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Nonlinear Analysis 56 (2004) 781 – 791

Nonlinear Analysis

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On nonlinear perturbations of a periodic elliptic problem in $\mathbb{R}^2$ involving critical growth☆

C.O. Alves[a,1], João Marcos do Ó[b,*], O.H. Miyagaki[c]

[a] *Departmento de Matemática e Estatística, Universidade Federal de Campina Grande, Campina Grande 58109-970, PB, Brazil*

[b] *Departamento de Matemática, CCEN, Universidade Federal de Paraíba, João Pessoa 58059-900, PB, Brazil*

[c] *Departamento de Matemática, Universidade Federal de Viçosa 36571-000, MG, Brazil*

Received 15 January 2003; accepted 29 June 2003

Abstract

We consider the equation $-\Delta u + V(x)u = f(x,u)$ for $x \in \mathbb{R}^2$ where $V : \mathbb{R}^2 \to \mathbb{R}$ is a positive potential bounded away from zero, and the nonlinearity $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ behaves like $\exp(\alpha|u|^2)$ as $|u| \to \infty$. We also assume that the potential $V(x)$ and the nonlinearity $f(x,u)$ are asymptotically periodic at infinity. We prove the existence of at least one weak positive solution $u \in H^1(\mathbb{R}^2)$ by combining the mountain-pass theorem with Trudinger–Moser inequality and a version of a result due to Lions for critical growth in $\mathbb{R}^2$.

MSC: 35J65; 35J20

Keywords: Variational methods; Trudinger–Mose inequality; Critical Sobolev exponents; Palais–Smale condition; Concentration–compactness principle; Mountain-pass theorem

1. Introduction

Coti-Zelati and Rabinowitz [5] studied a class of nonlinear Schrödinger equation in whole space $\mathbb{R}^N$, $N \geqslant 3$, of the form

$$-\Delta u + V(x)u = f(x,u) \text{ in } \mathbb{R}^N,\ u \in H^1(\mathbb{R}^N),\ u > 0, \tag{1.1}$$

☆ Research partially supported by CNPq/Brazil and PRONEX-MCT/Brazil.

* Corresponding author. Fax: +55-83-216-7117.

E-mail addresses: coalves@dme.ufpb.br (C.O. Alves), jmbo@mat.ufpb.br (J. Marcos do Ó), olimpio@mail.ufv.br (O.H. Miyagaki).

[1] Supported in part by the Millennium Institute for the Global Advancement of Brazilian Mathematics, IM-AGIMB.

doi:10.1016/j.na.2003.06.003

where the potential $V : \mathbb{R}^N \to \mathbb{R}$ and the nonlinearity $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfied the following periodicity conditions

$$V(x+p) = V(x),\ \ f(x+p,s) = f(x,s),\ \ \forall x \in \mathbb{R}^N,\ \ p \in \mathbb{Z}^N,\ \ s \in \mathbb{R}.$$

In their work, they obtained results of existence and multiplicity by applying the mountain-pass Theorem [12] together with a sort of Concentration Compactness Principle of Lions (see book [11]), when in addition to the periodicity condition, f had subcritical growth, that is, the growth of f in s must be like $|s|^{p-1}s,\ p < (N+2)/(N-2) \equiv 2^* - 1$. Later considering a small nonperiodic perturbations of V and f in the subcritical case or treating situation where f is a pure power like $|s|^{p-1}s$ with $p \geqslant 2^* - 1$ (critical and supercritical) their result were extended or complemented, for instance, by Montechiari [13], Alves-Carrion and Miyagaki [3], Zhu and Yang [15], Noussair-Swanson and Yang [14], and references therein. We recall that problem (1.1) in $\mathbb{R}$ or $\mathbb{R}^2$ when f behaves such as a power $|s|^{p-1}s,\ p < \infty$, it can be handled quite simply compared to the case $\mathbb{R}^N,\ N \geqslant 3$. On the other hand, in Admurth and Yadava [2] (see also [7]) motivated by Trudinger–Moser inequality, namely

$$\exp(\alpha|u|^2) \in L^1(\Omega),\ \ \forall u \in H_0^1(\Omega),\ \ \alpha > 0,$$

and

$$\sup_{|\nabla u|_{L^2(\Omega)} \leqslant 1} \int_\Omega \exp(\alpha|u|^2) \leqslant C_2 \leqslant \infty,\ \ \alpha \leqslant \alpha_2 = 4\pi,$$

where Ω is a bounded domain. They introduced the notion of criticality in $\mathbb{R}^2$: that is, $f(s)$ has a critical growth when it behaves like $\exp(\alpha|s|^2)$ as $s \to +\infty$. More exactly, there exists $\alpha_0 > 0$ such that

$$\lim_{s \to +\infty} \frac{f(x,s)}{\exp(\alpha s^2)} = 0, \quad \forall \alpha > \alpha_0,\ \ \lim_{s \to +\infty} \frac{f(x,s)}{\exp(\alpha s^2)} = +\infty, \quad \forall \alpha < \alpha_0.$$

We would like to mention, for instance, the papers by [1,2,6,7] which contain some results of existence for problem (1.1) in a bounded domain making use of this inequality. Afterwards, Cao in [4] proved a version of Moser–Trudinger inequality in whole space $\mathbb{R}^2$, which was improved by do Ó in [9];

$$\int_{\mathbb{R}^2} (\exp(\alpha|u|^2) - 1)\,\mathrm{d}x < +\infty, \quad \forall u \in H^1(\mathbb{R}^2),\ \ \alpha > 0.$$

Moreover, if $\alpha < 4\pi$ and $|u|_{L^2(\mathbb{R}^2)} \leqslant C$, there exists a constant $C_2 = C_2(C,\alpha)$ such that

$$\sup_{|\nabla u|_{L^2(\mathbb{R}^2)} \leqslant 1} \int_{\mathbb{R}^2} (\exp(\alpha|u|^2) - 1)\,\mathrm{d}x \leqslant C_2. \tag{1.2}$$

By combining this inequality with mountain-pass theorem do Ó and Souto in [10] studied problem (1.1) when the potential V satisfies some geometric condition. Same way do Ó in [9] got some existence result for problem (1.1) involving N-Laplacian operator but imposing a coercivity condition on the potential. On the other hand, Cao in [4] treated the situation where V and f are asymptotic to a constant function, without any coercivity condition on the potential.

We point out that in all the works mentioned above they made use of the so-called Ambrosetti–Rabinowitz condition, namely, there is a constant $\theta > 2$ such that $0 \leqslant \theta F(x,s) < sf(x,s)$ for all $s > 0$, $x \in \mathbb{R}^2$, where F is the primitive of f.

In this paper, by combining arguments used in [3,8–10] together with inequality (1.2) we will discuss the existence of solution for the critical periodic and asymptotic periodic problem (1.1) in $\mathbb{R}^2$, that is,

$$-\Delta u + V(x)u = f(x,u) \text{ in } \mathbb{R}^2,\ u \in H^1(\mathbb{R}^2),\ u > 0, \tag{1.3}$$

where $V : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function satisfying $V(x) \geqslant \tilde{V}_0 > 0$ for all x in $\mathbb{R}^2$.

Next we shall describe the conditions on the functions $V(x)$ and $f(x,s)$ in a more precise way.

(H_1) There exists a continuous 1-periodic function $V_0 : \mathbb{R}^2 \to \mathbb{R}$ such that
(1) $V_0(x) \geqslant V(x)$ for all $x \in \mathbb{R}^2$,
(2) $V_0(x) - V(x) \to 0$ as $|x| \to \infty$.
We assume that the nonlinearity $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following conditions:

(H_2) there exists a continuous 1-periodic function $f_0 : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ such that
(1) $0 \leqslant f_0(x,s) \leqslant f(x,s)$ for all $(x,s) \in \mathbb{R}^2 \times [0,+\infty)$,
(2) for all $\varepsilon > 0$, there exists $\eta > 0$ such that for $s \geqslant 0$ and $|x| \geqslant \eta$,

$$|f(x,s) - f_0(x,s)| \leqslant \varepsilon e^{4\pi s^2},$$

(H_3) $f(x,s) = o_1(s)$ near origin uniformly with respect to $x \in \mathbb{R}^2$;

(H_4) f has critical growth at $+\infty$, namely,

$$f(x,s) \leqslant Ce^{4\pi s^2} \text{ for all } (x,s) \in \mathbb{R}^2 \times [0,+\infty);$$

(H_5) there exist constants $\mu \geqslant \theta > 2$ such that

$$0 \leqslant \theta F_0(x,s) < sf_0(x,s) \quad \text{and} \quad 0 \leqslant \mu F(x,s) < sf(x,s), \text{ for all } (x,s) \in \mathbb{R}^2 \times (0,+\infty),$$

where the functions F_0 and F are the primitive of f_0 and f respectively;

(H_6) for each fixed $x \in \mathbb{R}^2$, the functions $s \to f_0(x,s)/s$ and $s \to f(x,s)/s$ are increasing;

(H_7) there are constants $p > 2$ and C_p such that

$$f_0(x,s) \geqslant C_p s^{p-1}, \text{ for all } (x,s) \in \mathbb{R}^2 \times [0,+\infty),$$

where

$$C_p > \left[\frac{\theta(p-2)}{p(\theta-2)}\right]^{(p-2)/2} S_p^p, \tag{1.4}$$

$$S_p = \inf_{u \in H^1(\mathbb{R}^2)\setminus\{0\}} \frac{\left(\int_{\mathbb{R}^2}(|\nabla u|^2 + V_1 u^2)\,dz\right)^{1/2}}{\left(\int_{\mathbb{R}^2}|u|^p\,dz\right)^{1/p}}$$

and

$$V_1 \doteq \max_{x \in \mathbb{R}^2} V_0(x);$$

(H_8) at least one of the nonnegative continuous functions

$$V_0(x) - V(x) \quad \text{and} \quad f(x,u) - f_0(x,u)$$

is positive on a set of positive measure.

Our first result is concerned on the existence of solutions for the following periodic critical problem:

$$-\Delta u + V_0(x)u = f_0(x,u) \text{ in } \mathbb{R}^2, \ u \in H^1(\mathbb{R}^2), \ u > 0. \tag{1.5}$$

Theorem 1.1. *In addition to* ($H_1 - 1$) *and* ($H_2 - 1$), *suppose that* (H_3)–(H_7) *hold, then* (1.5) *possesses a nontrivial weak solution* $u \in H^1(\mathbb{R}^2)$.

Then, using the above result we shall prove the existence of solutions for the asymptotic periodic problem (1.3) and we have the main theorem of this paper.

Theorem 1.2. *If* (H_1)–(H_8) *are satisfied, then* (1.3) *possesses a nontrivial weak solution* $u \in H^1(\mathbb{R}^2)$.

The above results complete the study made in [4] once that Cao considered the case where the potential and the nonlinearities are asymptotic to the constant functions with respect to x, and the behavior at infinity between the perturbed and unperturbed nonlinearities is near linear growth. Here we work with a general class of functions which are asymptotic to a nonautonomous periodic function, the behavior at infinity between the perturbed and unperturbed nonlinearities is exponential. This work also completes the study made in [9,10], in the sense that the potential V belongs to a class different from those treated by them.

To prove our main theorems we used variational methods. An important point is a version of a Lions' results for critical growth in $\mathbb{R}^2$. This results is crucial to establish some properties involving the Palais–Smale sequence.

Remark 1.3. We notice that in the proof of Theorem 1.1 we shall use only parts of assumptions (H_1)–(H_7) related with problem (1.5).

The organization of this paper is as follows: Section 2 contains preliminary results and the proof of Theorem 2. In Section 3, we prove our main result, Theorem 1.

Notation. In this paper we make use of the following notations:

L^p, L^p_{loc}, H^1, H^1_0 will be used to denote standard Lebesgue, or Sobolev spaces.

The usual norm in $L^p(\Omega)$ will be denoted by $|\cdot|_{L^p(\Omega)}$.

C denotes (possibly different) positive constants;

$B(p,R)$ denote the open ball with the radius R centered at point p of $\mathbb{R}^2$;

2. The periodic problem

The main objective of this section is to study the existence of solutions for the following periodic critical problem (1.5). Since we are interested in obtaining positive solutions, it is convenient to set

$$f_0(x,s) = f(x,s) \equiv 0 \text{ for all } (x,s) \in \mathbb{R}^2 \times (-\infty, 0].$$

Let E_0 denote the Sobolev space $H^1(\mathbb{R}^2)$ endowed with the equivalent norm

$$\|u\|_0 = \left\{ \int_{\mathbb{R}^2} [|\nabla u|^2 + V_0(x)|u|^2]\,\mathrm{d}x \right\}^{1/2}.$$

From assumptions (H_2) and (H_4), given $\varepsilon > 0$ there exist positive constants C_ε and $\beta > 1$ such that

$$F_0(x,s) \leqslant \varepsilon\frac{s^2}{2} + C_\varepsilon(\mathrm{e}^{\beta\pi s^2} - 1) \text{ for all } (x,s) \in \mathbb{R}^2 \times \mathbb{R}.$$

Thus, by a Trudinger–Moser inequality (see Lemma 1 in [9]), we have $F_0(x,u) \in L^1(\mathbb{R}^2)$ for all $u \in H^1(\mathbb{R}^2)$. Therefore, the functional

$$J_0(u) = \frac{1}{2}\int_{\mathbb{R}^2} [|\nabla u|^2 + V_0(x)|u|^2]\,\mathrm{d}x - \int_{\mathbb{R}^2} F_0(x,u)\,\mathrm{d}x$$

is well defined. Furthermore, using standard arguments we see that J_0 is C^1 functional on E_0 with

$$J_0'(u)\phi = \int_{\mathbb{R}^2} [\nabla u \nabla\phi + +V_0(x)u\phi]\,\mathrm{d}x - \int_{\mathbb{R}^2} f_0(x,u)\phi\,\mathrm{d}x, \text{ for all } \phi \in E_0.$$

Consequently, critical points of J_0 are precisely the weak solutions of problem (1.5).

The next result concerns the mountain-pass geometry of J_0. Its proof is a consequence of our assumptions (H_3)–(H_4), and it can be found in [9].

Lemma 2.1 (Mountain-pass geometry). *The functional J_0 satisfies the following conditions*:

(i) *there exist ρ, $\beta > 0$, such that $J_0(u) \geqslant \beta$, if $\|u\|_0 = \rho$ and*
(ii) *for any $u \in H^1(\mathbb{R}^2)\backslash\{0\}$ with $u \geqslant 0$ we have $J_0(tu) \to +\infty$ as $t \to +\infty$.*

Now, in view of Lemma 2.1, we can apply a version of Ambrosetti–Rabinowitz mountain-pass theorem without a compactness condition such as the one of Palais–Smale (see [12]), to get a Palais–Smale sequence of the functional J_0, that is, $(u_n) \subset E_0$ such that

$$J_0(u_n) \to c_0 \text{ and } J_0'(u_n) \to 0, \text{ as } n \to \infty,$$

where the mountain-pass level c_0 is characterized by

$$c_0 = \inf_{\gamma \in \Gamma_0} \max_{0 \leqslant t \leqslant 1} J_0(\gamma(t)) \tag{2.1}$$

and

$$\Gamma_0 = \{\gamma \in C([0,1],\ H^1(\mathbb{R}^2)) : J_0(\gamma(0)) \leqslant 0 \text{ and } J_0(\gamma(1)) \leqslant 0\}.$$

Lemma 2.2. *The mountain-pass level c_0 satisfies $c_0 \in [\beta, (\theta-2)/2\theta)$. Moreover, the $(PS)_{c_0}$ sequence (u_n) is bounded and its weak limit denoted by u_0 satisfies $J_0'(u_0)=0$.*

Proof. In view of Lemma 2.1 we see that $c_0 \geqslant \beta$. In order to prove the other inequality, we fix a positive radial function $v_p \in H^1(\mathbb{R}^2)$ such that

$$S_p = \frac{[\int_{\mathbb{R}^2} [|\nabla v_p|^2 + V_1 v_p^2]\,\mathrm{d}x]^{1/2}}{[\int_{\mathbb{R}^2} |v_p|^p\,\mathrm{d}x]^{1/p}}.$$

Notice that

$$\begin{aligned} c_0 &\leqslant \max_{t\geqslant 0} J_0(tv_p) \\ &\leqslant \max_{t\geqslant 0}\left[\frac{t^2}{2}\int_{\mathbb{R}^2}[|\nabla v_p|^2 + V_1 v_p^2]\,\mathrm{d}x - t^p C_p \int_{\mathbb{R}^2} v_p^p\,\mathrm{d}x\right] \\ &= \frac{(p-2)}{2p}\,\frac{S_p^{2p/(p-2)}}{C_p^{2/(p-2)}}. \end{aligned}$$

On the other hand by (H_7),

$$\frac{(p-2)}{2p}\,\frac{S_p^{2p/(p-2)}}{C_p^{2/(p-2)}} < \frac{(\theta-2)}{2\theta}.$$

So $c_0 < ((\theta-2)/2\theta)$.

Using well-known arguments it is not difficult to check that $\{u_n\}$ is a bounded sequence. Thus for a subsequence still denoted by (u_n) there is $u_0 \in E_0$ such that $u_n \rightharpoonup u_0$ in E_0, $u_n \to u_0$ in $L^s_{\mathrm{loc}}(\mathbb{R}^2)$ for all $s \geqslant 1$ and $u_n(x) \to u_0(x)$ almost everywhere in $\mathbb{R}^2$. Now, from (H_5),

$$\begin{aligned} c_0 &= \lim J_0(u_n) \\ &= \lim\left[J_0(u_n) - \frac{1}{\theta}J_0'(u_n)u_n\right] \\ &\geqslant \frac{\theta-2}{2\theta}\limsup_{n\to+\infty}\|u_n\|_0^2, \end{aligned}$$

which implies

$$\limsup_{n\to+\infty}\|u_n\|_0^2 = m \leqslant \frac{2\theta c_0}{\theta-2} < 1. \tag{2.2}$$

From Trudinger–Moser inequality, (see Lemma 1 in [9]) that there exists $\gamma, q > 1$ sufficiently close to 1 such that sequence

$$h_n(x) = \mathrm{e}^{4\pi\gamma u_n^2(x)} - 1$$

belongs to $L^q(\mathbb{R}^2)$ and there exists $C>0$ such that $|h_n|_q \leqslant C$ for all $n\in\mathbb{N}$. These informations above imply the following limit:

$$\int_{\mathbb{R}^2} f(x,u_n)v\,\mathrm{d}x \to \int_{\mathbb{R}^2} f(x,u_0)v\,\mathrm{d}x \quad \forall v\in E_0.$$

From these and keeping in mind that $J_0'(u_n)v = o_n(1)$ for all $v\in E_0$, taking the limit we prove that $J_0'(u_0)v=0$ for all $v\in E_0$. $\square$

The next proposition is a version of a Lions' results to critical growth in $\mathbb{R}^2$.

Proposition 2.3. *Let* $(u_n)\subset H^1(\mathbb{R}^2)$ *be a sequence with* $u_n \rightharpoonup 0$ *and*

$$\limsup_{n\to\infty}\|u_n\|^2 \leqslant m < 1.$$

If there exists $R>0$ *such that*

$$\liminf_{n\to\infty}\sup_{y\in\mathbb{R}^2}\int_{B_R(y)}|u_n|^2\,\mathrm{d}x = 0,$$

and (H_4)–(H_5) *hold, we have*

$$\int_{\mathbb{R}^2} F(x,u_n)\,\mathrm{d}x,\ \int_{\mathbb{R}^2} u_n f(x,u_n)\,\mathrm{d}x \to 0 \text{ as } n\to\infty.$$

Proof. By hypothesis

$$\liminf_{n\to+\infty}\sup_{y\in\mathbb{R}^2}\int_{B_R(y)}|u_n|^2\,\mathrm{d}x = 0. \tag{2.3}$$

Thus, by Lemma 8.4 in [11], we get

$$u_n \to 0 \text{ in } L^q(\mathbb{R}^2) \text{ for all } q\in(2,+\infty). \tag{2.4}$$

This fact together with Trudinger–Moser inequality (see Lemma 1 in [9]) imply that

$$\int_{\mathbb{R}^2}(\mathrm{e}^{4\pi\gamma u_n^2}-1)\,\mathrm{d}x \leqslant C,$$

if we choose $\gamma>1$ sufficiently close to 1. Therefore, by assumptions (H_2) and (H_4), given $\varepsilon>0$ there exist positive constants C_ε and q, $\gamma>1$ sufficiently close to 1 such that

$$\begin{aligned}\int_{\mathbb{R}^2} u_n f_0(x,u_n)\,\mathrm{d}x &\leqslant \varepsilon\int_{\mathbb{R}^2} u_n^2\,\mathrm{d}x + C_\varepsilon\int_{\mathbb{R}^2} u_n(\mathrm{e}^{4\pi\gamma u_n^2}-1)\,\mathrm{d}x\\ &\leqslant C\left\{\int_{\mathbb{R}^2}|u_n|^{q'}\,\mathrm{d}x\right\}^{1/q'}\left\{\int_{\mathbb{R}^2}(\mathrm{e}^{4\pi\gamma q u_n^2}-1)\,\mathrm{d}x\right\}^{1/q} + \varepsilon C\\ &= C\left\{\int_{\mathbb{R}^2}|u_n|^{q'}\,\mathrm{d}x\right\}^{1/q'} + \varepsilon C,\end{aligned}$$

which together with (2.4) leads to

$$\lim_{n\to+\infty}\int_{\mathbb{R}^2} u_n f_0(x,u_n)=0.$$

Thus, in view of assumption (H_5) we conclude that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} F_0(x,u_n)\,\mathrm{d}x=0.$$

Thus, the proof of Proposition 2.3 is complete. □

2.1. *Proof of Theorem 1.1*

Using Lemma 2.2 we have that the weak limit u_0 of the $(PS)_{c_0}$ sequence satisfies $J_0'(u_0)=0$, thus if u_0 is nontrivial the theorems is proved. If $u_0=0$, we have the following claim:

Claim 1. *There is a sequence $(y_n)\subset\mathbb{R}^2$, and $R,\ a>0$ such that*

$$\liminf_{n\to+\infty}\sup_{y\in\mathbb{R}^2}\int_{B_R(y_n)}|u_n|^2\,\mathrm{d}x>a. \tag{2.5}$$

This claim is true, because for the contrary case, using the version of Lions' proved at Proposition 2.3, we have

$$\int_{\mathbb{R}^2} u_n f(x,u_n)\,\mathrm{d}x\to 0,$$

which implies that $\{u_n\}$ converges strongly to zero. This limit is absurd, because it implies $c_0=0$. Thus the claim is proved.

It is clear that we may assume, without loss of generality, that $(y_n)\subset\mathbb{Z}^N$. Now, letting $\tilde u_n(x)=u_n(x-y_n)$, since V, $f(\cdot,s)$ and $F(\cdot,s)$ are 1-periodic functions, by a routine calculus we obtain

$$\|u_n\|_0=\|\tilde u_n\|_0,\ J_0(u_n)=J_0(\tilde u_n)\quad\text{and}\quad J_0'(\tilde u_n)\to 0.$$

Then, there exists $\tilde u_0\in E_0$ such that $\tilde u_n\rightharpoonup\tilde u_0$ weakly in E_0 and as before it follows that $J_0'(\tilde u_0)v=0$ for all $v\in E_0$. Now, by (2.5), taking a subsequence and R bigger we get

$$a^{1/2}\leqslant|\tilde u_n|_{L^2(B_R(0))}\leqslant|\tilde u_0|_{L^2(B_R(0))}+|\tilde u_n-\tilde u_0|_{L^2(B_R(0))}. \tag{2.6}$$

Thus, from the Rellich imbedding theorem we conclude that $\tilde u_0$ is nontrivial.

Since $f_0(x,s)=0$ for all $s\leqslant 0$ and $J_0'(u_0)v$ for all $v\in E_0$, choosing the teste function $v=u_0^-=\max\{-u_0,0\}\in E_0$, we conclude that $\|u_0^-\|_0=0$. Thus, u_0 is a nonnegative function. Now using that $u_0\geqslant 0$ is nontrivial and the maximum principle we conclude that $u_0>0$.

3. The nonperiodic problem: Proof of Theorem 1.2

Let E denote the Sobolev space $H^1(\mathbb{R}^2)$ endowed with the equivalent norm

$$\|u\| = \left\{ \int_{\mathbb{R}^2} [|\nabla u|^2 + V(x)|u|^2] \, \mathrm{d}x \right\}^{1/2}.$$

In this section, we are going to prove the existence of solutions of problem (1.3) as critical point of the associated C^1 functional on E given by

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^2} [|\nabla u|^2 + V(x)|u|^2] \, \mathrm{d}x - \int_{\mathbb{R}^2} F(x,u) \, \mathrm{d}x.$$

As in the last section, we may check that the functional energy J has the geometry of the mountain-pass theorem. Therefore applying the mountain-pass theorem without Palais–Smale condition together with the arguments from the last section, we obtain a bounded sequence $(v_n) \subset E$ such that

$$J(v_n) \to c_1 \text{ and } J'(v_n) \to 0 \text{ as } n \to +\infty,$$

where c_1 is the minimax level of the functional J given by

$$c_1 = \inf_{v \in E \setminus \{0\}} \max_{t \geqslant 0} J(tv).$$

Furthermore, from (H_7) we see that $c_1 \in (\beta', [\theta - 2)/2\theta)$ for positive constant β' and $v_n \rightharpoonup v_0$ in E (see the proof of Lemma 2.2). Therefore, v_0 is a critical point of functional J and $v_0 > 0$.

Using the same argument of Section 2, we get that

$$\limsup_{n \to +\infty} \|v_n\|^2 < 1,$$

which implies, for all n,

$$\int_{\mathbb{R}^2} (\mathrm{e}^{4\pi\gamma v_n^2} - 1) \, \mathrm{d}x \leqslant C, \tag{3.7}$$

if we take $\gamma > 1$ sufficiently close to 1. From now on, we shall be working in order to prove that v_0 is nontrivial.

Assertion 1. *v_0 is nontrivial.*

Proof. Assume for the sake of contradiction that v_0 is trivial. Thus, from (H_1) and the Sobolev compact embedding theorem,

$$\int_{\mathbb{R}^2} [V_0(x) - V(x)] v_n^2 \, \mathrm{d}x \to 0, \text{ as } n \to +\infty. \tag{3.8}$$

Next, we assume the following result, which will be proved later.

$$\lim_{n\to+\infty}\int_{\mathbb{R}^2}|F_0(x,v_n)-F(x,v_n)|\,\mathrm{d}x$$
$$=\lim_{n\to+\infty}\int_{\mathbb{R}^2}|[f_0(x,v_n)-f(x,v_n)]v_n|\,\mathrm{d}x=0 \tag{3.9}$$

From (3.8)–(3.9) we conclude that

$$|J_0(v_n)-J(v_n)|\to 0 \text{ and } \|J_0'(v_n)-J(v_n)'\|\to 0 \text{ as } n\to+\infty.$$

Therefore,

$$J_0(v_n)\to c_1 \text{ and } J_0'(v_n)\to 0 \text{ as } n\to+\infty.$$

As in Proposition 2.3, there is a sequence $(y_n)\subset Z^2$, and $R,\ a>0$ such that

$$\liminf_{n\to+\infty}\sup_{y\in\mathbb{R}^2}\int_{B_R(y_n)}|v_n|^2\,\mathrm{d}x>a$$

and letting $\tilde{v}_n(x)=v_n(x-y_n)$, since $V,\ f_0(\cdot,s)$ and $F_0(\cdot,s)$ are 1-periodic functions,

$$\|v_n\|_0=\|\tilde{v}_n\|_0,\ J_0(v_n)=J_0(\tilde{v}_n) \text{ and } J_0'(\tilde{v}_n)\to 0.$$

Then, there exists $\tilde{v}_0\in E_0$ such that $\tilde{v}_n\rightharpoonup\tilde{v}_0$ weakly in E_0 and $J_0'(\tilde{v}_0)=0$. Moreover, $J_0(\tilde{v}_0)\leqslant c_1$ since by Fatou's lemma,

$$\begin{aligned}J_0(\tilde{v}_0)&=J_0(\tilde{v}_0)-\tfrac{1}{2}J_0'(\tilde{v}_0)\tilde{v}_0\\&=\frac{1}{2}\int_{\mathbb{R}^2}[f_0(x,\tilde{v}_0)\tilde{v}_0-2F(x,\tilde{v}_0)]\\&\leqslant\liminf_{n\to+\infty}\frac{1}{2}\int_{\mathbb{R}^2}[f_0(x,\tilde{v}_n)\tilde{v}_n-2F(x,\tilde{v}_n)]\\&=\lim_{n\to+\infty}\left[J_0(\tilde{v}_n)-\tfrac{1}{2}J_0'(\tilde{v}_n)\tilde{v}_n\right]=c_1.\end{aligned}$$

Arguing as in (2.6) we conclude that $\tilde{v}_0$ is nontrivial and

$$c_1\geqslant J_0(\tilde{v}_0)=\max_{t\geqslant 0}J_0(t\tilde{v}_0)\geqslant c_0. \tag{3.10}$$

On the other hand, from $(\mathrm{H}_1),(\mathrm{H}_2)$, and (H_8),

$$c_1\leqslant\max_{t\geqslant 0}J(tu_0)=J(t_1u_0)<J_0(t_1u_0)\leqslant\max_{t\geqslant 0}J_0(tu_0)=J_0(u_0)=c_0,$$

that is, $c_1<c_0$, which is a contradiction with (3.10). Therefore, v_0 is nontrivial.

Next we prove (3.9). Using condition (H_2), given $\varepsilon>0$ there exists $\eta>0$ such that

$$\int_{|x|\geqslant\eta}|f(x,v_n)-f_0(x,v_n)||v_n|\,\mathrm{d}x\leqslant\varepsilon\int_{|x|\geqslant\eta}|(\mathrm{e}^{4\pi\gamma v_n^2}-1)v_n|\,\mathrm{d}x$$
$$\leqslant\varepsilon\left\{\int_{\mathbb{R}^2}|(e^{4\pi\gamma v_n^2}-1)|^q\,\mathrm{d}x\right\}^{1/q}\left\{\int_{\mathbb{R}^2}|v_n|^{q'}\right\}^{1/q'}\leqslant C\varepsilon,$$

where we have used that $h_n(x)=\mathrm{e}^{4\pi\gamma v_n^2(x)}-1$ belongs to $L^q(\mathbb{R}^2)$ with $\gamma,q>1,1/q+1/q'=1$, and $|h_n|_q\leqslant C,\ \forall n\in\mathbb{N}$.

On the other hand, using the Sobolev compact embedding theorem and (3.7),

$$\int_{|x|\leqslant\eta}|f(x,v_n)-f_0(x,v_n)||v_n|\,\mathrm{d}x\leqslant\left\{\int_{\mathbb{R}^2}|(\mathrm{e}^{4\pi\gamma v_n^2}-1)|^q\,\mathrm{d}x\right\}^{1/q}$$

$$\left\{\int_{|x|\leqslant\eta}|v_n|^{q'}\right\}^{1/q'}+\varepsilon\int_{\mathbb{R}^2}|v_n|^2.$$

Thus

$$\int_{|x|\leqslant\eta}|f(x,v_n)-f_0(x,v_n)||v_n|\,\mathrm{d}x\to 0 \text{ as } n\to\infty.$$

Finally, these facts together with assumptions (H_2)–(H_5) conclude our proof. □

References

[1] Adimurth, Existence of positive solutions of the semilinear Dirichlet problem with critical growth for the N-Laplacian, Ann. Scuola Norm. Sup. Pisa 17 (1990) 393–413.

[2] Adimurth, S.L. Yadava, Multiplicity results for semilinear elliptic equations in bounded domain of $\mathbb{R}^2$ involving critical exponent, Ann. Scuola. Norm. Sup. Pisa 17 (1990) 481–504.

[3] C.O. Alves, P.C. Carrião, O.H. Miyagaki, Nonlinear perturbations of a periodic elliptic problem with critical growth, J. Math. Anal. Appl. 260 (2001) 133–146.

[4] D.M. Cao, Nontrivial solution of semilinear elliptic equation with critical exponent in $\mathbb{R}^2$, Comm. Partial Diff. Equations 17 (1992) 407–435.

[5] V. Coti Zelati, P. Rabinowitz, Homoclinic type solutions for a semilinear elliptic PDE on $\mathbb{R}^N$, Comm. Pure Appl. Math. 10 (1992) 1217–1269.

[6] D.G. de Figueiredo, J.M. do Ó, B. Ruf, On an Inequality by N. Trudinger and J. Moser and related elliptic equations, Comm. Pure. Appl. Math. 55 (2002) 135–152.

[7] D.G. de Figueiredo, O.H. Miyagaki, B. Ruf, Elliptic equations in $\mathbb{R}^2$ with nonlinearities in the critical growth range, Calc. Var. 3 (1995) 139–153.

[8] J.M. do Ó, Semilinear Dirichlet problems for the N-Laplacian in $\mathbb{R}^N$ with nonlinearities in the critical growth range, Diff. Int. Eqns. 9 (1996) 967–979.

[9] J.M. do Ó, N-Laplacian equations in $\mathbb{R}^N$ with critical growth, Abstr. Appl. Anal. 2 (1997) 301–315.

[10] J.M. do Ó, M.A.S. Souto, On a class of nonlinear Schrödinger equations in $\mathbb{R}^2$ involving critical growth, J. Diff. Eqns. 174 (2001) 289–311.

[11] O. Kavian, Introduction à la théorie des points critiques et applications aux problèmes elliptiques, Springer, Paris, 1993.

[12] J. Mawhin, M. Willem, Critical Point Theory and Hamiltonian Systems, Springer, Berlin, Heidelberg, New York, 1989.

[13] P. Montecchiari, Multiplicity results for a class of semilinear elliptic equations on $\mathbb{R}^N$, Rend. Sem. Mat. Univ. Padova 95 (1996) 217–252.

[14] E.S. Noussair, C.A. Swanson, J. Yang, Quasilinear elliptic problem with critical exponents, Nonlinear Anal. TMA 20 (1993) 285–301.

[15] X.P. Zhu, J. Yang, The quasilinear elliptic equation on unbounded domain involving critical Sobolev exponent, J. Partial Differential Equations 2 (1989) 53–64.