_k}} = v$;
2. $\|Pv\|_{L^2(\mathbb{R}^2)} \leq c\|v\|_{L^2(B_{R_k})}$;
3. $\|Pv\|_{H^1(\mathbb{R}^2)} \leq c\|v\|_{H^1(B_{R_k})}$,

where $c > 0$ does not depend on k. The construction of P can be done as follows: noting that $d := \min_{[0,R_k]} u^*(r) = u^*(R_k)$, we continue $u^*(r)$ by $d(R_k + 1 - r)$ on $[R_k, R_k + 1]$, and then by zero on $[R_k + 1, +\infty)$. Easy calculations yield the stated properties.

Thus, we have

$$\begin{aligned}
\int_{Q_k} |u|^q [\exp(\beta u^2) - 1] dx &= \int_{B_{R_k}} |u^*|^q [\exp(\beta |u^*|^2) - 1] dx \\
&\leq \int_{\mathbb{R}^2} |Pu^*|^q [\exp(\beta |Pu^*|^2) - 1] dx \\
&\leq \int_{\{|x| \leq R_0\}} |u^*|^q \exp(\beta |u^*|^2) dx + \int_{\{|x| > R_0\}} |Pu^*|^q [\exp(\beta |Pu^*|^2) - 1] dx
\end{aligned} \tag{2.6}$$

where $0 < R_0 < R_k$ is a number to be determined later. Using the Hölder inequality, we estimate the first term as

$$\int_{B_{R_0}} |u^*|^q \exp(\beta |u^*|^2) dx \leq \left(\int_{B_{R_0}} \exp(r\beta |u^*|^2) dx \right)^{1/r} \left(\int_{B_{R_0}} |u^*|^{qs} dx \right)^{1/s},$$

where $1/r + 1/s = 1$ with s such that $qs = 2^*$. Let $v(x) = u^*(x) - u^*(R_0)$; then $v \in H_0^1(B_{R_0})$ and we can estimate

$$\int_{B_{R_0}} |\nabla v|^2 dx = \int_{B_{R_0}} |\nabla u^*|^2 dx \leq \int_{B_{R_k}} |\nabla u^*|^2 dx \leq \int_{Q_k} |\nabla u|^2 dx \leq \|u\|^2_{H^1(Q_k)}.$$

If we now take

$$\|u\|^2_{H^1(Q_k)} \leq \rho_1 \quad \text{with} \quad r\beta\rho_1 \leq 2\pi, \tag{2.7}$$

then

$$\begin{aligned} \int_{B_{R_0}} \exp(r\beta|u^*|^2)dx &\leq \exp(2r\beta|u^*(R_0)|^2) \int_{B_{R_0}} \exp(2r\beta v^2)dx \\ &\leq \exp(2r\beta|u^*(R_0)|^2) \int_{B_{R_0}} \exp\left(2r\beta\rho_1 \frac{v^2}{\|\nabla v\|^2_{L^2}}\right) dx \\ &\leq C(R_0), \end{aligned}$$

in view of the Trudinger-Moser inequality, cf. [17].

Thus, using the continuous imbedding $H^1(\mathbb{R}^2) \hookrightarrow L^{qs}(\mathbb{R}^2)$, we get

$$\int_{B_{R_0}} |u^*|^q \exp(\beta|u^*|^2)dx \leq C(R_0)\|u\|^q_{H_1(Q_k)}, \tag{2.8}$$

where $C(R_0) > 0$ does not depend on k.

To estimate the second term in (2.6) we use the following Radial Lemma (cf. [3, Lemma A.IV.])

$$|Pu^*(x)| \leq \frac{\|Pu^*\|_{L^2(\mathbb{R}^2)}}{\sqrt{\pi}|x|}, \quad \text{for } x \neq 0. \tag{2.9}$$

Writing

$$\int_{\{|x|\geq R_0\}} |Pu^*|^q(\exp(\beta|Pu^*|^2) - 1)dx = \sum_{k=1}^{\infty} \int_{\{|x|\geq R_0\}} |Pu^*|^q \frac{\beta^k |Pu^*|^{2k}}{k!} dx$$

we can estimate the single terms by (2.9) and Hölder

$$\begin{aligned} &\int_{\{|x|\geq R_0\}} |Pu^*|^q |Pu^*|^{2k} dx \\ &\leq \left(\frac{\|Pu^*\|_{L^2(\mathbb{R}^2)}}{\sqrt{\pi}}\right)^{2k} \int_{|x|\geq R_0} \frac{|Pu^*|^q}{|x|^{2k}} dx \\ &\leq \left(\frac{\|Pu^*\|_{L^2(\mathbb{R}^2)}}{\sqrt{\pi}}\right)^{2k} \left(\int_{|x|\geq R_0} \frac{1}{|x|^{2kr}} dx\right)^{\frac{1}{r}} \|Pu^*\|^q_{L^{sq}(\mathbb{R}^2)} \\ &\leq \left(\frac{\pi}{rk-1}\right)^{1/r} \left(\frac{\|Pu^*\|_{L^2(\mathbb{R}^2)}}{\sqrt{\pi} R_0^{1-1/rk}}\right)^{2k} \|Pu^*\|^q_{L^{sq}(\mathbb{R}^2)} \end{aligned} \tag{2.10}$$

We have $\|Pu^*\|^2_{L^2(\mathbb{R}^2)} \leq c_1\|u^*\|^2_{L^2(B_{R_k})} = c_1\|u\|^2_{L^2(Q_k)} \leq c_2\|u\|^2_{H^1(Q_k)} \leq c_2\ \rho_1$, where ρ_1 is as in (2.7). Hence, choosing $R_0 > 1$ such that $R_0^{1-1/rk} \geq c\ \rho_1/\sqrt{\pi}$, $\forall\ k \geq 1$, the last expression in (2.10) is bounded by

$$C\ \|Pu^*\|^q_{H^1(Q_k)} \ \text{ with }\ C\ \text{ independent of }\ k$$

Hence

$$\int_{\{|x|\geq R_0\}} |Pu^*|^q[\exp(\beta|Pu^*|^2) - 1]dx \leq C\|u\|^q_{H^1(Q_k)}. \tag{2.11}$$

Finally, from estimates (2.8) and (2.11) we complete the proof of the claim. □

2.3 Behavior of J_k near infinity

Lemma 2.6 *Let Y be a finite dimensional subspace of E_k. Then J_k is bounded above in Y, and moreover, given $m > 0$ there is an $R > 0$ such that*

$$J_k(u) \leq -m, \quad \forall\ u \in Y \quad \textit{with} \quad \|u\| \geq R.$$

Proof. By assumption (A_3) it follows that there exists a positive constant c such that

$$F(s) \geq c\ |s|^\mu,\ \mu > 2, \quad \forall\ (x, s) \in Q_k \times \mathbb{R}.$$

Thus, given $u \in Y - \{0\}$, for all $t \geq 0$,

$$J_k(tu) \leq t^2\ell_k(u) - c\,t^\mu\|u\|^\mu_{L^\mu} + d_1,$$

which implies that $J_k(tu) \to -\infty$ as $t \to +\infty$. From this fact together with compactness we deduce easily the result. □

2.4 Uniform bound of the minimax-levels

In the following Lemma we set $L^2_k := L^2(Q_k)$ and $W^{2,2}_k := W^{2,2}(Q_k)$. Then we have

Lemma 2.7 *There exists a constant $C > 0$ (independent of k), such that*

$$\|y\|_k \leq C\|y\|_{L^2_k},\ \forall\ y \in Y_k \tag{2.12}$$

and

$$\|y\|_\infty \leq C\|y\|_{L^2_k},\ \forall\ y \in Y_k \tag{2.13}$$

Proof. First note that there exists a constant c (independent of k) such that

$$\|y\|_\infty \le c\|y\|_{W^{2,2}_k} \text{ and } \|y\|_k \le c\|y\|_{W^{2,2}_k}.$$

Next, note that there exists a constant c_1 (independent of k) such that

$$\|y\|_{W^{2,2}_k} \le c_1\| -\Delta y + y\|_{L^2_k}$$

Finally, we show that there exists a constant c_2 (independent of k) such that

$$\| -\Delta y + y\|_{L^2_k} \le c_2\|y\|_{L^2_k}.$$

Let $\phi_{k,i}$ denote as before the normalized eigenvectors of $L = -\Delta + V$ on E_k; note that for $0 \le i \le \gamma_k - 1$ the corresponding eigenvalues $\lambda_{k,i}$ satisfy $\sigma_0 \le \lambda_{k,i} \le -a$, where $\sigma_0 = \min \Sigma$ (the spectrum of $-\Delta + V(x)$ on $H^1(\mathbb{R}^2)$) and $-a$ the lower bound of the spectral gap around 0 of $-\Delta + V(x)$. Thus, we have

$$\begin{aligned} \| -\Delta y + V(x)y\|^2_{L^2_k} &= \left\| \sum_{i=1}^{\gamma_k - 1} \alpha_{k,i}\lambda_{k,i}\phi_{k,i} \right\|^2_{L^2_k} \\ &= \sum_{i=1}^{\gamma_k-1} \lambda^2_{k,i}\alpha^2_{k,i} \le \max_{1\le i\le \gamma_k - 1} |\lambda_{k,i}|^2 \sum \alpha^2_{k,i} \\ &\le c\, \|y\|^2_{L^2_k}, \ \forall\, y \in Y_k. \end{aligned} \tag{2.14}$$

Finally, we have

$$\| -\Delta y + y\|_{L^2_k} - \|V(x)y + y\|_{L^2_k} \le \| -\Delta y + V(x)y\|_{L^2_k} \le c\, \|y\|_{L^2_k} \tag{2.15}$$

which yields

$$\| -\Delta y + y\|_{L^2_k} \le c\, \|y\|_{L^2_k} + \|V(x)y + y\|_{L^2_k} \le (c + |V|_\infty + 1)\|y\|_{L^2_k} \le c_2\|y\|_{L^2_k} \tag{2.16}$$

□

In order to prove the linking condition required by Theorem 2.1, we consider the following sequence of nonnegative functions

$$\omega_n(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} (\log n)^{1/2} & \text{if } |x| \le 1/n \\ \log \frac{1}{|x|} \big/ (\log n)^{1/2} & \text{if } 1/n \le |x| \le 1 \\ 0 & \text{if } |x| \ge 1 \end{cases} \tag{2.17}$$

Notice that $\omega_n \in H^1(\mathbb{R}^2)$, $\operatorname{supp} \omega_n \subset B_1(0)$, $\|\nabla\omega_n\|_{L^2} = 1$ and $\|\omega_n\|_{L^2} = O(1/(\log n)^{1/2})$ as $n \to \infty$. Hereafter, without loss of generality, we suppose that $B_2(0) \subset Q_k$ for k suitably large.

From the next lemma we see that the orthogonal projection T_k of ω_n onto Z_k is a nontrivial map for all n sufficiently large.

Lemma 2.8 $\|T_k\omega_n\|_k^2 = 1 + O(\frac{1}{\log n})$, *for all* k *and all* n.

Proof. Let $L = -\Delta + V$ on E_k. We have by (2.14)

$$\begin{aligned} \|S_k\omega_n\|_k^2 = & \ |(\omega_n, LS_k\omega_n)_{L_k^2}| \le c\ \|\omega_n\|_{L_k^2}\|L\ S_k\omega_n\|_{L_k^2} \\ & \le c\ \|\omega_n\|_{L_k^2}\|S_k\omega_n\|_{L_k^2} \le c\ \|\omega_n\|_{L_k^2}^2, \end{aligned} \tag{2.18}$$

and thus

$$\|S_k\omega_n\|_k^2 \le c\ \|\omega_n\|_{L_k^2}^2 \le \frac{c}{\log n}. \tag{2.19}$$

Therefore

$$\begin{aligned} 1 + O\left(\frac{1}{\log n}\right) &= \int |\nabla\omega_n|^2 + V(x)\omega_n^2 = \|T_k\omega_n\|_k^2 - \|S_k\omega_n\|_k^2 \\ &= \|T_k\omega_n\|_k^2 + O\left(\frac{1}{\log n}\right), \end{aligned} \tag{2.20}$$

and the result follows. □

Let $\overline{\omega}_n^k := \frac{T_k\omega_n}{\|T_k\omega_n\|_k}$, and define for $n \ge n_0$ the sets

$$Q_k(n) = \{v + s\overline{\omega}_n^k\ :\ v \in Y_k,\ \|v\| \le n,\ 0 \le s \le n\} \tag{2.21}$$

We prove

Lemma 2.9 *Let* $k \to \infty$; *then there exists* $n_0 > 0$ *such that for all* $n \ge n_0$

$$\lim_{k\to\infty} \max\{J_k(u) : u \in Q_k(n)\} < \frac{2\pi}{\alpha_0}.$$

Proof. Assume by contradiction that this is not the case. Thus there exists a sequence $n \to \infty$ such that

$$\lim_{k\to\infty} \max\{J_k(u) : u \in Q_k(n)\} \ge \frac{2\pi}{\alpha_0}.$$

and hence there exists $\delta_k(n) \to 0^+$ as $k \to \infty$, with

$$\max\{J_k(u)\ :\ u \in Q_k(n)\} \ge \frac{2\pi}{\alpha_0} - \delta_k$$

For each k, let $u_n^k = v_n^k + t_n^k\overline{\omega}_n^k$ be the point where this maximum is achieved. So,

$$\frac{1}{2}(|t_n^k|^2 - \|v_n^k\|_k^2) - \int_{Q_k} F(u_n)dx \ge \frac{2\pi}{\alpha_0} - \delta_k$$

which together with (A_3) implies that

$$|t_n^k|^2 \geq \frac{4\pi}{\alpha_0} + \|v_n^k\|_k^2 - \delta_k \tag{2.22}$$

By Lemma 2.7 there exists a number d (independent of k) such that

$$d\,\|v_n^k\|_k \geq |v_n^k|_\infty \tag{2.23}$$

and hence by (2.22)

$$d\,|t_n^k| \geq |v_n^k|_\infty \tag{2.24}$$

Since $\langle (J_k|_{Y_k\oplus\mathbb{R}\overline{\omega}_n^k})'(u_n^k), u_n^k\rangle = 0$ we get

$$|t_n^k|^2 - \|v_n^k\|_k^2 - \int_{Q_k} f(u_n^k)u_n^k = 0 \tag{2.25}$$

which implies

$$|t_n^k|^2 \geq \int_{Q_k} f(u_n^k)u_n^k. \tag{2.26}$$

By (2.17) and Lemma 2.8 we have for $x \in B_{1/n}(0)$, setting $M_n = \frac{1}{\sqrt{2\pi}}\sqrt{\log n}$

$$\frac{1}{M_n}\overline{\omega}_n^k(x) = \frac{1}{M_n}\frac{T_k\omega_n(x)}{\|T_k\omega_n\|_k} = \frac{1}{\|T_k\omega_n\|_k} - \frac{1}{M_n}\frac{S_k\omega_n(x)}{\|T_k\omega_n\|_k} \geq 1 - \frac{c}{\log n}. \tag{2.27}$$

Using (2.24) and (2.27) we can now estimate $u_n^k(x)$ as follows, for large n and $x \in B_{1/n}(0)$,

$$\begin{aligned} u_n^k(x) &= t_n^k M_n \left[\frac{v_n^k(x)}{t_n^k M_n} + \frac{\overline{\omega}_n^k(x)}{M_n}\right] \\ &\geq t_n^k M_n \left[-\frac{|v_n^k|_\infty}{t_n^k M_n} + 1 - \frac{c}{\log n}\right] \\ &\geq t_n^k M_n \left[-\frac{d}{M_n} + 1 - \frac{c}{\log n}\right] \end{aligned} \tag{2.28}$$

Then

$$u_n^k(x) \geq t_n^k M_n[1-\epsilon(n)] \quad \text{with} \quad \epsilon(n) := \frac{d}{M_n} + \frac{c}{\log n} \to 0.$$

By (A_5) it follows that there exists $s_1 > 0$ such that $f(s)s \geq e^{\alpha_0 s^2}$, $\forall s \geq s_1$. So from (2.25) we obtain, for large k and any $n \geq n_0$

$$\begin{aligned} |t_n^k|^2 &\geq \int_{B_{r/n}(0)} \exp[\alpha_0(1-\epsilon(n))^2|t_n^k|^2 M_n^2]dx \qquad (2.29) \\ &= \pi\frac{r^2}{n^2}\exp[\alpha_0(1-\epsilon(n))^2|t_n^k|^2 M_n^2] \\ &\geq \pi r^2 \exp\left[2\log n\left(\alpha_0(1-\epsilon(n))^2\frac{|t_n^k|^2}{4\pi} - 1\right)\right] \end{aligned}$$

from which follows that t_n^k is bounded in k, for all $n \geq n_0$. Assume that $t_n^k \to t_n^0$. From the last estimate we conclude that $|t_n^0|^2 \leq \frac{4\pi}{\alpha_0} \frac{1}{(1-\epsilon(n))^2}$. This, in conjunction with (2.22), implies that

$$\frac{4\pi}{\alpha_0} \frac{1}{(1-\epsilon(n))^2} \;\geq\; t_0^2 \;\geq\; \frac{4\pi}{\alpha_0} - \delta_k.$$

We now improve the estimate (2.28):

Claim: There exists some $c > 1$ such that

$$b_n^k := \frac{|v_n^k|_\infty}{t_n^k M_n} \leq \frac{c}{\log n} + \sqrt{\delta_k} \tag{2.30}$$

If the claim were not true, then we would have by the second line of (2.28)

$$u_n^k(x) \geq t_n^k M_n \left[-\frac{|v_n^k|_\infty}{t_n^k M_n} + 1 - \frac{c}{\log n} \right] \geq t_n^k M_n \left[1 - 2b_n^k \right] \tag{2.31}$$

Instead of (2.29) we can now write

$$|t_n^k|^2 \geq \pi \frac{r^2}{n^2} \exp[\alpha_0 |t_n^k|^2 M_n^2 (1 - 2b_n^k)^2] \tag{2.32}$$

$$\geq \pi r^2 \exp \left[\left[|t_n^k|^2 \frac{\alpha_0}{4\pi} (1 - 4b_n^k) - 1 \right] 2 \log n \right] \tag{2.33}$$

We now consider two cases:

Case 1: $|t_n^k|^2 > \frac{4\pi}{\alpha_0} + \frac{20\pi}{\alpha_0} b_n^k$. Then

$$\begin{aligned} |t_n^k|^2 \frac{\alpha_0}{4\pi}(1 - 4b_n^k) - 1 &\geq \left(\frac{4\pi}{\alpha_0} + \frac{20\pi}{\alpha_0} b_n^k \right) \frac{\alpha_0}{4\pi}(1 - 4b_n^k) - 1 \\ &= 1 + b_n^k - 20|b_n^k|^2 - 1 \geq \frac{1}{2} b_n^k \end{aligned}$$

for n sufficiently large such that $20 b_n^k \leq 1/2$, which implies by (2.32) that

$$|t_n^k|^2 \geq \pi r^2 \exp[b_n^k \log n]$$

and hence $b_n^k \leq \frac{c}{\log n}$, i.e., the claim (2.30).

Case 2: $|t_n^k|^2 \leq \frac{4\pi}{\alpha_0} + \frac{20\pi}{\alpha_0} b_n^k$. Then we obtain together with (2.22)

$$-\delta_k + \frac{4\pi}{\alpha_0} + \|v_n^k\|_k^2 \leq |t_n^k|^2 \leq \frac{4\pi}{\alpha_0} + \frac{c|v_n^k|_\infty}{M_n} = \frac{4\pi}{\alpha_0} + \frac{c|v_n^k|_\infty}{\sqrt{\log n}}$$

and hence by (2.23)

$$-\delta_k + \frac{1}{d^2}|v_n^k|_\infty^2 \leq \frac{c|v_n^k|_\infty}{\sqrt{\log n}}$$

i.e.,

$$|v_n^k|_\infty \leq \frac{c\, d^2}{\sqrt{\log n}} + \sqrt{\delta_k} \tag{2.34}$$

Thus, the claim (2.30) is proved.

Using (2.30), the estimate (2.28) can now be improved: for $x \in B_{1/n}(0)$ we have by (2.22)

$$\begin{aligned} |u_n^k|^2(x) &\geq |t_n^k|^2 M_n^2 \left[1 - \frac{c}{\log n} - \sqrt{\delta_k}\right]^2 \\ &\geq \frac{2}{\alpha_0}\log n - d - c\sqrt{\delta_k}\log n, \end{aligned} \tag{2.35}$$

and thus, for $k(n)$ sufficiently large

$$|u_n^{k(n)}|^2(x) \geq \frac{2}{\alpha_0}\log n - (d+1).$$

By (A_5) we may choose a number $n_1 \geq n_0$ such that

$$f(s)\; s\; e^{-\alpha_0 s^2} \geq \beta := e^{d+1}\frac{8}{\alpha_0} \quad \text{for} \quad s \geq s_1.$$

Finally, we estimate more precisely (2.26): Let n_1 such that $\frac{2}{\alpha_0}\log n_1 d(d+1) \geq s_1^2$; then, for $n \geq n_1$, using (2.35), we get

$$\begin{aligned} |t_n^k|^2 &\geq \beta \int_{B_{1/n}(0)} e^{\alpha_0 |u_n^k|^2}\,dx \\ &\geq \beta e^{-d-1}\int_{B_{1/n}(0)} e^{2\log n}\,dx = \beta e^{-d-1}\pi \\ &= \frac{8\pi}{\alpha_0}. \end{aligned} \tag{2.36}$$

But this contradicts

$$\lim_{k\to\infty}|t_n^k|^2 \leq \frac{4\pi}{\alpha_0}\frac{1}{1-\epsilon(n)},$$

for n sufficiently large. Thus, Lemma 2.9 is proved. □

2.5 Palais-Smale sequences

In view of Lemmas 2.4, 2.6 and 2.9, we can apply Theorem 2.1 to obtain a Palais-Smale sequence $(u_m^k) \subset E_k$, i.e., satisfying

$$J(u_m^k) = \frac{1}{2}\int_{Q_k} [\mid \nabla u_m^k \mid^2 + V(x)(u_m^k)^2] dx - \int_{Q_k} F(u_m^k) dx \to c_k, \tag{2.37}$$

where

$$c_k = \inf_{\gamma \in \Gamma_k} \max_{u \in Q_k(n)} I(\gamma(u)) \in [\sigma, 2\pi/\alpha_0 - \delta),$$

and

$$\begin{aligned} |\langle J'(u_m^k), \phi\rangle| &= \left| \int_{Q_k} \left[\nabla u_m^k \nabla \phi + V(x) u_m^k \phi\right] dx - \int_{Q_k} f(u_m^k)\phi dx \right| \\ &\leq \epsilon_m \|\phi\|_k, \quad \forall\ \phi \in E_k, \end{aligned} \tag{2.38}$$

with $\epsilon_m \to 0$ as $m \to \infty$; we recall that $Q_k(n)$ is given in (2.21).

Proposition 2.10 *There is a positive constant C independent of k and m such that $\|u_m^k\|_k \leq C$.*

Proof. Setting $\phi = u_m^k$ in (2.38) we get

$$\left| \|T_k u_m^k\|_k^2 - \|S_k u_m^k\|_k^2 - \int_{Q_k} f(u_m^k) u_m^k dx \right| \leq \epsilon_m \|u_m^k\|_k, \tag{2.39}$$

while (2.37) can be written as

$$\|T_k u_m^k\|_k^2 - \|S_k u_m^k\|_k^2 - 2\int_{Q_k} F(u_m^k) dx = 2c_k + \delta_m^k, \tag{2.40}$$

with $\delta_m^k \to 0$ as $m \to +\infty$. From (2.39) and (2.40) we get

$$\left| 2\int_{Q_k} F(u_m^k) dx - \int_{Q_k} f(u_m^k) u_m^k dx \right| \leq 2c_k + \delta_m^k + \epsilon_m \|u_m^k\|_k,$$

which yields, using assumption (A_3)

$$\left| \int_{Q_k} f(u_m^k) u_m^k dx - \frac{2}{\mu}\int_{Q_k} f(u_m^k) u_m^k dx \right| \leq 2c_k + \delta_m^k + \epsilon_m \|u_m^k\|_k.$$

Thus, we have

$$\int_{Q_k} f(u_m^k) u_m^k dx \leq \frac{\mu}{\mu - 2}\left[2c_k + \delta_m^k + \epsilon_m \|u_m^k\|_k\right] \leq c + \epsilon_m \|u_m^k\|_k \tag{2.41}$$

Setting $\phi = T_k u_m^k$ we get by (2.38)

$$\|T_k u_m^k\|_k^2 \leq \int_{Q_k} |f(u_m^k)||T_k u_m^k| dx + \epsilon_m \|T_k u_m^k\|_k.$$

and by setting $y_k = \frac{T_k u_m^k}{\|T_k u_m^k\|}$ we obtain

$$\|T_k u_m^k\|_k \leq \int_{Q_k} c\,|f(u_m^k)||y_k| + \epsilon_m. \tag{2.42}$$

We now rely on the following inequality □

Lemma 2.11

$$t\ s \leq \begin{cases} (e^{t^2}-1) + s\ (\log^+ s)^{1/2}, & \textit{for all } t \geq 0 \textit{ and } s \geq e^{1/4} \\ (e^{t^2}-1) + \frac{1}{4}s^2, & \textit{for all } t \geq 0 \textit{ and } 0 \leq s \leq e^{1/4} \end{cases}$$

Proof. For $s > 0$ given, consider $\max_{t\geq 0}\{ts - (e^{t^2}-1)\}$. Let t_s denote the (unique) point where the maximum is attained. Then $s = 2t_s e^{t_s^2}$. Consider the following cases:

$t_s \geq \frac{1}{2}$: then $s \geq e^{t_s^2}$, which implies $(\log^+ s)^{1/2} \geq t_s$. Thus

$$\max_{t\geq 0}\{ts - (e^{t^2}-1)\} = t_s s - (e^{t_s^2}-1) \leq t_s s \leq (\log^+ s)^{1/2}\ s$$

$0 \leq t_s \leq \frac{1}{2}$ and $s \geq e^{1/4}$: then $t_s s \leq \frac{s}{2}$ and $\frac{s}{2} \leq s(\log^+ s)^{1/2}$ iff $s \geq e^{1/4}$. Hence, the first inequality is proved.

The second inequality holds in fact always (without restrictions on s): indeed,

$$ts \leq t^2 + \frac{1}{4}s^2 \leq (e^{t^2}-1) + \frac{1}{4}s^2$$

Hence, the Lemma is proved. □

We now continue with the estimate of (2.42). Note that by assumption (A_4), for $\beta > \alpha_0$ there exists a constant $c > 0$ such that $|f(r)| \leq ce^{\beta r^2}$, for all $r \in \mathbb{R}$. Hence, we can estimate (2.42) by using the above inequality with $t = |y_k(x)|$ and $s = |\frac{1}{c}f(u_m^k(x))|$:

$$\int_{Q_k} |f(u_m^k)||y_k| \leq c\int_{Q_k}(e^{y_k^2}-1) + c\int_{Q_k}\frac{1}{c}|f(u_m^k)|\ \left[\log^+\left(\frac{1}{c}|f(u_m^k)|\right)\right]^{1/2}$$
$$+ \frac{c}{4}\int_{Q_k\cap\{x:|f(u_m^k)|\leq e^{1/4}\}}\frac{1}{c^2}[f(u_m^k)]^2$$

The first term on the right is uniformly bounded, arguing as in (2.5). For the second term, we have

$$\left[\log^+\left(\frac{1}{c}|f(u_m^k)|\right)\right]^{1/2} \leq [\log(e^{\beta(u_m^k)^2})]^{1/2} = |u_k^m|\sqrt{\beta} \tag{2.43}$$

For estimating the third term, we use that by (A_6) there exist constants c and $s_0 > 0$ such that $|f(s)| \leq c|s|$ for $|s| \leq s_0$, and hence

$$|f(s)|^2 \leq cf(s)s, \text{ for } \{s \in \mathbb{R} : |s| \leq s_0 \text{ and } |f(s)| \leq e^{1/4}\},$$

while for $|s| > s_0$

$$|f(s)|^2 \leq \frac{e^{1/4}}{s_0} f(s)s, \text{ for } \{s \in \mathbb{R} : |s| > s_0 \text{ and } |f(s)| \leq e^{1/4}\}.$$

Hence the third term can be estimated as

$$\frac{c}{4}\int_{Q_k\cap\{x:|f(u_m^k)|\leq e^{1/4}\}} \frac{1}{c^2}[f(u_m^k)]^2 \leq d\int_{Q_k} f(u_m^k)u_m^k. \tag{2.44}$$

So, the estimate (2.42) becomes, joining (2.43), (2.43) and (2.44),

$$\|T_k u_m^k\|_k \leq \int_{Q_k} f(u_m^k)\; y_k + \epsilon_m \leq c + \epsilon_m\|u_m^k\|_k + \epsilon_m. \tag{2.45}$$

Repeating now the same argument as for (2.42), by setting $\phi = S_k u_m^k$ and $z_k = S_k u_m^k / \|S_k u_m^k\|$ we have

$$\|S_k u_m^k\|_k \leq \int_{Q_k} f(u_m^k)\; z_k + \epsilon_m, \tag{2.46}$$

and then arguing as above

$$\|S_k u_m^k\|_k \leq \int_{Q_k} f(u_m^k) z_k + \epsilon_m \leq c + \epsilon_m\|u_m^k\|_k + \epsilon_m. \tag{2.47}$$

Joining the estimates (2.45) and (2.47) we finally obtain

$$\|u_m^k\|_k \leq C,$$

where C is a constant independent of k.

□

Next, we show:

Proposition 2.12 *The Palais-Smale sequence* (u_m^k) *contains a subsequence, still denoted by* (u_m^k)*, which converges to a nontrivial critical point* u_k *of* J_k *with* $J(u_k) = c_k$*.*

Proof. By Proposition 2.10 we may assume that

$$\begin{aligned} &u_m^k \rightharpoonup u_k \quad \text{in} \quad E_k \\ &u_m^k \to u_k \quad \text{in} \quad L^q(Q_k), \ \forall \ q \geq 1 \\ &u_m^k(x) \to u_k(x) \text{ a.e. in} \quad Q_k \end{aligned} \tag{2.48}$$

We now use the following result; for the proof we refer to Lemma 2.1 in [9].

Lemma 2.13 $f(u_m^k) \to f(u_k)$ *in* $L^1(Q_k)$*.*

First, we prove that u_k is a critical pont of J_k. It follows from assumption (A_3) and Lemma 2.13, using the generalized Lebesgue dominated convergence theorem, that also $F(u_m^k) \to F(u_k)$ in $L^1(Q_k)$. This fact together with (2.37) and (2.48) imply that

$$\lim_{m\to\infty} \int_{Q_k} |\nabla u_m^k|^2 dx = -\int_{Q_k} V(x)(u_k)^2 dx + 2c_k + 2\int_{Q_k} F(u_k) dx \tag{2.49}$$

Also, it follows from Lemma 2.13 and (2.38) that

$$\int_{Q_k} (\nabla u_k \nabla \phi + V(x) u_k \phi) dx = \int_{Q_k} f(u_k)\phi dx, \quad \forall \phi \in E_k.$$

Hence u_k is a critical point of J_k.

Next, we prove that u_k is nontrivial. Assume for the sake of contradiction that $u_k \equiv 0$. From (2.49) we get

$$\lim_{m\to\infty} \int_{Q_k} |\nabla u_m^k|^2 dx = 2c_k. \tag{2.50}$$

Using this, and that $c_k < 2\pi/\alpha_0$, we can choose $q > 1$ sufficiently close to 1 and $\beta > \alpha_0$ sufficiently close to α_0 such that $q\beta \|\nabla u_m^k\|_{L^2}^2 < 4\pi$. Hence, from assumption (A_4) and using the Trudinger-Moser inequality, we obtain

$$\int_{Q_k} |f(u_m^k)|^q dx \leq C \int_{Q_k} \exp(q\beta (u_m^k)^2) dx \leq C.$$

Now, using this estimate, from (2.38) with $\phi = u_m^k$ we have

$$\lim_{m\to\infty} \int_{Q_k} |\nabla u_m^k|^2 dx = 0.$$

But this contradicts (2.50), since $c_k > 0$. Consequently u_k is a nontrivial critical point of J_k.

Finally, we prove that $J_k(u_k) = c_k$. We argue by contradiction, assuming that $J_k(u_k) < c_k$; this implies that

$$\int_{Q_k} |\nabla u_k|^2 dx < \left[2c_k + 2\int_{Q_k} F(u_k)dx - \int_{Q_k} V(x)(u_k)^2 dx\right]. \tag{2.51}$$

Let

$$v_m^k = \frac{u_m^k}{\|\nabla u_m^k\|_{L^2(Q_k)}}$$

and

$$v_k = \frac{u_k}{[2c_k + 2\int_{Q_k} F(u_k)dx - \int_{Q_k} V(x)|u_k|^2 dx]^{1/2}}.$$

Since $v_m^k \rightharpoonup v_k \neq 0$ in $H^1(Q_k)$ and $\|\nabla v_k\|_{L^2(Q_k)} < 1$, it follows by a result of P.-L. Lions [16] (see also [2]) that

$$\sup \int_{Q_k} \exp(p\,|v_m^k|^2)dx < \infty, \quad \forall\, p < \frac{4\pi}{1 - \|\nabla v_k\|^2_{L^2(Q_k)}}.$$

Notice that since $J_k(u_k) > 0$ and $c_k < 2\pi/\alpha_0$ we have

$$\frac{\alpha_0}{2\pi} < \frac{1}{c_k - J_k(u_k)},$$

which implies that we may choose $q > 1$ and $\beta > \alpha_0$ such that for some $\delta > 0$ and $\epsilon_m \to 0$

$$\begin{aligned} q\beta\|\nabla u_m^k\|^2_{L^2(Q_k)} &\leq \frac{2\pi}{c_k - J_k(u_k)}\|\nabla u_m^k\|^2_{L^2(Q_k)} - \delta \\ &= 4\pi\frac{[c_k + \int_{Q_k} F(u_m^k)dx - \frac{1}{2}\int_{Q_k} V(x)|u_m^k|^2 dx] + \epsilon_m}{c_k - J_k(u_k)} - \delta \\ &\leq 4\pi\frac{[c_k + \int_{Q_k} F(u_k)dx - \frac{1}{2}\int_{Q_k} V(x)|u_k|^2 dx]}{c_k - J_k(u_k)} - \frac{\delta}{2} \end{aligned}$$

for m sufficiently large. Now note that

$$\frac{c_k + \int_{Q_k} F(u_k)dx - \frac{1}{2}\int_{Q_k} V(x)|u_k|^2 dx}{c_k - J_k(u_k)} = \frac{1}{1 - \|\nabla v_k\|^2_{L^2(Q_k)}}$$

since

$$
\begin{aligned}
&(1-\|\nabla v_k\|^2_{L^2(Q_k)})\left[c_k+\int_{Q_k}F(u_k)dx-\frac{1}{2}\int_{Q_k}V(x)|u_k|^2dx\right]\\
&\quad = c_k-J_k(u_k)+\frac{1}{2}\|\nabla u_k\|^2_{L^2(Q_k)}\\
&\qquad -\|\nabla v_k\|^2_{L^2(Q_k)}\left[c_k+\int_{Q_k}F(u_k)dx-\frac{1}{2}\int_{Q_k}V(x)|u_k|^2dx\right]\\
&\quad = c_k-J_k(u_k)+\frac{1}{2}\|\nabla u_k\|^2_{L^2(Q_k)}-\|\nabla v_k\|^2_{L^2(Q_k)}\frac{\frac{1}{2}\|\nabla u_k\|^2_{L^2(Q_k)}}{\|\nabla v_k\|^2_{L^2(Q_k)}}.
\end{aligned}
$$

Thus, we have shown that

$$
q\beta\|\nabla u_m^k\|^2_{L^2(Q_k)}\le\frac{4\pi}{1-\|\nabla v_k\|^2_{L^2(Q_k)}}-\frac{\delta}{2}.
$$

By assumption (A_4) there exists a constant c such that

$$
\int_{Q_k}|f(u_m^k)|^q\le c\int_{Q_k}e^{q\beta|u_m^k|^2}=c\int_{Q_k}e^{q\beta\|\nabla u_m^k\|^2_{L^2}|v_m^k|^2},
$$

and the last integral is bounded by the above considerations, and hence the L^q norm of $f(u_m^k)$ is bounded. Setting $\phi=u_m^k$ in (2.38), and concluding by Hölder and the above estimate that $\int_{Q_k}f(u_m^k)u_m^k\to 0$, we get

$$
\lim_{m\to\infty}\|\nabla u_m^k\|_{L^2(Q_k)}=\|\nabla u_k\|_{L^2(Q_k)},\quad\text{and hence } u_m^k\to u_k\ \text{ in } E_k.
$$

This is impossible in view of (2.49) and (2.51). □

3 Proof of the Theorem

We have proved in the last section that for every k suitably large, we have that $u_k\in E_k$ is a critical point of J_k with

$$
J_k(u_k)=c_k\in(\sigma,2\pi/\alpha_0-\delta)\ \text{ and }\ \|u_k\|_k\le C.
$$

So, up to subsequence, we can assume that

$$
u_k\rightharpoonup u\quad\text{in}\quad H^1_{\text{loc}}(\mathbb{R}^2),\quad u_k(x)\to u(x)\quad\text{a.e. in }\mathbb{R}^2.
$$

For fixed $\phi\in C_0^\infty(\mathbb{R}^2)$, we can take k suitably large such that the support of ϕ is contained in Q_k and so

$$
\int_{\mathbb{R}^2}\nabla u_k\nabla\phi+V(x)u_k\phi=\int_{\mathbb{R}^2}f(u_k)\phi. \tag{3.52}
$$

Now, using again Lemma 2.1 in [9], we have that for any fixed bounded domain Ω of $\mathbb{R}^2$,

$$f(u_k) \to f(u) \text{ in } L^1(\Omega) \text{ as } k \to \infty. \tag{3.53}$$

Thus, taking the limit in (3.52) with $\Omega = supp(\phi)$, we see that u is a weak solution of problem (1.2). Moreover, since $\|u_k\|_k \leq C$ for all $k \in \mathbb{N}$, we conclude that $u \in H^1(\mathbb{R}^2)$ and u is a critical point of the C^1-functional $J : H^1(\mathbb{R}^2) \to \mathbb{R}$ given by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)u^2)dx - \int_{\mathbb{R}^2} F(u)dx$$

at level $c = \lim_{k\to\infty} c_k$ (taking a subsequence). Thus, from Lemma 2.9, we see that $c < 2\pi/\alpha_0$.

If u is nontrivial we have finished. Next, we prove that there exists a nontrivial solution. To this end we make use of the following result (to be proved later).

Proposition 3.1 *There are constants $r, \eta > 0$ and a sequence of vectors $\xi_k \in \mathbb{R}^2$ such that*

$$\liminf_{k\to\infty} \int_{K_r(\xi_k)} |u_k|^2\, dx \geq \eta, \tag{3.54}$$

where $K_r(\xi)$ is the closed cube with edge length r centered a the point ξ.

Using Proposition 3.1, we can find a sequence of integer vectors $b_k \in \mathbb{Z}^2$ and a positive number r_1 such that the sequence $\widetilde{u}_k(x)$, defined by $\widetilde{u}_k(x) = u_k(x+b_k)$, satisfies

$$\int_{K_{r_1}(0)} |\widetilde{u}_k|^2\, dx \geq \eta/2. \tag{3.55}$$

Since $V(x)$ and $f(s)$ are $1-$periodic functions in x_1 and x_2, by an easy computation we obtain

$$\|u_k\|_k = \|\widetilde{u}_k\|_k, \quad J_k(u_k) = J_k(\widetilde{u}_k) \quad \text{and} \quad J_k'(\widetilde{u}_k) = 0.$$

Then,

$$\widetilde{u}_k \rightharpoonup \widetilde{u} \quad \text{in} \quad H^1_{\text{loc}}(\mathbb{R}^2), \quad \widetilde{u}_k(x) \to \widetilde{u}(x) \quad \text{a.e. in } \mathbb{R}^2$$

and, as before, it follows that $\widetilde{u}$ is a weak solution of problem (1.2). Furthermore, $\widetilde{u}$ is nontrivial in view of (3.55) and the Sobolev embedding Theorem.

Before proving Proposition 3.1 we state the following auxiliary result whose proof can be found in [19].

Lemma 3.2 *Let Q_n be the cube of edge length $l_n \to \infty$ as $n \to \infty$ centered at the origin, and $K_r(\xi)$ as in Proposition* 3.1. *Let $(u_n) \subset H^1_{\text{loc}}(\mathbb{R}^N)$ be a sequence of l_n-periodic functions such that $\|u_n\|_{H^1(Q_n)} \le c$ for some constant independent of n. Suppose that there is $r > 0$ such that*

$$\liminf_{n\to\infty} \left(\sup_{\xi} \int_{K_r(\xi)} |u_n|^2 \, dx \right) = 0. \tag{3.56}$$

Then

$$\lim_{n\to+\infty} \|u_n\|_{L^q(Q_n)} = 0, \ \forall \ q \in (2, \infty).$$

Proof of Proposition 3.1: Suppose that (3.54) does not hold. Thus by virtue of Lemma 3.2 we have that

$$\lim_{n\to+\infty} \|u_n\|_{L^q(Q_n)} = 0, \ \forall \ q \in (2, \infty). \tag{3.57}$$

From assumptions (A_4) and (A_6), we have that for every $\epsilon > 0$, $\beta > \alpha_0$ and $q > 2$ there exists a constant $C_1 = C_1(\epsilon, \beta, q) > 0$ such that

$$f(s)s \le \epsilon s^2 + C_1|s|^q[\exp(\beta s^2) - 1], \quad \forall \ s \in \mathbb{R} \ , \tag{3.58}$$

which together with assumption (A_3) implies that

$$\mu F(s) \le \epsilon s^2 + C_1|s|^q[\exp(\beta s^2) - 1], \quad \forall \ s \in \mathbb{R}. \tag{3.59}$$

Claim 3.3 *The following limits hold:*

$$\lim_{k\to\infty} \int_{Q_k} V(x) u_k^2 dx = 0 \tag{3.60}$$

and

$$\lim_{k\to\infty} \int_{Q_k} F(u_k) dx = 0. \tag{3.61}$$

From Claim 3.3 and the fact that $J_k(u_k) = c_k \to c < 4\pi/\alpha_0$, we conclude that for large k

$$\int_{Q_k} |\nabla u_k|^2 dx < \frac{4\pi}{\alpha_0}.$$

From this, taking $\beta > \alpha_0$ sufficiently close to α_0 and $1/r + 1/s = 1$ with $s > 1$ sufficiently close to 1, we see that

$$\begin{aligned} \int_{Q_k} f(u_k) u_k dx \ &\le \ \varepsilon \|u_k\|^2_{L^2(Q_k)} + \|u_k\|^q_{L^{qs}(Q_k)} \int_{Q_k} [\exp(r\beta u_k^2) - 1] dx \\ &\le \ \varepsilon \|u_k\|^2_{L^2(Q_k)} + C\|u_k\|^q_{L^{qs}(Q_k)}. \end{aligned}$$

Thus, taking the limit as $k \to \infty$, using (3.57), and then taking the limit as $\varepsilon \to 0$ we have

$$\int_{Q_k} f(u_k)u_k dx \to 0 \text{ as } k \to \infty. \tag{3.62}$$

On the other hand we see that

$$\|T_k u_k\|_k^2 - \|P_k u_k\|_k^2 - \int_{Q_k} f(u_k)u_k dx = 0,$$

and

$$\|T_k u_k\|_k^2 - \|P_k u_k\|_k^2 - 2\int_{Q_k} F(u_k)dx = 2c_k,$$

then

$$\int_{Q_k} f(u_k)u_k dx - 2\int_{Q_k} F(u_k)dx = 2c_k, \tag{3.63}$$

which together with Claim 3.3 and (3.62) implies that $c_k \to 0$, which is a contradiction with the fact that $c = \lim_{k\to\infty} c_k \geq \sigma > 0$.

Verification of Claim 3.3: The proof of (3.60) is the same as that of (24) of [5]. We proceed to prove (3.61). Using the same kind of argument and notations as in the proof of Claim 2.5, we have

$$\begin{aligned}
\int_{Q_k} F(u_k)\, dx &= \int_{B_{R_k}} F((u_k)^*)\, dx \\
&= \int_{B_{R_0}} F((u_k)^*)\, dx + \int_{R_0\leq|x|\leq R_k} F((u_k)^*)\, dx \\
&\leq \int_{B_{R_0}} F((u_k)^*)\, dx + \int_{R_0\leq|x|} F(P(u_k)^*)\, dx
\end{aligned}$$

where $R_0 > 0$ is a number to be determined later. From Lemma 2.1 in [9] we see that for all fixed $R_0 > 0$,

$$\int_{B_{R_0}} F((u_k)^*)\, dx \to 0.$$

On the other hand, from (3.59)

$$\int_{R_0\leq|x|} F(P(u_k)^*)dx \leq \epsilon\|u_k\|_{L^2}^2 + C_1\int_{R_0\leq|x|} |\, P(u_k)^* \,|^q\, [\exp(\beta \mid P(u_k)^* \mid^2) - 1]dx$$

so, using the Radial Lemma (2.9) and proceeding as in the estimate following (2.9) (choosing R_0 such that $R_0^{1-1/rk} \geq \sup_k \|u_k\|_{L^2}/\sqrt{\pi}$, $\forall\ k \geq 1$) we get

$$\int_{R_0 \leq |x|} F(P(u_k)^*)\ dx \leq \epsilon \|u_k\|_{L^2}^2 + C_1 \|u_k\|_{L^{qs}}^q,$$

Finally, taking the limit as $k \to \infty$, using (3.57), and then taking the limit as $\varepsilon \to 0$ we obtain (3.61).

This completes the proof of Theorem 1.2.

We end the paper with the Example 1.1 mentioned in the introduction

Proposition 3.4 *The function* $f(s) = \text{sign}(s)\ (e^{s^2} - 1)$ *satisfies conditions* (A_3)*–*(A_6)*, with* $\alpha_0 = 1$*.*

Proof. Conditions $(A_4) - (A_6)$ are trivially satisfied.

Condition (A_3): Consider $\tilde{F}(s) = \frac{1}{|s|}(e^{s^2} - s^2 - 1)$. One checks, using Taylor series, that

$$\tilde{F}'(s) \geq \frac{3}{2} f(s),\ \forall\ s \in \mathbb{R}^+,$$

and hence by integration $\tilde{F}(s) \geq \frac{3}{2} F(s),\ \forall\ s \in \mathbb{R}$.

Again using Taylor series one shows that

$$s\ f(s) \geq 2\tilde{F}(s),\ \forall s \in \mathbb{R},$$

and then we get

$$s\ f(s) \geq 3F(s),\ \forall\ s \in \mathbb{R}.$$

Hence, the first part of condition (A_3) is satisfied with $\mu = 3$. The second part of condition (A_3) follows now easily: indeed, one has trivially

$$|f(s)| \geq \tilde{F}(s),\ \text{for } |s| \geq 1,$$

and hence by the above

$$\frac{2}{3}|f(s)| \geq F(s),\ \text{for } |s| \geq 1.$$

□

Acknowledgments

Parts of this work were done while the first author was visiting the Università degli Studi di Milano, and while the second author was visiting the Universidade Federal do Paraíba in João Pessoa. The authors would like to express their gratitude for the hospitality.

References

[1] ADIMURTHI, Existence of Positive solutions of the semilinear Dirichlet problem with critical growth for the N-Laplacian, *Ann. Sc. Norm. Sup. Pisa*, **17** (1990), 393–413.

[2] ADIMURTHI and S.L. YADAVA, Critical Exponent problem in $\mathbb{R}^2$ with Newmann boundary condition, *Comm. Partial Diff. Eq.* **15** (1990), 461–501.

[3] H. BERESTYCKI and P.-L. LIONS, Nonlinear scalar field equations, I. Existence of ground state, *Arch. Rat. Mech. Anal.* **82** (1983), 313–346.

[4] D.M. CAO, Nontrivial solution of semilinear elliptic equation with critical exponent in $\mathbb{R}^2$, *Comm. Partial Diff. Eq.* **17** (1992), 407–435.

[5] J. CHABROWSKI and JIANFU YANG, On Schrödinger equation with periodic potencial and critical Sobolev exponent, *Topol. Meth. Nonl. Anal.* **12** (1998), 245–261.

[6] V. COTI-ZELATI and P.H. RABINOWITZ, Homoclinic type solutions for a semilinear elliptic PDE on $\mathbb{R}^N$, *Comm. Pure Appl. Math.* **45** (1992), 1217–1269.

[7] DJAIRO G. DE FIGUEIREDO, Lectures on the Ekeland variational principle with applications and detours. Lectures notes, College on Variational Problem in Analysis, Trieste (1988).

[8] D.G. DE FIGUEIREDO, JOÃO MARCOS DO Ó and B. RUF, On an inequality by N. Trudinger and J. Moser and related elliptic equations, *Comm. Pure Appl. Math.* **55** (2002), 1–18.

[9] D.G. DE FIGUEIREDO, O.H. MIYAGAKI and B. RUF, Elliptic equations in $\mathbb{R}^2$ with nonlinearities in the critical growth range, Calc. Var. **3** (1995), 139–153.

[10] D.G. DE FIGUEIREDO and B. RUF, Existence and non-existence of radial solutions for elliptic equations with critical exponent in $\mathbb{R}^2$, *Comm. Pure Appl. Math.* **68** (1995), 639–655.

[11] JOÃO MARCOS DO Ó, Quasilinear elliptic equations with exponential nonlinearities, *Comm. Appl. Nonlin. Anal.* **2** (1995), 63–72.

[12] JOÃO MARCOS DO Ó, Semilinear dirichlet problems for the N-Laplacian in $\mathbb{R}^N$ with nonlinearities in the critical growth range, *Diff. Int. Equ.* **9** (1996), 967–979.

[13] JOÃO MARCOS DO Ó, N-Laplacian equations in $\mathbb{R}^N$ with critical growth, *Abstract Appl. Anal.* **2** (1997), 301–315.

[14] O. KAVIAN, Introduction à la théorie des points critiques et applications aux problèmes elliptiques, Springer-Verlag, Paris, 1993.

[15] W. KRYSZEWSKI and A. SZULKIN, Generalized linking theorem with an application to a semilinear Schrödinger equation, *Advances in Diff. Eq.* **3** (1998), 441–472.

[16] P.-L. LIONS, The concentration-compactness principle in the calculus of variations, Part I, *Revista Matemática Iberoamericana*, **1** (1985), 145–201.

[17] J. MOSER, A sharp form of an inequality by N. Trudinger, *Ind. Univ. Math. J.* **20** (1971), 1077–1092.

[18] A.A. PANKOV, Semilinear elliptic equations in $\mathbb{R}^N$ with nonstabilizing coefficients, *Ukraine Math. J.* **41** (1989), 1075–1078.

[19] A.A. PANKOV and K. PFLÜGER, On a semilinear Schrödinger equation with periodic potential, *Nonlin. Anal.* **33** (1998), 593–609.

[20] P.H. RABINOWITZ, Minimax Methods in Critical Point Theory with Applications to Differential Equations, CBMS Regional Conf. Ser. in Math. No. 65 AMS, 1986.

[21] P.H. RABINOWITZ, A note on a semilinear elliptic equation on $\mathbb{R}^N$, Nonlinear Analysis, a tribute in honour of Giovanni Prodi, A. Ambrosetti and A. Marino, eds., Quaderni Scuola Normal Superior Pisa (1991), 307–318.

[22] M. REED and B. SIMON, Methods of Mathematical Physics IV, Academic Pess, New York, 1978.

[23] C. TROESTLER and M. WILLEM, Nontrivial solution of a semilinear Schrödinger equation, *Comm. Partial Diff. Eq.* **21** (1996), 1431–1449.

Received 11 July 2003, accepted 11 March 2004

Published Online First 16 February 2006