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SEMILINEAR DIRICHLET PROBLEMS FOR THE N-LAPLACIAN IN $\mathbb{R}^N$ WITH NONLINEARITIES IN THE CRITICAL GROWTH RANGE

JOÃO MARCOS B. DO Ó[1]

IMECC - UNICAMP, Caixa Postal 6065, 13083-970, Campinas, SP-Brasil

(Submitted by: Reza Aftabizadeh)

Abstract. The goal of this paper is to study the existence of solutions for the following class of problems for the N-Laplacian

$$u \in W_0^{1,N}(\Omega), \quad u \geq 0 \text{ and } \quad -\Delta_N u \equiv -\mathrm{div}(|\nabla u|^{N-2}\nabla u) = f(x,u) \quad \text{in} \quad \Omega,$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ with $N \geq 2$ and the nonlinearity $f(x,u)$ behaves like $\exp(\alpha|u|^{\frac{N}{N-1}})$ when $|u| \to \infty$.

1. Introduction. Let Ω be a bounded smooth domain in $\mathbb{R}^N$ with $N \geq 2$. In this paper we study the existence of solutions for the following class of semilinear elliptic problems:

$$\begin{gathered} u \in W_0^{1,N}(\Omega), \quad u \geq 0 \text{ and} \\ -\Delta_N u \equiv -\mathrm{div}(|\nabla u|^{N-2}\nabla u) = f(x,u) \quad \text{in} \quad \Omega, \end{gathered} \tag{1}$$

where the nonlinearity $f(x,u)$ has the maximal growth on u which allows us to treat problem (1) variationally in the Sobolev space $W_0^{1,N}(\Omega)$. Here this maximal growth is given by Trudinger-Moser inequality (cf. [13], [17]) which says that

$$\exp(\alpha|u|^{\frac{N}{N-1}}) \in L^1, \quad \forall\, u \in W_0^{1,N}(\Omega), \quad \forall \alpha > 0$$

and

$$\sup_{\|u\|_{W_0^{1,N}} \leq 1} \int \exp(\alpha|u|^{\frac{N}{N-1}}) \leq C(N) \in \mathbb{R}, \quad \text{if} \quad \alpha \leq \alpha_N,$$

where $\alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$ and ω_{N-1} is the $(N-1)$-dimensional surface of the unit sphere. Therefore, from this result we have naturally associated notions of criticality and subcriticality, namely: we say that a function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ has subcritical growth on Ω if

$$\lim_{|u|\to\infty} \frac{|f(x,u)|}{\exp(\alpha|u|^{\frac{N}{N-1}})} = 0, \quad \text{uniformly on } x \in \Omega, \ \forall \alpha > 0 \tag{2}$$

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and f has critical growth on Ω if there exists $\alpha_0 > 0$ such that

$$\lim_{|u|\to\infty} \frac{|f(x,u)|}{\exp(\alpha |u|^{\frac{N}{N-1}})} = 0, \quad \text{uniformly on } x \in \Omega, \ \forall \alpha > \alpha_0 \tag{3}$$

and

$$\lim_{|u|\to\infty} \frac{|f(x,u)|}{\exp(\alpha |u|^{\frac{N}{N-1}})} = +\infty, \quad \forall \alpha < \alpha_0. \tag{4}$$

Here we will consider problems where the nonlinearities have critical growth. There is an extensive bibliography on this subject. See for example, Atkinson-Peletier ([7]), Carleson-Chang ([8]), Adimurthi et al. ([1–4]), de Figueiredo et al. ([10–11]) and references therein. Our paper is closely related to works of Adimurthi ([1–2]) and to the recent work of de Figueiredo et al. ([10]). Indeed, by using the Mountain Pass Lemma without the Palais-Smale condition, we improve the existence conditions in [1, 2, 10] and therefore extend the results to more general nonlinearities. We observe that problems involving nonlinearities with subcritical growth, have been studied recently in [10] and by the author in [12].

In order to study the existence of solutions of problem (1) we are going to impose the following conditions on the function f.

(F_1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous.

(F_2) $\exists\, R > 0$, $\exists\, M > 0$ such that $\forall u \geq R$, $\forall x \in \Omega$,

$$0 < F(x,u) = \int_0^u f(x,t)\,dt \leq M f(x,u).$$

(F_3) $f(x,u) \geq 0$, $\forall (x,u) \in \Omega \times [0,+\infty)$ and $f(x,0) = 0\ \forall x \in \Omega$.

Consider the following nonlinear eigenvalue problem:

$$-\Delta_p u = \lambda |u|^{p-2} u, \quad u \in W_0^{1,p}(\Omega).$$

It is well known (cf. [6]) that there exists a smallest eigenvalue $\lambda_1(p) > 0$ and an associated eigenfunction $\psi_1 > 0$ in Ω that solves this problem. Moreover, we recall that $\lambda_1(p)$ can be variationally characterized as

$$\lambda_1(p) = \inf\Big\{\int |\nabla u|^p\,dx : u \in W_0^{1,p}(\Omega),\ \int |u|^p\,dx = 1\Big\}.$$

We define

$$\mathcal{M} = \lim_{n\to\infty} n \int_0^1 \exp n(t^{\frac{N}{N-1}} - t)\,dt.$$

We see, after some computation, that $\mathcal{M}$ is a real number greater or equal to 2. We denote by d the inner radius of the set Ω; that is, d equals the radius of the largest open ball contained in Ω.

Now we state the main results of this paper.

Theorem 1. *Assume that f has critical growth on Ω and satisfies (F_1), (F_2) and (F_3). Furthermore assume that*

$$(F_4) \qquad \limsup_{u\to 0^+} \frac{NF(x,u)}{|u|^N} < \lambda_1(p), \qquad \textit{uniformly on } x\in\Omega,$$

(F_5)

$$\lim_{u\to+\infty} uf(x,u)\exp\big(-\alpha_0|u|^{\frac{N}{N-1}}\big) \geq \beta_0 > \big(\frac{N}{d}\big)^N \frac{1}{\mathcal{M}\alpha_0^{N-1}} \quad \textit{uniformly on } x\in\Omega.$$

Then, problem (1) *has a nontrivial solution.*

Remark. In [1], results analogous to our Theorem 1 have been proved, but under more restrictive conditions. For instance (among other), it was assumed that f is C^1 and satisfies

$$\frac{\partial f}{\partial u}(x,u) > \frac{f(x,u)}{u} \quad \forall u\in\mathbb{R}\backslash\{0\}, \quad \forall x\in\Omega.$$

In [10], studying problem (1) to the Laplacian, instead of this last condition they assumed the following less restrictive hypothesis:

$$0 < F(x,u) \leq \frac{1}{2}uf(x,u) \quad \forall u\in\mathbb{R}\backslash\{0\}, \quad \forall x\in\Omega.$$

It should be noted that these last two assumptions do not seem natural in the context of critical growth in bounded domains, since they imply restrictions on the growth of the nonlinearities in the whole line.

2. The variational formulation. Under the hypothesis that f is continuous and has critical growth, as defined in (3) and (4), we see that given $\beta > \alpha_0$, there exists $C > 0$ such that

$$|f(x,u)| \leq C\exp(\beta|u|^{\frac{N}{N-1}}), \quad \forall (x,u)\in\Omega\times\mathbb{R}. \tag{5}$$

Consequently, using the Trudinger-Moser inequality and standard arguments (cf. [16]), we see that the functional $I : W_0^{1,N}(\Omega)\to\mathbb{R}$ given by

$$I(u) = \frac{1}{N}\int|\nabla u|^N - \int F(x,u)\,dx \tag{6}$$

is well defined and belongs to $C^1(W_0^{1,N}(\Omega),\mathbb{R})$ with

$$I'(u)v = \int |\nabla u|^{N-2}\nabla u\nabla v - \int f(x,u)v\,dx, \qquad \forall v\in W_0^{1,N}(\Omega). \tag{7}$$

We remark that, since we seek nonnegative solutions, it is convenient to define

$$f(x,u) = 0 \quad \text{on} \quad \Omega\times(-\infty,0].$$

Thus, conditions (F_1), (F_2) and (F_3) imply that

(a) $F(x,u)\geq 0$, $\forall (x,u)\in\Omega\times\mathbb{R}$;

(b) there is a positive constant C such that $\forall u\geq R$, $\forall x\in\Omega$

$$F(x,u)\geq C\exp(\frac{1}{M}u); \tag{8}$$

(c) $\exists\, R_0 > 0$, $\exists\,\theta > N$ such that for $\forall |u|\geq R_0$, $\forall x\in\Omega$

$$\theta F(x,u)\leq uf(x,u). \tag{9}$$

Lemma 1. *Assume* (F_1), (F_2), (F_3) *and* (5). *Then* $I(tu) \to -\infty$ *as* $t \to +\infty$, *for all* $u \in W_0^{1,N}(\Omega)\backslash\{0\}$ *with* $u \geq 0$.

Proof. It follows from (8) that, for $p > N$, there exists a positive constant C such that $\forall u \geq 0$,

$$F(x, u) \geq cu^p - d. \tag{10}$$

Choosing any $u \in W_0^{1,N}(\Omega)\backslash\{0\}$ with $u \geq 0$, (10) leads to

$$I(tu) \leq \frac{t^N}{N} \int |\nabla u|^N - Ct^p \int |u|^p + C.$$

Since $p > N$, we obtain $I(tu) \to -\infty$ as $t \to +\infty$.

Lemma 2. *Suppose* (F_1), (F_4) *and* (5). *Then there exist* δ, $\rho > 0$ *such that*

$$I(u) \geq \delta \quad \textit{if} \quad \|u\|_{W_0^{1,N}} = \rho.$$

Proof. Using (F_1), (F_4) and (5) we can choose $\lambda < \lambda_1(N)$ such that

$$F(x, u) \leq \frac{1}{N}\lambda |u|^N + C\exp(\beta |u|^{\frac{N}{N-1}})|u|^q, \quad \forall (x, u) \in \Omega \times \mathbb{R},$$

for $q > N$. Now, using the Hölder inequality and the Trudinger-Moser inequality we obtain

$$\begin{aligned}\int \exp(\beta r |u|^{\frac{N}{N-1}})|u|^q &\leq \Big\{\int \exp\big[\beta r \|u\|_{W_0^{1,N}}^{\frac{N}{N-1}} \big(\frac{|u|}{\|u\|_{W_0^{1,N}}}\big)^{\frac{N}{N-1}}\big]\Big\}^{1/r} \Big\{\int |u|^{sq}\Big\}^{1/s} \\ &\leq C(N)\Big\{\int |u|^{sq}\Big\}^{1/s}\end{aligned}$$

if $\|u\|_{W_0^{1,N}} \leq \sigma$, where $\beta r \sigma^{\frac{N}{N-1}} < \alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$ and $1/r + 1/s = 1$. Thus, using the variational characterization of the first eigenvalue and Sobolev embedding, these last two estimates imply

$$I(u) \geq \frac{1}{N}\Big(1 - \frac{\lambda}{\lambda_1(N)}\Big)\|u\|_{W_0^{1,N}}^N - C\|u\|_{W_0^{1,N}}^q.$$

Since $\lambda < \lambda_1(N)$ and $N < q$ we may choose $\rho > 0$ such that $I(u) \geq \delta$ if $\|u\|_{W_0^{1,N}} = \rho$. $\square$

Now consider the following sequence of nonnegative functions:

$$\tilde{M}_n(x) = \omega_{N-1}^{-\frac{1}{N}} \begin{cases} (\log n)^{\frac{N-1}{N}} & \text{if} \quad |x| \leq \frac{1}{n} \\ \log|x|^{-1}/(\log n)^{\frac{1}{N}} & \text{if} \quad \frac{1}{n} \leq |x| \leq 1 \\ 0 & \text{if} \quad |x| \geq 1. \end{cases}$$

Let $x_0 \in \Omega$ and $r > 0$ be such that the ball $B(x_0, r)$ of radius r centered at x_0 is contained in Ω. Therefore the functions $M_n(x, x_0, r) = \tilde{M}_n(\frac{x-x_0}{r})$ belong to $W_0^{1,N}(\Omega)$, $\|M_n(\cdot, x_0, r)\|_{W_0^{1,N}} = 1$ and the support of $M_n(x, x_0, r)$ is contained in $B(x_0, r)$.

Lemma 3. *Assume (F_1), (F_2), (F_3) and suppose there exists $r > 0$ such that*

$$\lim_{u\to+\infty} uf(x,u)\exp\big(-\alpha_0|u|^{\frac{N}{N-1}}\big) \geq \beta_0 > \Big(\frac{N}{r}\Big)^N \frac{1}{\mathcal{M}\alpha_0^{N-1}}, \tag{11}$$

uniformly for almost every $x \in \Omega$ and $B(x_0, r) \subset \Omega$ for some $x_0 \in \Omega$. Then there exists n such that

$$\max\{I(tM_n) : t \geq 0\} < \frac{1}{N}\Big(\frac{\alpha_N}{\alpha_0}\Big)^{N-1},$$

where $M_n(x) = M_n(x, x_0, r)$.

Proof. Suppose for the sake of contradiction that for all n we have

$$\max\{I(tM_n) : t \geq 0\} \geq \frac{1}{N}\Big(\frac{\alpha_N}{\alpha_0}\Big)^{N-1}.$$

In view of Lemma 1, for all n there exists $t_n > 0$ such that

$$I(t_nM_n) = \max\{I(tM_n) : t \geq 0\}.$$

Thus,

$$I(t_nM_n) = \frac{1}{N}t_n^N - \int F(x, t_nM_n) \geq \frac{1}{N}\Big(\frac{\alpha_N}{\alpha_0}\Big)^{N-1},$$

and using that $F(x, u) \geq 0$ we obtain

$$t_n^N \geq \Big(\frac{\alpha_N}{\alpha_0}\Big)^{N-1}. \tag{12}$$

Since $\frac{d}{dt}I(tM_n) = 0$ for $t = t_n$, it follows that

$$t_n^N = \int t_nM_n f(x, t_nM_n). \tag{13}$$

On the other hand from (11), given $\epsilon > 0$ there exists $R_\epsilon > 0$ such that

$$uf(x,u) \geq (\beta_0 - \epsilon)\exp\big(\alpha_0|u|^{\frac{N}{N-1}}\big), \qquad \forall u \geq R_\epsilon. \tag{14}$$

Thus, from these last two facts we obtain

$$\begin{aligned} t_n^N &\geq (\beta_0 - \epsilon)\int_{B(x_0,r/n)} \exp\big(\alpha_0|t_nM_n|^{\frac{N}{N-1}}\big) \\ &= (\beta_0 - \epsilon)\frac{\omega_{N-1}}{N}\Big(\frac{r}{n}\Big)^N \exp\big(\alpha_0 t_n^{\frac{N}{N-1}}\omega_{N-1}^{-\frac{1}{N-1}}\log n\big) \\ &= (\beta_0 - \epsilon)\frac{\omega_{N-1}}{N}r^N \exp\Big[\Big(\frac{\alpha_0 t_n^{\frac{N}{N-1}}}{\alpha_N} - 1\Big)N\log n\Big] \end{aligned}$$

for large n, which implies that (t_n) is a bounded sequence. Moreover, using (12) we obtain

$$t_n^N \to \Big(\frac{\alpha_N}{\alpha_0}\Big)^{N-1}, \quad \text{as } n \to \infty. \tag{15}$$

Let

$$A_n = \{x \in B(x_0, r) : t_n M_n \geq R_\epsilon\} \quad \text{and} \quad B_n = B(x_0, r) \backslash A_n.$$

Using (13) and (14), we can estimate

$$\begin{aligned} t_n^N \geq (\beta_0 - \epsilon) \int_{B(x_0,r)} \exp\big(\alpha_0 |t_n M_n|^{\frac{N}{N-1}}\big) + \int_{B_n} t_n M_n f(x, t_n M_n) \\ - (\beta_0 - \epsilon) \int_{B_n} \exp\big(\alpha_0 |t_n M_n|^{\frac{N}{N-1}}\big). \end{aligned}$$

Note that $M_n(x) \to 0$ for almost every $x \in B(x_0, r)$ and the characteristic functions $\chi_{B_n} \to 1$ almost everywhere in $B(x_0, r)$. Therefore, in view of the Lebesgue Dominated Convergence Theorem, we have

$$\int_{B_n} t_n M_n f(x, t_n M_n) \to 0 \quad \text{and} \quad \int_{B_n} \exp\big(\alpha_0 |t_n M_n|^{\frac{N}{N-1}}\big) \to \frac{\omega_{N-1}}{N} r^N, \quad \text{as } n \to \infty.$$

Note also that

$$\int_{B(x_0,r)} \exp\big(\alpha_0 |t_n M_n|^{\frac{N}{N-1}}\big) \geq \int_{B(x_0,r)} \exp\big(\alpha_N |M_n|^{\frac{N}{N-1}}\big) = r^N \int_{B(x_0,1)} \exp\big(\alpha_N |\tilde{M}_n|^{\frac{N}{N-1}}\big)$$

and denoting the last integral by I_n, we have

$$\begin{aligned} I_n &= \int_{|x| \leq \frac{1}{n}} \exp\Big[\frac{\alpha_N}{\omega_{N-1}^{\frac{1}{N-1}}} (\log n)\Big] + \int_{\frac{1}{n} \leq |x| \leq 1} \exp\Big[\frac{\alpha_N}{\omega_{N-1}^{\frac{1}{N-1}}} \frac{(\log |x|^{-1})^{\frac{N}{N-1}}}{(\log n)^{\frac{1}{N-1}}}\Big] \\ &= \frac{\omega_{N-1}}{N} \frac{1}{n^N} \exp\big[N(\log n)\big] + \omega_{N-1} \int_{\frac{1}{n}}^1 s^{N-1} \exp\Big[N \frac{(\log |s|^{-1})^{\frac{N}{N-1}}}{(\log n)^{\frac{1}{N-1}}}\Big] ds \\ &= \frac{\omega_{N-1}}{N} \Big\{1 + \int_0^1 N \log n \exp\big[N \log n \big(\tau^{\frac{N}{N-1}} - \tau\big)\big] d\tau\Big\}, \end{aligned}$$

where in the last integral we have used the change of variable $\tau = \log |s|^{-1} / \log n$. Thus, passing to limits and using (15), we obtain

$$\begin{aligned} \Big(\frac{\alpha_N}{\alpha_0}\Big)^{N-1} &\geq (\beta_0 - \epsilon) \frac{\omega_{N-1}}{N} r^N \lim_{n \to \infty} N \log n \int_0^1 \exp\big[N \log n \big(\tau^{\frac{N}{N-1}} - \tau\big)\big] d\tau \\ &= (\beta_0 - \epsilon) \frac{\omega_{N-1}}{N} r^N \mathcal{M}, \quad \forall \epsilon > 0, \end{aligned}$$

which implies

$$\beta_0 \leq \left(\frac{N}{r}\right)^N \frac{1}{\mathcal{M}\alpha_0^{N-1}},$$

but this contradicts (11).

Remark. In order to prove that a Palais-Smale sequence converges to a solution of problem (1) we need to establish the following preliminary lemma. We observe that an analogous result to our next lemma has been proved in [18, 19, 14], where f is assumed to have polynomial growth. Recently in [15] a similar result has been proved for nonlinearities f that behave like $\exp(\alpha|u|^{\frac{N}{N-1}})$ as $|u| \to \infty$, under the further assumption that $f(x,u)u \geq NF(x,u)$. Inspired by the results mentioned, we have achieved the following version which does not use this further assumption.

Lemma 4. *Let $(u_n) \subset W_0^{1,N}(\Omega)$ be a Palais-Smale sequence; i.e.,*

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0 \quad \text{in} \quad W^{-1,N'}(\Omega) \quad \text{as} \quad n \to \infty.$$

Then (u_n) has a subsequence, still denoted by (u_n), and $u \in W_0^{1,N}(\Omega)$ such that

$$\begin{cases} f(x,u_n) \to f(x,u) & \text{in} \quad L^1(\Omega) \\ |\nabla u_n|^{N-2}\nabla u_n \rightharpoonup |\nabla u|^{N-2}\nabla u & \text{weakly in} \quad \left(L^{N/(N-1)}(\Omega)\right)^N. \end{cases}$$

Proof. Let $(u_n) \subset W_0^{1,N}(\Omega)$ be a Palais-Smale sequence; i.e.,

$$\frac{1}{N}\int |\nabla u_n|^N - \int F(x,u_n) \to c, \tag{16}$$

$$\left| \int |\nabla u_n|^{N-2}\nabla u_n . \nabla v \, dx - \int f(x,u_n)v \, dx \right| \leq \varepsilon_n \|v\|_{W_0^{1,N}}, \quad \forall v \in W_0^{1,N}(\Omega), \tag{17}$$

where $\varepsilon_n \to 0$ as $n \to \infty$. Multiplying (16) by θ, subtracting (17), with $v = u_n$, from the expression obtained, we come to the conclusion that

$$\int |\nabla u_n|^N - \int (\theta F(x,u_n) - f(x,u_n)u_n) \leq C + \varepsilon_n \|u_n\|_{W_0^{1,N}}.$$

From this inequality, using (9) we easily find that (u_n) is a bounded sequence in $W_0^{1,N}(\Omega)$. Consequently $|\nabla u_n|^{N-2}\nabla u_n$ is bounded in $\left(L^{N/(N-1)}(\Omega)\right)^N$ and $\int F(x,u_n) \leq C$, $\int f(x,u_n)u_n \leq C$. Moreover, passing to a subsequence, we may assume that

$$\begin{aligned} &u_n \rightharpoonup u \text{ in } W_0^{1,N}(\Omega), \quad u_n \to u \text{ in } L^q(\Omega), \ \forall q \geq 1, \\ &u_n(x) \to u(x) \text{ almost everywhere in } \Omega. \end{aligned} \tag{18}$$

Now we have that $f(x,u_n) \to f(x,u)$ in $L^1(\Omega)$, as a consequence of the following result of convergence, whose proof can be found in [10].

Lemma 5. *Let (u_n) in $L^1(\Omega)$ such that $u_n \to u$ in $L^1(\Omega)$ and let f be a continuous function. Then $f(x, u_n) \to f(x, u)$ in $L^1(\Omega)$, provided that $f(x, u_n(x)) \in L^1(\Omega)$ $\forall n$ and $\int |f(x, u_n(x))u_n(x)| \leq C_1$.*

Now we are going to prove the second assertion in Lemma 4. Note that without loss of generality we may assume that

$$|\nabla u_n|^N \to \mu \quad \text{in} \quad \mathcal{D}'(\Omega) \quad \text{and}$$
$$|\nabla u_n|^{N-2}\nabla u_n \rightharpoonup V \quad \text{weakly in} \quad \left(L^{N/(N-1)}(\Omega)\right)^N,$$

where μ is a nonnegative regular measure.

Let $\sigma > 0$ and $\mathcal{A}_\sigma = \{x \in \bar{\Omega} : \forall r > 0,\ \mu(B_r(x) \cap \bar{\Omega}) \geq \sigma\}$. We claim that $\mathcal{A}_\sigma$ is a finite set. Suppose for the sake of contradiction that there exists a sequence of distinct points (x_k) in $\mathcal{A}_\sigma$. Since $\forall r > 0$, $\mu(B_r(x_k) \cap \bar{\Omega}) \geq \sigma$, we have that $\mu(\{x_k\}) \geq \sigma$, which implies that $\mu(\mathcal{A}_\sigma) = \infty$, contradicting

$$\mu(\mathcal{A}_\sigma) = \lim_{k\to\infty} \int_{\mathcal{A}_\sigma} |\nabla u_n|^N \leq C.$$

Therefore $\mathcal{A}_\sigma = \{x_1, \ldots, x_m\}$.

Let $u \in W^{1,N}(\mathbb{R}^N)$, have compact support in $\mathbb{R}^N$. We know (cf. [9]) that there are positive constants C_1, C_2 depending only on N such that

$$\int_\Omega \exp\Big(C_1\Big(\frac{|u|}{\|\nabla u\|_{L^N}}\Big)^{\frac{N}{N-1}}\Big)\, dx \leq C_2|\mathrm{supp}(u)|. \tag{19}$$

Assertion 1. *Using estimate* (5), *if we choose $\sigma > 0$ such that $\sigma^{1/(N-1)}\beta < C_1$, then*

$$\lim_{n\to\infty} \int_K f(x, u_n(x))u_n(x) = \int_K f(x, u(x))u(x),$$

for any relatively compact subset K of $\bar{\Omega}\backslash\mathcal{A}_\sigma$.

Proof. Let $x_0 \in K$ and $r_0 > 0$ such that $\mu(B_{r_0}(x_0) \cap \bar{\Omega}) < \sigma$. Consider φ a C^∞ function such that $0 \leq \varphi(x) \leq 1$, $\varphi \equiv 1$ in $B_{r_0/2}(x_0) \cap \bar{\Omega}$ and $\varphi \equiv 0$ in $\bar{\Omega}\backslash(B_{r_0}(x_0) \cap \bar{\Omega})$. Thus

$$\lim_{n\to\infty} \int_{B_{r_0}(x_0)\cap\bar{\Omega}} |\nabla u_n|^N \varphi = \int_{B_{r_0}(x_0)\cap\bar{\Omega}} \varphi\, d\mu < \mu(B_{r_0}(x_0) \cap \bar{\Omega}) < \sigma.$$

Therefore

$$\int_{B_{r_0/2}(x_0)\cap\bar{\Omega}} |\nabla u_n|^N \leq (1 - \varepsilon)\sigma,$$

for large n, if we take ε sufficiently small. Now using this last estimate and the Trudinger-Moser inequality given in (19), from (5) we obtain that

$$\int_{B_{r_0/2}(x_0)\cap\bar{\Omega}} |f(x, u_n(x))|^q\, dx \leq C, \tag{20}$$

if we choose $q > 1$ sufficiently close to 1. Notice that

$$\begin{aligned}&\int_{B_{r_0/2}(x_0)\cap\bar{\Omega}} |f(x,u_n)u_n - f(x,u)u|\,dx\\ &\leq \int_{B_{r_0/2}(x_0)\cap\bar{\Omega}} |f(x,u_n) - f(x,u)||u|\,dx + \int_{B_{r_0/2}(x_0)\cap\bar{\Omega}} |f(x,u_n)||u_n - u|\,dx.\end{aligned}$$

Since $f(x,u_n) \to f(x,u)$ in $L^1(\Omega)$, then $f(x,u_n)v \to f(x,u)v$ in $L^1(\Omega)$, $\forall v \in \mathcal{D}(\Omega)$ and consequently using the argument of density we obtain

$$\lim_{n\to\infty} \int_{B_{r_0/2}(x_0)\cap\bar{\Omega}} |f(x,u_n) - f(x,u)||u|\,dx = 0.$$

On the other hand, using Hölder's inequality and estimate (20) we have

$$\int_{B_{r_0/2}(x_0)\cap\bar{\Omega}} |f(x,u_n)||u_n - u|\,dx \leq C \int_{B_{r_0/2}(x_0)\cap\bar{\Omega}} |u_n - u|^{q'} \to 0.$$

Thus

$$\int_{B_{r_0/2}(x_0)\cap\bar{\Omega}} |f(x,u_n)u_n - f(x,u)u|\,dx \to 0,$$

and Assertion 1 follows by compactness.

Let $\varepsilon_0 > 0$ be fixed small enough such that $B(x_i,\varepsilon_0) \cap B(x_j,\varepsilon_0) = \phi$ if $i \neq j$ and $\Omega_{\varepsilon_0} = \{x \in \bar{\Omega} : \|x - x_j\| \geq \varepsilon_0,\ j = 1,\dots,m\}$.

Assertion 2. $\int_{\Omega_{\varepsilon_0}} (|\nabla u_n|^{N-2}\nabla u_n - |\nabla u|^{N-2}\nabla u)(\nabla u_n - \nabla u) \to 0$ *as* $n \to \infty$.

Proof. Let $0 < \varepsilon < \varepsilon_0$, $\varphi \in C^\infty(\mathbb{R}^N, [0,1])$ such that $\varphi \equiv 1$ in $B_{1/2}(0)$ and $\varphi \equiv 0$ in $B_1(0)$ and we define $\psi_\varepsilon(x) = 1 - \sum_{j=1}^m \varphi(\frac{x-x_j}{\varepsilon})$. Notice that $0 \leq \psi_\varepsilon \leq 1$, $\psi_\varepsilon \equiv 1$ in $\bar{\Omega}_\varepsilon = \bar{\Omega}\backslash \bigcup_{j=1}^m B(x_j,\varepsilon)$, $\psi_\varepsilon \equiv 0$ in $\bigcup_{j=1}^m B(x_j,\varepsilon/2)$ and $\psi_\varepsilon u_n$ is a bounded sequence in $W_0^{1,N}(\Omega)$, for each ε.

Using (17) with $v = \psi_\varepsilon u_n$ we have

$$\int_\Omega \left[|\nabla u_n|^N \psi_\varepsilon + u_n|\nabla u_n|^{N-2}\nabla u_n \nabla\psi_\varepsilon - \psi_\varepsilon f(x,u_n)u_n\right] \leq \varepsilon_n \|\psi_\varepsilon u_n\|_{W_0^{1,N}}. \tag{21}$$

Similarly, using (17) with $v = \psi_\varepsilon u$ we obtain

$$\begin{aligned}&\int_\Omega \left[-|\nabla u_n|^{N-2}\psi_\varepsilon \nabla u_n \cdot \nabla u - |\nabla u_n|^{N-2} u \nabla u_n \nabla\psi_\varepsilon + \psi_\varepsilon f(x,u_n)u\right]\\ &\quad\leq \varepsilon_n \|\psi_\varepsilon u\|_{W_0^{1,N}}.\end{aligned} \tag{22}$$

Now, since the function $g : \mathbb{R}^N \to \mathbb{R}$, $g(v) = |v|^N$ is strictly convex we have that

$$0 \leq (|\nabla u_n|^{N-2}\nabla u_n - |\nabla u|^{N-2}\nabla u)(\nabla u_n - \nabla u)$$

and consequently

$$0 \le \int_{\bar{\Omega}\varepsilon_0} \big(|\nabla u_n|^{N-2}\nabla u_n - |\nabla u|^{N-2}\nabla u\big)\big(\nabla u_n - \nabla u\big)$$
$$\le \int_{\Omega} \big(|\nabla u_n|^{N-2}\nabla u_n - |\nabla u|^{N-2}\nabla u\big)\big(\nabla u_n - \nabla u\big)\psi_\varepsilon,$$

which can be written as

$$0 \le \int_{\Omega} \big[|\nabla u_n|^N \psi_\varepsilon - |\nabla u_n|^{N-2}\psi_\varepsilon \nabla u_n \nabla u - |\nabla u|^{N-2}\psi_\varepsilon \nabla u \nabla u_n + |\nabla u|^N \psi_\varepsilon\big].$$

Thus, from (21) and (22) we obtain

$$0 \le \int_{\Omega} \big[-|\nabla u_n|^{N-2} u_n \nabla u_n \nabla \psi_\varepsilon + \psi_\varepsilon f\big(x, u_n\big)u_n\big] + \varepsilon_n \|\psi_\varepsilon u_n\|_{W_0^{1,N}}$$
$$+ \Big[\int_{\Omega} |\nabla u_n|^{N-2} u \nabla u_n \nabla \psi_\varepsilon - \psi_\varepsilon f\big(x, u_n\big)u\Big] + \varepsilon_n \|\psi_\varepsilon u\|_{W_0^{1,N}}$$
$$+ \int_{\Omega} \big[|\nabla u|^N \psi_\varepsilon - |\nabla u|^{N-2}\psi_\varepsilon \nabla u \nabla u_n\big];$$

that is,

$$0 \le \int_{\Omega} |\nabla u_n|^{N-2}\nabla u_n \nabla \psi_\varepsilon\big(u - u_n\big) + \int_{\Omega} \psi_\varepsilon |\nabla u|^{N-2}\nabla u\big(\nabla u - \nabla u_n\big)$$
$$+ \int_{\Omega} \psi_\varepsilon f\big(x, u_n\big)\big(u_n - u\big) + \varepsilon_n \|\psi_\varepsilon u_n\|_{W_0^{1,N}} + \varepsilon_n \|\psi_\varepsilon u\|_{W_0^{1,N}}. \tag{23}$$

Now we estimate each integral in (23) separately. Note that for arbitrary $\delta > 0$, using the interpolation inequality $ab \le \delta a^{N/N-1} + C_\delta b^N$, with $C_\delta = \delta^{1-N}$, we have

$$\int_{\Omega} |\nabla u_n|^{N-2}\nabla u_n \nabla \psi_\varepsilon\big(u - u_n\big) \le \delta \int_{\Omega} |\nabla u_n|^N + C_\delta \int_{\Omega} |\nabla \psi_\varepsilon|^N |u - u_n|^N$$
$$\le \delta C + C_\delta \Big(\int_{\Omega} |\nabla \psi_\varepsilon|^{rN}\Big)^{1/r} \Big(\int_{\Omega} |u - u_n|^{sN}\Big)^{1/s}.$$

Thus, since $u_n \to u$ in $L^{sN}\big(\Omega\big)$ and δ is arbitrary we obtain that

$$\lim_{n\to\infty} \sup \int_{\Omega} |\nabla u_n|^{N-2}\nabla u_n \nabla \psi_\varepsilon\big(u - u_n\big) \le 0. \tag{24}$$

Now, using that $u_n \rightharpoonup u_0$ in $W_0^{1,N}(\Omega)$ we have

$$\int_{\Omega} \psi_\varepsilon |\nabla u|^{N-2}\nabla u\big(\nabla u - \nabla u_n\big) \to 0 \text{ as } n \to \infty. \tag{25}$$

On the other hand

$$\int_\Omega \psi_\varepsilon f(x, u_n)(u_n - u) \to 0 \text{ as } n \to \infty, \tag{26}$$

since

$$\int_\Omega \psi_\varepsilon f(x, u_n)(u_n - u) = \int_\Omega \psi_\varepsilon f(x, u_n)u_n - \int_\Omega \psi_\varepsilon f(x, u)u + \int_\Omega \psi_\varepsilon f(x, u)u - \int_\Omega \psi_\varepsilon f(x, u_n)u,$$

and applying Assertion 1 to the function $g(x, u) = \psi_\varepsilon(x) f(x, u)$ and $K = \bar{\Omega}_{\varepsilon/2}$ we have that

$$\int_\Omega \psi_\varepsilon f(x, u_n)u_n = \int_{\bar{\Omega}_{\varepsilon/2}} \psi_\varepsilon f(x, u_n)u_n \to \int_{\bar{\Omega}_{\epsilon/2}} \psi_\varepsilon f(x, u)u = \int_\Omega \psi_\varepsilon f(x, u)u \text{ as } n \to \infty,$$

and using that $f(x, u_n) \to f(x, u)$ in L^1 we obtain

$$\int_\Omega \psi_\varepsilon f(x, u_n)u \to \int_\Omega \psi_\varepsilon f(x, u)u \text{ as } n \to \infty.$$

Therefore, from (23) using (24)–(26), we come to the conclusion that Assertion 2 holds.

Finally using Assertion 2, because ε_0 is arbitrary, we obtain that

$$\nabla u_n \to \nabla u \quad \text{almost everywhere in} \quad \Omega.$$

This result and the fact that the sequence $(|\nabla u_n|^{N-2}\nabla u_n)$ is bounded in $L^{N/(N-1)}(\Omega)$, imply, passing to a subsequence, that

$$|\nabla u_n|^{N-2}\nabla u_n \rightharpoonup |\nabla u|^{N-2}\nabla u \quad \text{in} \quad L^{N/(N-1)}(\Omega).$$

Thus, we have completed the proof of Lemma 4.

3. Proof of the existence result. First we recall the following version of Mountain Pass Lemma (cf. [5]).

Theorem 2. *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$. Suppose there exist a neighbourhood U of 0 in E and a positive constant α which satisfy the following conditions*:

I_1) $I(0) = 0$,

I_2) *$I(u) \geq \alpha$ on the boundary of U*,

I_3) *There exists an $e \notin U$ such that $I(e) < \alpha$.*

Then, for the constant

$$c = \inf_{\gamma \in \Gamma} \max_{u \in \gamma} I(u) \geq \alpha$$

there exists a sequence (u_n) *in* E *such that*

$$I(u_n) \to c \quad \textit{and} \quad I'(u_n) \to 0,$$

where $\Gamma = \{g \in C([0, 1], E) : g(0) = 0, g(1) = e\}$.

In view of Lemma 1 and Lemma 2, we can apply the Mountain-Pass Lemma to obtain a positive level c and a Palais-Smale sequence (u_n) in $W_0^{1,N}(\Omega)$, i.e., satisfying (16) and (17). Thus, as in Lemma 4, we have that (u_n) is a bounded sequence in $W_0^{1,N}(\Omega)$ and consequently

$$\int F(x, u_n) \leq C \quad \text{and} \quad \int f(x, u_n)u_n \leq C.$$

So for a subsequence still denoted by (u_n), we may assume that

$$\begin{aligned} &u_n \rightharpoonup u_0 \text{ in } W_0^{1,N}(\Omega), \; u_n \to u_0 \text{ in } L^q(\Omega), \; \forall q \geq 1, \\ &u_n(x) \to u_0(x) \text{ almost everywhere in } \Omega. \end{aligned}$$

Now from (F_2) and Lemma 4, using the generalized Lebesgue Dominated Convergence Theorem, we have

$$F(x, u_n) \to F(x, u_0) \text{ in } L^1(\Omega). \tag{27}$$

So, from (16) and (27) we obtain

$$\lim_{n\to\infty} \int |\nabla u_n|^N = N\Big(c + \int F(x, u_0)\Big). \tag{28}$$

Notice that (17) and Lemma 4 imply that

$$\int |\nabla u_0|^{N-2} \nabla u_0 \cdot \nabla w - \int f(x, u_0)w = 0, \quad \forall w \in \mathcal{D}(\Omega).$$

By using an argument of density, this identity holds for all w in $W_0^{1,N}(\Omega)$. Hence u_0 is a weak solution of problem (1).

Finally it only remains to prove that u_0 is nontrivial. Assume for the sake of contradiction that $u_0 \equiv 0$. From (28) we get

$$\lim_{n\to\infty} \int |\nabla u_n|^N = Nc.$$

Thus given $\epsilon > 0$, we have $\|u_n\|^N_{W_0^{1,N}} \leq Nc + \epsilon$, for large n. Since from Lemma 3 the level $c < \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}$ we have $q\alpha_0 \|u_n\|^{\frac{N}{N-1}}_{W_0^{1,N}} < \alpha_N$ if we choose $q > 1$ sufficiently close to 1 and ϵ sufficiently small. Then, from the Trudinger-Moser inequality using estimate (5) with $\beta = q\alpha_0$, we obtain

$$\int |f(x, u_n(x))|^q \, dx \leq C \int \exp(q\alpha_0 |u_n|^{\frac{N}{N-1}}) \, dx$$
$$\leq C \int \exp\big[q\alpha_0 \|u_n\|_{W_0^{1,N}}^{\frac{N}{N-1}} \big(\frac{u_n}{\|u_n\|_{W_0^{1,N}}}\big)^{\frac{N}{N-1}}\big] \, dx \leq C.$$

Now using this estimate, from (17) with $v = u_n$ we have $u_n \to 0$ in $W_0^{1,N}(\Omega)$. But this is impossible in view of $\lim_{n\to\infty} \int |\nabla u_n|^N = Nc$ and $c \neq 0$. Consequently $u_0 \not\equiv 0$.

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