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On a class of semilinear Schrödinger equations involving critical growth and discontinuous nonlinearities☆

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Abstract

In this paper, we deal with a class of Schrödinger equation in $\mathbb{R}^N$ involving critical Sobolev exponent and jump discontinuities. The basic tool employed here is an approximation technique with periodic functions and variational arguments based on a linking theorem for locally Lipschitz functionals.

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1. Introduction

The main purpose of this paper is to establish the existence of solution for the Schrödinger equation

$$-\Delta u + V(x)u = K(x)|u|^{2^*-2}u + f(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $2^* = 2N/(N-2)$, $N \geqslant 3$, is the critical Sobolev exponent and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is given by

$$f(x,t) = \rho(x)t^{p-1}H(t-a),$$

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where H is the Heaviside function, $a > 0$ and $p \in (2, 2^*)$. We assume that V, ρ and K are continuous and 1-periodic functions in each variable. Furthermore, ρ is nonnegative and K is positive in $\mathbb{R}^N$.

We notice that by a solution for (1.1) we mean a function $u \in W^{1,s}_{\mathrm{loc}}(\mathbb{R}^N)$, for some $s > 1$, verifying, in an appropriate weak sense, the following inequalities:

$$f(x, u(x) - 0) \leqslant -\Delta u + V(x)u - K(x)|u(x)|^{2^*-2}u(x) \leqslant f(x, u(x) + 0), \tag{1.2}$$

where

$$f(x, t + 0) = \lim_{s \downarrow t} f(x, s) \quad \text{and} \quad f(x, t - 0) = \lim_{s \uparrow t} f(x, s).$$

Throughout this paper we will be using the following assumptions:

(h_1) 0 is in the spectral gap of the operator $-\Delta + V$,

(h_2) $0 < \max_{B_1(0)} K = K(0)$ and $K(x) = K(0) + O(|x|)$ for $x \in B_1(0)$.

The main result of this paper is stated as follows:

Theorem 1.1. *Suppose* (h_1)–(h_2) *hold. Furthermore assume that there is* $0 < r \leqslant 1$ *such that*

(h_3) $\rho(x)(|x|^\alpha + 1) \geqslant 1$ *for all* $x \in B_r(0)$,

where α *is a positive real number verifying*

$$\alpha > \begin{cases} \max\{2, N - 1 - p(N-2)\} & \text{if } 2 < p < (N+2)/(N-2), \\ p(N-2) - N & \text{if } (N+2)/(N-2) < p < 2N/(N-2). \end{cases}$$

Then, for each $a > 0$ *fixed, there is a solution* $u = u_a$ *of* (1.1).

Remark 1.2. Assumptions like (h_1)–(h_2) are quite natural and have already appeared in the papers [12,14,25].

Furthermore, it should be remarked that in the proof of Theorem 1.1, in place of (h_3) we use the technical assumption

$$\int_{B_{\sqrt{\varepsilon}}(0)} \rho(x) \left(\frac{\varepsilon}{|x|^2 + \varepsilon^2} \right)^{p(N-2)/2} \mathrm{d}x \geqslant O(\varepsilon^\tau),$$

where $\tau < \min\{(N-2)/2,\ N - p(N-2)/2\}$, which included the family of functions ρ satisfying (h_3).

Remark 1.3. The set defined by

$$\Lambda_a(u) = \{x \in \mathbb{R}^N,\ u(x) = a\}$$

has a great importance relating to the regularity of the solution u. In fact, if the Lebesgue measure of $\Lambda_a(u)$ is zero, then u is a solution in the *almost everywhere sense*, that means, u satisfies

$$-\Delta u(x) + V(x)u(x) = K(x)|u(x)|^{2^*-2}u(x) + f(u(x)), \tag{1.3}$$

almost everywhere in $\mathbb{R}^N$. Now, by applying Stampacchia theorem in the set $\Lambda_a(u)$ (see [28]), we obtain the relation

$$K(x)a^{2^*-2} \leqslant V(x) \leqslant K(x)a^{2^*-2} + \rho(x)a^{p-2}, \tag{1.4}$$

which represents a condition involving K, V, ρ, and a. Therefore, if the set characterized by condition (1.4) has measure zero, then the set $\Lambda_a(u)$ also has measure zero. We can deduce that u satisfies (1.3). Thus, a natural assumption to get a solution in the almost everywhere sense is the following:

$$meas(\{x \in \mathbb{R}^N : K(x)a^{2^*-2} \leqslant V(x) \leqslant K(x)a^{2^*-2} + \rho(x)a^{p-2}\}) = 0.$$

We notice that we can present a simple case where this hypothesis holds, for instance in condition

$$\sup_{x\in\mathbb{R}^N} V(x) \leqslant \sup_{x\in\mathbb{R}^N} K(x)a^{2^*-2}.$$

An equation of type (1.1) is related to the so-called Grad-Schafranov equation of Plasma Physics and obstacle problems. For the background and related results on some typical models involving discontinuous nonlinearities we refer the reader to [3–5,10,11,15–17,20]. There is an extensive bibliography dealing with semilinear Schrödringer equations with periodic potential. At first, let us recall the so-called definite case, that is, when V is strictly positive. In [24], Pankov using the Nehari variational principle, proved the existence of ground states, i.e., solutions having smallest energy among all nontrivial solutions. Rabinowitz in [26], under less restrictive assumptions on $f(x,s)$, has obtained a result of existence but not necessarily a ground state. Moreover, in [18], Coti Zelati and Rabinowitz have proved the existence of infinitely many solutions under some additional technical assumptions.

When it is the case that V is indefinite and 0 lies in a gap of the spectrum, $H^1(\mathbb{R}^N)$ is the direct sum of two infinite dimensional subspaces where the quadratic part of the variational functional is negative and positive, respectively. Thus it is not possible to use the Leray–Schauder degree like in the proof of the Benci–Rabinowitz mountain pass theorem (see [6]). This class of problems, under the additional assumption that the primitive F is strictly convex, has been explored by many authors including [1,9,19,22]. This assumption has allowed them to solve the problem via a reduction method by applying the mountain-pass theorem.

In recent papers Troestler and Willem [30], and Kryszewski and Szulkin [21] have proved a result of existence for this class based on the generalized linking theorem. This linking theorem requires the construction of a new degree theory. This approach has been simplified by Pankov and Pflüger [25] by using the approximation technique with periodic functions. Later, Chabrowski and Jianfu [12], used this same approach in dealing with a periodic semilinear Schrödringer equation and critical Sobolev exponent. In this paper, we also apply this technique to obtain an existence result for Eq. (1.1). The crucial point in the approach presented here lies in the fact that the approximation technique of [25] can be combined with the methods developed in [13] to determine the range for level sets of the energy functional for which the Palais–Smale condition holds. This allows us to obtain an approximating sequence by applying a linking theorem for local Lipschitz functionals.

This paper is composed of three sections. In the next section we shall prove preliminary results and the main result in the third section.

Notation. In this paper we make use of the following notation:

- $c, c_1, c_2, \ldots$ denote (possibly different) positive constants;
- $B_R(p)$ denotes the open ball with the radius R centered at point p of $\mathbb{R}^N$;
- $L^p(\Omega)$, $1 \leqslant p \leqslant \infty$, denote Lebesgue spaces; the norm in $L^p(\Omega)$ is denoted by $|u|_p$;
- S is the optimal constant to the Sobolev embedding, $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, that is,

  $$S = \inf\{|\nabla u|_2^2 \colon u \in D^{1,2}(\mathbb{R}^N) \text{ and } |u|_{2^*} = 1\},$$

  where $D^{1,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ in the norm $\|u\| := (\int_{\mathbb{R}^N} |\nabla u|^2 \, dx)^{1/2}$. It is known (see [29]) that the optimal constant S is attained by the functions

  $$\psi_{\varepsilon,x_o}(x) := \left(\frac{c_N \varepsilon}{(\varepsilon^2 + |x - x_o|^2)} \right)^{(N-2)/2}, \quad \text{where } c_N := (N(N-2))^{1/2}. \tag{1.5}$$

2. Preliminary results

To prove Theorem 1.1 we will combine variational methods applied to locally lipschitzian functionals and an approximation technique as in [12,25]. As starting point, we solve the problem

$$\begin{cases} -\Delta u + V(x)u = K(x)|u|^{2^*-2}u + f(x,u) & \text{in } Q_k, \\ u \in H^1_{\mathrm{per}}(Q_k) \end{cases} \tag{$1.1_{\mathrm{a,k}}$}$$

where Q_k is a cube in $\mathbb{R}^N$ with length of edge $k \in \mathbf{N}$, $L^2_{\mathrm{per}}(Q_k)$ is the space of k-periodic functions of $L^2(Q_k)$, and

$H^1_{\mathrm{per}}(Q_k) = H^1(Q_k) \cap L^2_{\mathrm{per}}(Q_k)$.

The proof of the result of existence for problem $(1.1)_{a,k}$ will be based on the next critical point theorem and its proof follows the same kind of ideas as those used in the proof of an analogous result for differential functionals (see [2,7]).

In what follows let X be a Banach space, $\Phi \in Lip_{\mathrm{loc}}(X, \mathbb{R})$ means that the functional Φ is locally lipschitzian from X to $\mathbb{R}$ and we denote by $\partial\Phi$ the generalized gradient at the point $u \in X$ of Φ (see [16]).

Theorem 2.1. *Let $X = Y \oplus Z$ with $\dim Y < \infty$. Let $R > R_1 > 0$ and $z \in Z$ such that $\|z\| = R_1$. Define*

$$M = \{u = y + tz,\ \|u\| \leqslant R,\ t \geqslant 0,\ y \in Y\},$$

$$\Gamma = \{\gamma \in \mathscr{C}(M, X); \gamma|_{\partial M} = id\} \quad \text{and} \quad c = \inf_{\gamma \in \Gamma} \max_{u \in M} I(\gamma(u)), \tag{2.1}$$

where $I \in Lip_{\mathrm{loc}}(X; \mathbb{R})$ verifying

$$\inf_{\substack{\|u\| = R_1 \\ u \in Z}} I(u) > \max_{u \in \partial M} I(u). \tag{2.2}$$

Then there exists a sequence $u_n \in X$ such that

$$I(u_n) \to c \text{ and } \min_{\mu \in \partial I(u_n)} \|\mu\|_{X'} \to 0, \text{ both of limits taken when } n \to \infty. \tag{2.3}$$

The variational functional associated with $(1.1)_{a,k}$ is defined by

$$J_{a,k}(u) = \frac{1}{2} \int_{Q_k} (|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x - \frac{1}{2^*} \Psi_k(u) - \Phi_{a,k}(u), \quad u \in H^1_{\mathrm{per}}(Q_k),$$

where

$$\Phi_{a,k}(u) = \int_{Q_k} \int_0^u f(x,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}x \quad \text{and} \quad \Psi_k(u) = \int_{Q_k} K(x)|u|^{2^*-1} u(x)\,\mathrm{d}x.$$

Using standard arguments (see [16]) we can find that $\Phi_{a,k} \in Lip_{\mathrm{loc}}(L^s(Q_k), \mathbb{R})$ for $2 \leqslant s \leqslant 2^*$ and $\Phi_{a,k}|_{H^1_{\mathrm{per}}(Q_k)} \in Lip_{\mathrm{loc}}(H^1_{\mathrm{per}}(Q_k), \mathbb{R})$. Furthermore, if $\mu \in \partial \Phi_{a,k}(u)$ then

$$f(x, u(x) - 0) \leqslant \mu(x) \leqslant f(x, u(x) + 0), \tag{2.4}$$

in the weak sense.

We recall that the operator $-\Delta + V$ on $L^2_{\mathrm{per}}(Q_k)$ has discrete spectrum with eigenvalues $\lambda_{k,1} \leqslant \cdots \lambda_{k,i} \leqslant \cdots \to \infty$ and there is a finite $\gamma(k)$ minimum of $\{i\colon \lambda_{k,i} > 0\}$. Moreover, every eigenvalue $\lambda_{k,i}$ is contained in the spectrum of $-\Delta + V$ on the whole space and then if $(\alpha,\ \beta)$, $\alpha > 0$ is the spectral gap around 0, we find that $\lambda_{k,i} \notin (\alpha,\ \beta)$ for all $k, i \in \mathbf{N}$. We denote by $\phi_{k,i}$ the corresponding eigenfunctions. Since every function $u \in H^1_{\mathrm{per}}(Q_k)$ is, by periodicity, also in $H^1_{\mathrm{per}}(Q_{mk})$ for every natural number m, we claim that every eigenvalue of $-\Delta + V$ on $L^2_{\mathrm{per}}(Q_k)$ is also an eigenvalue of this operator on $L^2_{\mathrm{per}}(Q_{mk})$ (see [27]).

Furthermore, the space $H^1_{\mathrm{per}}(Q_k)$ can be decomposed in the direct sum of the spaces Y_k, finite dimensional, and Z_k both generated by the eigenfunctions corresponding to negative and positive eigenvalues, respectively.

The quadratic part of $J_{a,k}$,

$$\ell_k(u) = \int_{Q_k} (|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x, \quad u \in H^1_{\mathrm{per}}(Q_k)$$

is positive on Z_k and negative on Y_k. We may define a new scalar product $(\cdot,\cdot)_k$ on $H^1_{\text{per}}(Q_k)$ and a corresponding norm $\|\cdot\|_k$ such that

$$\int_{Q_k} (|\nabla y|^2 + V(x)y^2)\,\mathrm{d}x = -\|y\|_k^2 \quad \text{for } y \in Y_k,$$

$$\int_{Q_k} (|\nabla z|^2 + V(x)z^2)\,\mathrm{d}x = \|z\|_k^2 \quad \text{for } z \in Z_k.$$

Let $P_k : H^1_{\text{per}}(Q_k) \to Y_k$ and $T_k : H^1_{\text{per}}(Q_k) \to Z_k$ be the orthogonal projections of $H^1_{\text{per}}(Q_k)$ onto Y_k and Z_k, respectively. Using these projections we can write the variational functional $J_{a,k}$ by the formula

$$J_{a,k}(u) = \frac{1}{2}(\|T_k u\|_k^2 - \|P_k u\|_k^2) - \frac{1}{2^*}\Psi_k(u) - \Phi_{a,k}(u), \quad u \in H^1_{\text{per}}(Q_k).$$

In order to prove our main result of this section, we begin stating some basic lemmas. Set

$$M_{k,R}(z_0) = \{u = y + tz_0,\ \|u\|_k \leqslant R,\ t \geqslant 0,\ y \in Y_k\} \tag{2.5}$$

for some fixed $z_0 \in Z_k$ and $R > 0$, to be determined later, and

$$\Gamma_k = \{\gamma \in \mathscr{C}(M_{k,R}(z_0), H^1_{\text{per}}(Q_k));\ \gamma|_{\partial M_{k,R}} = id\}. \tag{2.6}$$

We notice that the set $\partial J_{a,k}(u)$ is weakly* compact (see [16]) and, as a consequence, the minimum of $\{\|\mu\|_k,\ \mu \in \partial J_{a,k}(u)\}$ is attained by some $\mu_n^k \in \partial J_{a,k}(u_n^k)$. We will use this fact in the next lemma.

Lemma 2.2. *If $u_n \in H^1_{\text{per}}(Q_k)$ is a sequence verifying*

$$J_{a,k}(u_n^k) \to c^k \quad \textit{with } 0 < c^k < \frac{S^{N/2}}{N|K|_\infty^{(N-2)/2}} \tag{2.7}$$

and

$$\mu_n^k \to 0 \quad \textit{as } n \to \infty, \tag{2.8}$$

then u_n^k is relatively compact in $H^1_{\text{per}}(Q_k)$.

Proof. First we prove that the sequence u_n^k is bounded in $H^1_{\text{per}}(Q_k)$. Let μ_n^k and $\sigma_n^k \in \partial\Phi_{a,k}(u_n^k)$ such that

$$\mu_n^k = \ell_k'(u_n^k) - \Psi_k'(u_n^k) - \sigma_n^k. \tag{2.9}$$

We have for $\|v\|_k = 1$ that $|\langle \mu_n^k, v\rangle| \leqslant \|\mu_n^k\|_k$, as $n \to \infty$, so that we can write

$$|\langle \mu_n^k, u_n^k\rangle| = \varepsilon_n \|u_n^k\|_k \quad \text{with } \varepsilon_n \to 0.$$

Using (2.4) with $u_-(x) = \max\{-u(x), 0\}$ as a function test we get

$$0 = \int_{u_n^k > a} \rho(x)(u_n^k)^{p-1} u_{n-}^k \, dx \leqslant \int_{Q_k} \sigma_n^k u_{n-}^k \, dx \leqslant \int_{u_n^k \geqslant a} \rho(x)(u_n^k)^{p-1} u_{n-}^k \, dx = 0$$

and then

$$\langle \mu_n^k, u_{n-}^k \rangle = 0.$$

Consequently, using again (2.4) with $u_+(x) = \max\{u(x), 0\}$ as a test function, we obtain

$$\begin{aligned} J_{a,k}(u_n^k) - \frac{1}{2}\langle \mu_n^k, u_n^k \rangle &= \frac{1}{N}\int_{Q_k} K(x)|u|^{2^*} \, dx + \frac{1}{2}\langle \sigma_n^k, u_n^k \rangle - \Phi_{a,k}(u_n^k) \\ &= \frac{1}{N}\int_{Q_k} K(x)|u|^{2^*} \, dx + \frac{1}{2}\langle \sigma_n^k, u_{n+}^k \rangle \\ &\quad - \frac{1}{p}\int_{Q_k} \rho(x)((u_n^k)^p - a^p) H(u_n^k - a) \, dx \\ &\geqslant \frac{1}{N}\int_{Q_k} K(x)|u|^{2^*} \, dx + \left(\frac{1}{2} - \frac{1}{p}\right)\int_{Q_k} \rho(x)(u_n^k)^p H(u_n^k - a) \, dx. \end{aligned}$$

This fact combined with (2.7) infer the following crucial inequalities:

$$\frac{1}{N}\int_{Q_k} K(x)|u|^{2^*} \, dx \leqslant c^k + o_n(1) + \frac{\varepsilon_n}{2}\|u_n^k\|_k \tag{2.10}$$

and

$$\frac{p-2}{2p}\int_{Q_k} \rho(x)(u_n^k)^p H(u_n^k - a) \, dx \leqslant c^k + o_n(1) + \frac{\varepsilon_n}{2}\|u_n^k\|_k. \tag{2.11}$$

On the other hand, we have

$$\begin{aligned} J_{a,k}(u_n^k) - \frac{1}{2^*}\langle \mu_n^k, u_n^k \rangle \geqslant{}& \frac{1}{N}(\|T_k u_n^k\|_k^2 - \|P_k u_n^k\|_k^2) \\ &+ \left(\frac{1}{2^*} - \frac{1}{p}\right)\int_{Q_k} \rho(x)(u_n^k)^p H(u_n^k - a) \, dx. \end{aligned}$$

Denoting $T_k(u_n^k) = z_n$ and $P_k(u_n^k) = y_n$ one obtains

$$\begin{aligned} \frac{1}{N}\|z_n\|_k^2 \leqslant{}& \frac{1}{N}\|y_n\|_k^2 + \left(\frac{1}{p} - \frac{1}{2^*}\right)\int_{Q_k} \rho(x)(u_n^k)^p H(u_n^k - a) \, dx \\ &+ c^k + \frac{\varepsilon_n}{2^*}\|u_n^k\|_k + o_n(1), \end{aligned}$$

so that from (2.11) follows

$$\frac{1}{N}\|z_n\|_k^2 \leqslant \frac{1}{N}\|y_n\|_k^2 + \frac{(2^*-p)2p}{2^*p(p-2)}\left(c^k + \frac{\varepsilon_n}{2}\|u_n^k\|_k\right) + c^k + \frac{\varepsilon_n}{2^*}\|u_n^k\|_k + o_n(1).$$

Now, since $\|u_n^k\|_k^2 = \|z_n\|_k^2 + \|y_n\|_k^2$ and $\|y_n\|_k^2 \leqslant c_1|u_n^k|_2^2$ holds, and using (2.7), one gets

$$\frac{1}{N}\|u_n^k\|_k^2 - c_2\|u_n^k\|_k - c_3 \leqslant c_4|u_n^k|_2^2 \tag{2.12}$$

for large n. We notice that, from (2.12), it is sufficient to prove that the L^2 norm of u_n^k on Q_k is bounded to obtain the same result for $\|u_n^k\|_k$, for each fixed k. Suppose, by contradiction, taking a subsequence if necessary, that $|u_n^k|_2^2 \to \infty$ as $n \to \infty$ and define $v_n = u_n^k/|u_n^k|_2$. Thus, one has $|v_n|_2 = 1$ and $\|v_n\|_k \leqslant c$. In fact, by letting n_1 such that $|u_n^k|_2 \geqslant 1$ for $n \geqslant n_1$, and from (2.12), it follows

$$\frac{1}{N}\|v_n\|_k^2 - c_2\|v_n\|_k - c_3 \leqslant \frac{1}{|u_n^k|_2^2}\left(\frac{1}{N}\|u_n^k\|_k^2 - c_2\|u_n^k\|_k - c_3\right) \leqslant c_4,$$

which implies that $\|v_n\|_k$ is bounded.

Now, we take $\phi \in C_0^\infty(Q_k)$ and use (2.9) to obtain

$$\int_{Q_k}(\nabla u_n^k \nabla\phi + V(x)u_n^k\phi)\,\mathrm{d}x = \int_{Q_k}K(x)|u_n^k|^{2^*-1}\phi\,\mathrm{d}x + \langle\sigma_n^k,\phi\rangle + o_n(1). \tag{2.13}$$

To proceed further, we shall estimate the two terms on the right-hand side using inequalities (2.10) and (2.11) as follows:

$$\begin{aligned}&\int_{Q_k}K(x)|u_n^k|^{2^*-1}|\phi|\,\mathrm{d}x\\ &\quad\leqslant \left(\int_{Q_k}(K(x)|u_n^k|^{2^*-1}\,\mathrm{d}x)^{2^*/(2^*-1)}\right)^{(2^*-1)/2^*}|\phi|_{2^*}\\ &\quad\leqslant |\phi|_{2^*}|K|_\infty^{1/2^*}N^{(2^*-1)/2^*}(c^k + o_n(1) + \varepsilon_n\|u_n^k\|_k)^{(2^*-1)/2^*}.\end{aligned} \tag{2.14}$$

On the other hand, from (2.4) we have

$$\begin{aligned}|\langle\sigma_n^k,\phi\rangle| &\leqslant \int_{u_n^k \geqslant a}\rho(x)(u_n^k)^{p-1}|\phi|\,\mathrm{d}x\\ &\leqslant \int_{u_n^k > a}\rho(x)(u_n^k)^{p-1}|\phi| + a^{p-1}\int_{Q_k}\rho(x)|\phi|\,\mathrm{d}x.\end{aligned} \tag{2.15}$$

Besides, using (2.11) we get

$$\begin{aligned}&\int_{u_n^k > a}\rho(x)(u_n^k)^{p-1}|\phi|\\ &\quad\leqslant |\rho|_\infty^{1/p}|\phi|_p\left(\frac{2p}{p-2}\right)^{(p-1)/p}\left(c^k + o_n(1) + \frac{\varepsilon_n}{2}\|u_n^k\|_k\right)^{(p-1)/p}.\end{aligned} \tag{2.16}$$

Consequently from (2.13)–(2.16) follow that

$$\left|\int_{Q_k} (\nabla v_n \nabla \phi + V(x) v_n \phi)\,\mathrm{d}x\right|$$
$$\leqslant \frac{1}{|u_n^k|_2}\left(\int_{Q_k} K(x)|u_n^k|^{2^*}|\phi|\,\mathrm{d}x + |\langle \sigma_n^k, \phi\rangle| + o_n(1)\right)$$
$$\leqslant \frac{c}{|u_n^k|_2}(c(k) + o_n(1) + \tilde{\varepsilon}_n \|u_n^k\|_k^{(2^*-1)/2^*} + \hat{\varepsilon}_n \|u_n^k\|_k^{(p-1)/p}),$$

where $\tilde{\varepsilon}_n$, $\widehat{\varepsilon_n} \to 0$ and $c(k)$ is a constant which depends on k. This implies that

$$\left|\int_{Q_k} (\nabla v_n \nabla \phi + V(x) v_n \phi)\,\mathrm{d}x\right|$$
$$\leqslant o_n(1) + \tilde{\varepsilon}_n |u_n^k|_2^{-1/2^*} \|v_n\|_k^{(2^*-1)/2^*} + \hat{\varepsilon}_n |u_n^k|_2^{-1/p} \|v_n\|_k^{(p-1)/p}$$
$$= o_n(1), \tag{2.17}$$

where here we use that $\|v_n\|_k$ is bounded. Therefore, there exists $v_k \in H^1_{\mathrm{per}}(Q_k)$ such that $v_n \rightharpoonup v^k$ in $H^1_{\mathrm{per}}(Q_k)$ and $v_n \to v^k$ in $L^2(Q_k)$. Since $|v_n|_2 = 1$ one has $|v^k|_2 = 1$. Consequently, $v^k \not\equiv 0$. But, from (2.17) it would hold

$$\int_{Q_k} (\nabla v^k \nabla \phi + V(x) v^k \phi)\,\mathrm{d}x = 0, \qquad \forall \phi \in C_0^\infty(Q_k),$$

which contradicts assumption (h_1). This proved that $|u_n^k|_2$ is bounded. As a consequence, the norm $\|u_n^k\|_k$ is as well bounded.

Now, taking subsequence if necessary, we have a function $u_k \in H^1_{\mathrm{per}}(Q_k)$, $u_n^k \rightharpoonup u_k$ in $H^1_{\mathrm{per}}(Q_k)$ and $u_n^k \to u_k$ in $L^s(Q_k)$, $2 \leqslant s < 2^*$.

Next, we will prove that the convergence of u_n^k to u_k is a strong one. Indeed, let

$$w_n^k = u_n^k - u_k \text{ and } 0 \leqslant l = \lim_{n\to\infty} \int_{Q_k} |\nabla w_n^k|^2\,\mathrm{d}x.$$

Then, we have $w_n^k \rightharpoonup 0$ in $H^1_{\mathrm{per}}(Q_k)$, $w_n^k \to 0$ in $L^s(Q_k)$ for all $2 \leqslant s < 2^*$, and

$$\langle \mu_n^k, w_n^k\rangle \geqslant \int_{Q_k} |\nabla w_n^k|^2 + \int_{Q_k} \nabla u_k \nabla w_n^k\,\mathrm{d}x - |V|_\infty |u_n^k|_2 |w_n^k|_2$$
$$- |K|_\infty \int_{Q_k} |u_n^k|^{2^*-1} w_n^k - \int_{u_n^k > a} \rho(x)|u_n^k|^{p-1} w_{n+}^k\,\mathrm{d}x.$$

Thus,

$$l + o_n(1)$$
$$\leqslant |K|_\infty \int_{Q_k} |u_n^k|^{2^*-1} w_n^k\,\mathrm{d}x + \int_{u_n^k \geqslant u} \rho(x)(u_n^k - u_k)\,\mathrm{d}x$$

$$\leqslant |K|_\infty \int_{Q_k} |u_n^k|^{2^*-1} w_n^k \,\mathrm{d}x + \int_{u_n^k \geqslant u} \rho(x)(u_n^k)^p \,\mathrm{d}x - \int_{u_n^k \geqslant u} \rho(x)(u_n^k)^{p-1} u_k \,\mathrm{d}x$$

$$= o_n(1) + |K|_\infty \left(\int_{Q_k} |u_n^k|^{2^*} \,\mathrm{d}x - \int_{Q_k} |u_n^k|^{2^*-1} u_k \,\mathrm{d}x \right)$$

$$= o_n(1) + |K|_\infty \left(\int_{Q_k} |u_k|^{2^*} \,\mathrm{d}x + \int_{Q_k} |w_n^k|^{2^*} \,\mathrm{d}x + o_n(1) - \int_{Q_k} |u_n^k|^{2^*-1} u_k \,\mathrm{d}x \right)$$

$$= o_n(1) + |K|_\infty \int_{Q_k} |w_n^k|^{2^*} \,\mathrm{d}x$$

$$\leqslant o_n(1) + |K|_\infty \left(S^{-1} \int_{Q_k} |\nabla w_n^k|^2 \,\mathrm{d}x \right)^{2^*/2},$$

where here we used the Brézis–Lieb lemma (see [8], Theorem 1). As a consequence, taking limit

$$\frac{S^{N/2}}{|K|_\infty^{(N-2)/2}} \leqslant l. \tag{2.18}$$

On the other hand, since we have

$$|\langle \sigma_n^k,\ \phi \rangle| = \left| -\langle \mu_n^k,\ \phi \rangle + \int_{Q_k} (\nabla u_n^k \nabla \phi + V(x) u_n^k \phi) \,\mathrm{d}x - \int_{Q_k} K(x) |u_n^k|^{2^*-1} \phi \,\mathrm{d}x \right|$$

$$\leqslant \varepsilon_n \|\phi\|_k + c|(u_n^k,\ \phi)| + |K|_\infty |u_n^k|_{L^{2^*}} |\phi|_{L^{2^*}}$$

$$\leqslant c\|\phi\|_k$$

for each test function ϕ, there is $\sigma_0^k \in H^1_{\mathrm{per}}(Q_k)$ such that $\sigma_n^k \rightharpoonup \sigma_0^k$ in $H^1_{\mathrm{per}}(Q_k)$, and $\sigma_n^k \to \sigma_0^k$ in L^s for all $2 \leqslant s < 2^*$.

Now, we will show that the following estimate holds to be true:

$$J_{a,k}(u_n^k) \geqslant \frac{1}{2} \int_{Q_k} |\nabla w_n^k|^2 + \frac{1}{2} \int_{Q_k} K(x) |w_n^k|^{2^*-1} |u_k|$$
$$- \frac{1}{p} \langle \sigma_n^k,\ w_n^k \rangle - \frac{1}{2} \int_{Q_k} K(x) |w_n^k|^{2^*} + o_n(1). \tag{2.19}$$

In fact,

$$J_{a,k}(u_n^k) \geqslant \frac{1}{2} \int_{Q_k} |\nabla w_n^k|^2 \,\mathrm{d}x - \frac{1}{2} \int_{Q_k} |\nabla u_k|^2 \,\mathrm{d}x + \int_{Q_k} \nabla u_n^k \nabla u_k \,\mathrm{d}x$$
$$+ \frac{1}{2} \int_{Q_k} V(x) w_{n^k}^2 \,\mathrm{d}x - \frac{1}{2} \int_{Q_k} V(x) u_k^2 + \int_{Q_k} V(x) u_n^k u_k \,\mathrm{d}x$$

$$
\begin{aligned}
&-\frac{1}{p}\int_{u_n^k>a}\rho(x)(u_n^k)^p\,\mathrm{d}x-\frac{1}{2^*}\int_{Q_k}K(x)|u_n^k|^{2^*}\,\mathrm{d}x\\
&\geqslant \frac{1}{2}\int_{Q_k}|\nabla w_n^k|^2\,\mathrm{d}x+\frac{1}{2}\int_{Q_k}|\nabla u_k|^2\,\mathrm{d}x+\frac{1}{2}\int_{Q_k}V(x)u_k^2\,\mathrm{d}x\\
&\quad-\frac{1}{p}\langle\sigma_n^k,\ u_n^k\rangle-\frac{1}{2^*}\int_{Q_k}K(x)|u_n^k|^{2^*}\,\mathrm{d}x+o_n(1)\\
&=\frac{1}{2}\int_{Q_k}|\nabla w_n^k|^2\,\mathrm{d}x+\frac{1}{2^*}\int_{Q_k}K(x)|u_n^k|^{2^*-1}|u_k|\,\mathrm{d}x\\
&\quad-\frac{1}{p}\langle\sigma_n^k,\ w_n^k\rangle-\frac{1}{2^*}\int_{Q_k}K(x)|u_n^k|^{2^*}\,\mathrm{d}x+o_n(1),
\end{aligned}
$$

where here we have used that the equality

$$
\begin{aligned}
\langle\mu_n^k,|u_k|\rangle=&\int_{Q_k}(\nabla u_n^k\nabla|u_k|\,\mathrm{d}x+V(x)u_n^k|u_k|)\,\mathrm{d}x\\
&-\int_{Q_k}K(x)|u_n^k|^{2^*-1}|u_k|\,\mathrm{d}x-\langle\sigma_n^k,|u_k|\rangle
\end{aligned}
$$

holds. Consequently,

$$
\lim_{n\to\infty}\int_{Q_k}K(x)|u_n^k|^{2^*-1}|u_k|\,\mathrm{d}x=\int_{Q_k}(\nabla|u_k|\,\mathrm{d}x+V(x)|u_k|^2)\,\mathrm{d}x-\langle\sigma_0^k,\ |u_k|\rangle.
$$

Hence, from this and (2.19), we conclude

$$
\begin{aligned}
J_{a,k}(u_n^k)-\frac{1}{2^*}\langle\mu_n^k,w_n^k\rangle\geqslant&\frac{1}{N}\int_{Q_k}|\nabla w_n^k|^2\,\mathrm{d}x+\frac{1}{2}\int_{Q_k}K(x)|u_n^k|^{2^*}|u_k|\,\mathrm{d}x\\
&+\left(\frac{1}{2^*}-\frac{1}{p}\right)\langle\sigma_n^k,w_n^k\rangle+o_n(1)\\
=&\frac{1}{N}\int_{Q_k}|\nabla w_n^k|^2\,\mathrm{d}x+o_n(1).
\end{aligned}\tag{2.20}
$$

Finally, taking limit when $n\to\infty$ in (2.20) it would be seen, from (2.18), that

$$
c^k\geqslant\frac{l}{N}\geqslant\frac{S^{N/2}}{N|K|_\infty^{(N-2)/2}},
$$

which contradicts that $0<c^k<S^{N/2}/N|K|_\infty^{(N-2)/2}$. This proved the lemma. □

In the next lemma we shall check the linking condition (2.2) of Theorem 2.1.

Take $r_0>0$ such that $B_{2r_0}(x_0)\subset Q_1$, where x_0 is a center of the cube Q_1. Let $\zeta\in C_0^\infty(\mathbb{R}^N,[0,1])$ be a cut-off function such that $\zeta\equiv 1$ in $B_{r_0}(x_0)$ and $\zeta\equiv 0$

in $\mathbb{R}^N \backslash B_{2r_0}(x_0)$. For each $\varepsilon > 0$ we set $\phi_\varepsilon(x) = \zeta(x)\psi_{\varepsilon,0}(x)$ (see (1.5)) and extending as a periodic function we have $\phi_\varepsilon \in H^1_{\text{per}}(Q_k)$. Let $\varphi_\varepsilon \doteq \phi_\varepsilon / k^N$ and denote $M_{k,R}(\varepsilon)$ the set $\{y + tT_k\varphi_\varepsilon\colon \|y + tT_k\varphi_\varepsilon\|_k \leqslant R,\ t \geqslant 0,\ y \in Y_k\}$, defined in (2.5).

Lemma 2.3. *There exist $R > R_1 > 0$, independent of k, such that*

$$\inf_{\substack{\|u\|_k = R_1 \\ u \in Z_k}} J_{a,k}(u) \geqslant \sup_{u \in \partial M_{k,R}(\varepsilon)} J_{a,k}(u).$$

Proof. Since ρ is continuous and periodic we have, for $z \in Z_k$, that

$$|\Phi_{a,k}(z)| \leqslant c \int_{Q_k} |z(x)|^p \, \mathrm{d}x \leqslant c\|z\|_k^p,$$

where c depends only on $|\rho|_\infty$. Thus,

$$J_{a,k}(z) \geqslant \frac{1}{2}\|z\|_k^2 - \frac{|K|_\infty}{2^*}\|z\|_k^{2^*} - c\|z,\|_k^p$$

and since $p > 2$, we obtain $R_1 > 0$, independent of k, such that if $\|z\|_k = R_1$ then $J_{a,k}(z) \geqslant \alpha > 0$.

On the other hand, notice that if $u \in \partial M_{k,R}(\varepsilon)$ and $t = 0$, then $J_{a,k}(u) \leqslant 0$. Hence, let $R = \|y + tT_k\varphi_\varepsilon\|_k$ with $t > 0$. Therefore,

$$\begin{aligned} &J_{a,k}(y + tT_k\varphi_\varepsilon) \\ &\quad = -\frac{1}{2}\|y\|_k^2 + \frac{t^2}{2}\|tT_k\varphi_\varepsilon\|_k^2 - \frac{1}{2^*}\int_{Q_k} K(x)|y + tT_k\varphi_\varepsilon|^{2^*}\,\mathrm{d}x - \Phi_{a,k}(u) \\ &\quad \leqslant -\frac{1}{2}\|y\|_k^2 + \frac{t^2}{2}\|tT_k\varphi_\varepsilon\|_k^2 - \frac{1}{2^*}\inf_{x \in \mathbb{R}^N} K(x) \int_{Q_k} |y + tT_k\varphi_\varepsilon|^{2^*}\,\mathrm{d}x, \end{aligned}$$

where here we used (h_2). In accordance with (see [14])

$$|tT_k\varphi_\varepsilon|_{L^{2^*}} \leqslant c|y + tT_k\varphi_\varepsilon|_{L^{2^*}}$$

we obtain

$$J_{a,k}(y + tT_k\varphi_\varepsilon) \leqslant -\frac{1}{2}\|y\|_k^2 + \frac{t^2}{2}\|tT_k\varphi_\varepsilon\|_k^2 - ct^{2^*}|tT_k\varphi_\varepsilon|_{L^{2^*}}^{2^*}.$$

Moreover, since $\|y + tT_k\varphi_\varepsilon\|_k^2 = \|y\|_k^2 + t\|T_k\varphi_\varepsilon\|_k^2$ we infer $\|y + tT_k\varphi_\varepsilon\|_k \to \infty$ if $\|y\|_k^2 \to \infty$ or $t \to \infty$. Therefore, $J_{a,k}(y + tT_k\varphi_\varepsilon) \to -\infty$ when $\|y\|_k^2 \to \infty$ or $t \to \infty$, which proved the lemma. □

In the next step we will be using assumption (h_3) to get appropriate estimates for the minimax levels.

Lemma 2.4. *For each $a > 0$ we have u^k critical point of $J_{a,k}$ at minimax level c^k given by*

$$c^k = \inf_{\gamma \in \Gamma_k} \max_{u \in M_{k,R}(\varepsilon)} J_{a,k}(\gamma(u)).$$

Furthermore, $0 < c^k < S^{N/2}/N|K|_\infty^{(N-2)/2}$.

Proof. Here we will use some ideas from [13]. From Lemma 5 in [13] there exists ε_0 such that $T_k\varphi_\varepsilon \not\equiv 0$ for each $0 < \varepsilon \leqslant \varepsilon_0$. Now, let

$$\tilde{M}_k(\varepsilon) = \{x = y + tT_k\varphi_\varepsilon,\ \ y \in Y_k \text{ and } t \geqslant 0\}.$$

We are going to prove that for each $a > 0$ and ρ verifying (h_3) it holds

$$\sup_{u \in \tilde{M}_k(\varepsilon)} J_{k,a} < \frac{S^{N/2}}{N|K|_\infty^{(N-2)/N}}.$$

In fact, let $s \geqslant 0$, $u \not\equiv 0$ and define

$$I(u) = \frac{1}{2}\int_{Q_k} (|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x - \frac{1}{2^*}\int_{Q_k} K(x)|u|^{2^*}\,\mathrm{d}x.$$

Then, we have

$$I(u) \geqslant J_{a,k}(u) \tag{2.21}$$

and

$$\max_{s \geqslant 0} I(su) \leqslant \frac{1}{N}\frac{(\int_{Q_k}(|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x)^{N/2}}{(\int_{Q_k} K(x)|u|^{2^*}\,\mathrm{d}x)^{(N-2)/2}}. \tag{2.22}$$

Next we will be using the following estimates with respect to φ_ε (see [13]):

$$|\nabla\varphi_\varepsilon|_2^2 = S^{N/2} + O(\varepsilon^{N-2}),$$

$$|\nabla\varphi_\varepsilon|_1 = O(\varepsilon^{(N-2)/2}),$$

$$|\varphi_\varepsilon|_{2^*}^{2^*} = S^{N/2} + O(\varepsilon^N),$$

$$|\varphi_\varepsilon|_{2^*-1}^{2^*-1} = O(\varepsilon^{(N-2)/2}),$$

$$|\varphi_\varepsilon|_1 = O(\varepsilon^{(N-2)/2}).$$

Set $\|u\|_{2^*,K}^{2^*} = \int_{Q_k} K(x)|u|^{2^*}\,\mathrm{d}x$, and $u = u^- + tT_k\varphi_\varepsilon$, with $P_k u = u^-$, $t \geqslant 0$. Thus

$$\begin{aligned}\|T_k\varphi_\varepsilon\|_{2^*,K}^2 &= (\|\varphi_\varepsilon\|_{2^*,K}^2 + O(\varepsilon^{N-2}))^{(N-2)/N} \\ &\leqslant |K|_\infty^{(N-2)/N} S^{(N-2)/2} + O(\varepsilon^{(N-2)^2/N}) \end{aligned} \tag{2.23}$$

and

$$\left|\int_{Q_k} |\nabla \varphi_\varepsilon|^2 \, dx - \int_{Q_k} |\nabla (T_k \varphi_\varepsilon)|^2 \, dx\right| + \int_{Q_k} |\nabla \varphi_\varepsilon|^2 \, dx$$

$$= O(\varepsilon^{N-2}) + S^{N/2} + O(\varepsilon^N). \tag{2.24}$$

Now, by using (h_2) and the previous estimates we obtain

$$\begin{aligned}
\|T_k \varphi_\varepsilon\|^2_{2^*,K} &= (\|T_k \varphi_\varepsilon\|^{2^*}_{2^*,K})^{2/2^*} \\
&= (\|\varphi_\varepsilon\|^2_{2^*,K} + O(\varepsilon^{N-2}))^{(N-2)/N} \\
&\leqslant (K(0) S^{N/2} + O(\varepsilon) + O(\varepsilon^{N-2}))^{(N-2)/N} \\
&= |K|_\infty^{(N-2)/N} S^{(N-2)/2} + O(\varepsilon^{(N-2)^2/N}).
\end{aligned} \tag{2.25}$$

Thus, setting $\|u\|^{2^*}_{2^*,K} = 1$ and taking into account (2.21)–(2.25) in the equation

$$\int_{Q_k} (|\nabla u|^2 + V(x) u^2) \, dx = -\|u^-\|_k^2 + \frac{|\nabla (t T_k \varphi_\varepsilon)|_2^2}{|t T_k \varphi_\varepsilon|^2_{2^*}} |t T_k \varphi_\varepsilon|^2_{2^*}$$

$$+ t^2 \int_{Q_k} V(x) (t T_k \varphi_\varepsilon)^2 \, dx,$$

we deduce

$$\int_{Q_k} (\nabla u|^2 + V(x) u^2) \, dx = \frac{S^{N/2}}{N |K|_\infty^{(N-2)/2}} \|T_k \varphi_\varepsilon\|^N_{2^*,K} + t^2 c \varepsilon^{N(N-2)/2}. \tag{2.26}$$

Now, we have that t is bounded and if

$$\|u^-\|^{2^*}_{2^*,K} \leqslant 2 c_1 t^{2^*} \varepsilon^{N(N-2)/(N+2)}$$

then

$$\|t T_k \varphi_\varepsilon\|^{2^*}_{2^*,K} \leqslant 1 + c \varepsilon^{N-2},$$

since it holds that

$$\begin{aligned}
1 = \|u\|^{2^*}_{2^*,K} &\geqslant \|t T_k \varphi_\varepsilon\|^{2^*}_{2^*,K} + \frac{1}{2} \|u^-\|^{2^*}_{2^*,K} - c_1 t^{2^*} \varepsilon^{N(N-2)(N+2)} \\
&\geqslant t^{2^*} \|\varphi_\varepsilon\|^{2^*}_{2^*,K} + \frac{1}{2} \|u^-\|^{2^*}_{2^*,K} - c_2 t^{2^*} \varepsilon^{N-2} - c_1 t^{2^*} \varepsilon^{N(N-2)/(N+2)}.
\end{aligned}$$

Otherwise we get

$$\|t T_k \varphi_\varepsilon\|^{2^*}_{2^*,K} \leqslant 1.$$

Hence, in any case, we have

$$\|tT_k\varphi_\varepsilon\|_{2^*,K}^{2^*} \leqslant 1 + O(\varepsilon^{N-2}). \tag{2.27}$$

Now we estimate the part related to the discontinuity f, namely, the expression involving the primitive $F(x,v) = \rho(x)H(v-a)v^p$:

$$\begin{aligned}
&\left|\int_{Q_k} F(x,u^- + tT_k\varphi_\varepsilon)\,\mathrm{d}x - \int_{Q_k} F(x,u^-)\,\mathrm{d}x - \int_{Q_k} F(x,tT_k\varphi_\varepsilon)\,\mathrm{d}x\right| \\
&\quad = \left|\int_{Q_k}\int_0^{tT_k\varphi_\varepsilon} \rho(x)H(u^- + \sigma - a)(u^- + \sigma)^{p-1}\,\mathrm{d}x\right. \\
&\qquad \left. - \int_{Q_k}\int_0^{tT_k\varphi_\varepsilon} \rho(x)H(\sigma - a)\sigma^{p-1}\,\mathrm{d}x\right| \\
&\quad \leqslant c\left(\int_{Q_k} |tT_k\varphi_\varepsilon||u^-|^{p-1}\,\mathrm{d}x + 2\int_{Q_k} |tT_k\varphi_\varepsilon|^p\,\mathrm{d}x\right) \\
&\quad \leqslant c\left(t|u^-|_\infty |T_k\varphi_\varepsilon|_{L^1} + 2t^p\left|\int_{Q_k} |tT_k\varphi_\varepsilon|^p - |\varphi_\varepsilon|^p\,\mathrm{d}x\right| + 2t^p\int_{Q_k} |\varphi_\varepsilon|^p\,\mathrm{d}x\right) \\
&\quad \leqslant c(\varepsilon^{(N-2)/2} + \varepsilon^{N-p(N-2)/2}),
\end{aligned}$$

where c is independent of ε since (2.27) holds. Analogously one gets

$$\left|\int_{Q_k} F(x,tT_k\varphi_\varepsilon)\,\mathrm{d}x - \int_{Q_k} F(x,\varphi_\varepsilon)\,\mathrm{d}x\right| \leqslant c\varepsilon^{(N-2)/2}.$$

Consequently, going back to (2.26) and joint up the previous facts, we get

$$J_{a,k}(su) \leqslant \frac{S^{N/2}}{N|K|_\infty^{-(N-2)/2}} + c(\varepsilon^{(N-2)/2} + \varepsilon^{N-p(N-2)/2}) - \int_{Q_k} F(x,\varphi_\varepsilon)\,\mathrm{d}x. \tag{2.28}$$

Let $\varepsilon_1 \leqslant \varepsilon_0$ such that the inequality $a < c_N/\varepsilon_1^{(N-2)/2}$ holds for $a > 0$ fixed. Hence, for all $\varepsilon \leqslant \varepsilon_1$ we get $a < c_N/\varepsilon^{(N-2)/2}$. Thus the positive radius $r(\varepsilon)$ defined by

$$r(\varepsilon) = \left(\frac{c_N\varepsilon}{a^{2/(N-2)}} - \varepsilon^2\right)^{1/2}$$

is well defined and less than r given by (h_3). Furthermore, the following inclusion holds:

$$B_{r(\varepsilon)}(0) \subset \{x \in Q_k,\ \ \varphi_\varepsilon > a\}.$$

Hence, by using (h_3), we can conclude

$$\int_{\varphi_\varepsilon > a} \rho(x)(\varphi_\varepsilon^p(x) - a^p)\,\mathrm{d}x$$

$$\geqslant \int_{B_{r(\varepsilon)}(0)} \left(\rho(x) \left(\frac{c_N \varepsilon}{|x|^2 + \varepsilon^2} \right)^{p(N-2)/2} \right) \mathrm{d}x - a^p \|\rho\|_\infty r(\varepsilon)^N \mu_N$$

$$= O(\varepsilon^\tau) - O(\varepsilon^{N/2}),$$

where μ_N is the volume of the unit ball in $\mathbb{R}^N$. Now, using this estimate together with (2.28) and taking ε small enough, the conclusion of the lemma readily follows. □

Lemma 2.5. *Any critical point u_k of $J_{a,k}$ satisfies $\|u_k\|_k \leqslant c$ (independent of k).*

Proof. We have $M_{k,R}(\varepsilon) \subset M_{k+1}(\varepsilon)$. In fact, since we can write

$$T_k \varphi_\varepsilon(x) = \sum_{j=k+1}^{\infty} \alpha_j^\varepsilon e_j(x)$$

with $\alpha_j^\varepsilon = \int_{\mathbb{R}^N} \varphi_\varepsilon(x) e_j(x)\,\mathrm{d}x$ then

$$T_{k+1}\varphi_\varepsilon(x) = T_k \varphi_\varepsilon(x) - \left(\int_{\mathbb{R}^N} \varphi_\varepsilon(x) e_k(x)\,\mathrm{d}x \right) e_k(x)$$

so that, if $y \in Y_k$ and $t > 0$ one gets

$$y + tT_{k+1}\varphi_\varepsilon(x) = y + tT_k\varphi_\varepsilon(x) - t \left(\int_{\mathbb{R}^N} \varphi_\varepsilon(x) e_k(x)\,\mathrm{d}x \right) e_k(x).$$

Hence, if $u = y + tT_k\varphi_\varepsilon \in M_{k,R}(\varepsilon)$, then

$$u = y + t \left(\int_{\mathbb{R}^N} \varphi_\varepsilon(x) e_k(x)\mathrm{d}x \right) e_k(x) + tT_{k+1}\varphi_\varepsilon(x) = \tilde{y} + tT_{k+1}\varphi_\varepsilon(x), \text{ with } \tilde{y} \in Y_{k+1}.$$

Therefore, for $h \in \Gamma_k$ (defined in (2.6)) we have

$$\sup_{u \in M_{k+1}(\varepsilon)} J_{a,k+1}(h(u)) \geqslant \sup_{u \in M_{k,R}(\varepsilon)} J_{a,k+1}(h(u))$$

On the other hand, if $u \in M_{k,R}(\varepsilon)$, one has

$$J_{a,k}(h(u)) \geqslant J_{a,k+1}(h(u)) \quad \text{and} \quad \Gamma_k \subset \Gamma_{k+1},$$

then

$$\inf_{h \in \Gamma_{k+1}} \sup_{u \in M_{k+1}(\varepsilon)} J_{a,k+1}(h(u)) \leqslant \inf_{h \in \Gamma_k} \sup_{u \in M_{k,R}(\varepsilon)} J_{a,k}(h(u)),$$

which proved

$$c^{k+1} \leqslant c^k \leqslant \cdots \leqslant c^1 < \frac{S^{N/2}}{N|K|_\infty^{(N-2)/2}}.$$

Finally, using the same arguments of the proof of Lemma 2.4, we can establish a uniform bound for $\|u_k\|_k$. □

Remark 2.6. As a consequence of these lemmas and from Theorem 1.2, we have already proved, up to this moment, that for each k the functional $J_{a,k}$ associated to $(1,1)_{a,k}$ has a critical point u_k at level $c^k \in (0, S^{N/2}/N|K|_\infty^{(N-2)/2})$ and $\|u_k\|_k \leqslant c$, for all $k \in \mathbf{N}$.

Proposition 2.7. *There is a sequence $\xi_k \in \mathbb{R}^N$ and $s, \eta > 0$ such that*

$$\limsup_{k\to\infty} |u_k|^2_{L^2(Q_s(\xi_k))} \geqslant \eta,$$

where $Q_s(\xi_k))$ is a cube with edge length s and centered at ξ_k.

The proof of this proposition follows immediately from the next auxiliary lemmas.

Lemma 2.8. *There exists $\varepsilon > 0$ independent of k such that $\|u_k\|_k \geqslant \varepsilon$ and $J_{a,k}(u_k) \geqslant \varepsilon$ hold for each nontrivial critical point u_k of $J_{a,k}$.*

Proof. Since (α, β) is in the spectral gap, there exists $c = c(\alpha, \beta) > 0$ such that

$$|\ell_k(u)| \geqslant c|u|^2_{L^2(Q_k)}, \quad u \in H^1_{\text{per}}(Q_k).$$

Therefore for $\varepsilon < 1$ we get

$$\begin{aligned} |\ell_k(u_k)| &= \varepsilon \left| \int_{Q_k} (|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x \right| + (1-\varepsilon) \left| \int_{Q_k} (|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x \right| \\ &\geqslant \varepsilon \int_{Q_k} \left(|\nabla u|^2 - \varepsilon \max_{x\in Q_k} V(x) \right) |u|_{L^2(Q_k)}\,\mathrm{d}x + (1-\varepsilon)c|u|_{L^2(Q_k)} \\ &= \varepsilon\|u\|_k^2 + \left((1-\varepsilon)c - \varepsilon \max_{x\in Q_k} V(x) \right) |u|^2_{L^2(Q_k)}. \end{aligned}$$

Thus, taking ε small enough, we obtain

$$|\ell_k(u)| \geqslant c_1\|u\|_k^2, \quad u \in H^1_{\text{per}}(Q_k). \tag{2.29}$$

Let u_k be a nontrivial critical point of $J_{a,k}$. Then, by using (2.29), one gets

$$c_1\|u\|_k^2 \leqslant |\ell_k(u)| \leqslant c_2\|u_k\|_k^{2^*} + c_3\|u_k\|^p.$$

So that, since the polinomium $p(t) = c_2 tk^{2^*-2} + c_3 t^{p-2} - c_1$ is nonnegative for $t \geqslant \varepsilon_1$ for some $\varepsilon_1 > 0$, the conclusion readily follows.

Finally, using the fact u_k is a critical point of $J_{a,k}$, we have $\sigma_k \in \partial\Phi_{a,k}(u_k)$ such that $0 = \ell'_k(u_k) - \Psi'_k(u_k) - \sigma_k$. Therefore

$$\begin{aligned} J_{a,k}(u_k) &\geqslant \frac{1}{2}\,\ell_k(u_k) - \frac{1}{2^*} \int_{Q_k} K(x)|u_k|^{2^*}\,\mathrm{d}x - \frac{1}{p}\langle \sigma_k, |u_k|\rangle \\ &= \frac{1}{2}\left(\ell_k(u_k) - \int_{Q_k} K(x)|u_k|^{2^*}\,\mathrm{d}x - \langle \sigma_k, |u_k|\rangle \right) \end{aligned}$$

$$+\left(\frac{1}{2}-\frac{1}{2^*}\right)\int_{Q_k} K(x)|u_k|^{2^*}\,\mathrm{d}x+\left(\frac{1}{2}-\frac{1}{p}\right)\langle\sigma_k,|u_k|\rangle$$

$$\geqslant \min\{1/N,(p-2)/2p\}|\ell_k(u_k)|\geqslant\|u_k\|_k^2,$$

and the proof is completed. □

Remark 2.9. The same arguments could be used to prove the result above for J the functional associated with (1.1).

Next we shall use a modification of the well-known concentration-compactness lemma of Lions [23].

Lemma 2.10. *Let Q_n be the cube of edge length $l_n\to\infty$ as $n\to\infty$ centered at the origin, and $K_r(\xi)$ be the closed cube with the edge length r centered at the point ξ. Let $u_n\in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ of l_n-periodic functions such that $\|u_k\|_{H^1(Q_n)}\leqslant c$ for some constant independent of n. Suppose that there is $r>0$ such that*

$$\liminf_{n\to\infty}\left(\sup_{\xi}\int_{K_r(\xi)}|u_n|^2\,\mathrm{d}x\right)=0.$$

Then $\|u_n\|_{L^q(Q_n)}\to 0$ as $n\to\infty$ for $q\in(2,2^*)$.

Proof. For the proof see [25]. □

Lemma 2.11. *Let u_k be a sequence verifying*

$$J_{a,k}(u_k)=c^k<\frac{S^{N/2}}{N|K|_\infty^{(N-2)/2}}\quad \textit{and}\quad \min_{\mu\in\partial J_{a,k}(u_k)}\|\mu\|\to 0\quad \textit{as } k\to\infty.$$

Then either

(1) *$\|u_k\|_k\to 0$ when $k\to\infty$, or*
(2) *there is a sequence $\xi_k\in\mathbb{R}^N$, and $s,\eta>0$ such that*

$$\lim_{k\to\infty}|u_k|^2_{L^2(Q_s(\xi_k))}\geqslant\eta.$$

Proof. Suppose that (ii) does not hold. By concentration-compactness arguments (see Lemma 2.10) one has

$$|u_k|_{L^q}\to 0\quad \text{for } 2<q<2^*.$$

Following [13] we have

$$\int_{Q_k} V(x)u_k^2\to 0.$$

On the other hand, it holds

$$|\langle \sigma_k, u_k\rangle| \leqslant c\int_{Q_k} |u_k|\,\mathrm{d}x \to 0$$

and

$$|\Phi_{a,k}(u_k)| \leqslant c\int_{Q_k} |u_k|^p\,\mathrm{d}x \to 0.$$

Thus

$$J_{a,k}(u_k) = \frac{1}{2}\int_{Q_k} |\nabla u_k|^2\,\mathrm{d}x - \frac{1}{2^*}\int_{Q_k} K(x)|u_k|^{2^*}\,\mathrm{d}x + o_k(1)$$

and

$$0 = \int_{Q_k} |\nabla u_k|^2\,\mathrm{d}x - \int_{Q_k} K(x)|u_k|^{2^*}\,\mathrm{d}x + o_k(1).$$

Consequently,

$$c^k = \frac{1}{N}\int_{Q_k} K(x)|u_k|^{2^*}\,\mathrm{d}x$$

and

$$\begin{aligned}\int_{Q_k} |\nabla u_k|^2\,\mathrm{d}x + o_k(1) &\geqslant S\|u_k\|_{2^*}^2 + o_k(1)\\ &\geqslant S|K|_\infty^{2/2^*}\left(\int_{Q_k} K(x)|u_k|^{2^*}\,\mathrm{d}x\right)^{2/2^*} + o_k(1).\end{aligned} \tag{2.30}$$

Therefore,

$$l \geqslant S^{N/2}|K|_\infty^{-(N-2)/2}.$$

This and (2.30) imply that

$$\lim_{k\to\infty} c^k \geqslant \frac{S^{N/2}}{N|K|_\infty^{(N-2)/2}},$$

which is a contradiction.

Finally, since u_k verifies $0 = l_k'(u_k) - \Psi_k'(u_k) - \sigma_k$, we obtain

$$0 = -\|z_k\|^2 - \int_{Q_k} K(x)|u_k|^{2^*-2}u_k z_k\,\mathrm{d}x - \langle \sigma_k, z_k\rangle$$

and

$$0 = \|y_k\|^2 - \int_{Q_k} K(x)|u_k|^{2^*-2}u_k y_k\,\mathrm{d}x - \langle \sigma_k, y_k\rangle,$$

where $z_k = T_k u_k$ and $y_k = P_k u_k$. So that, since

$$
\begin{aligned}
|\langle \sigma_k, z_k \rangle| &\leqslant \int_{u_k \geqslant a} \rho(x) u_k^{p-1} |z_k| \, \mathrm{d}x \\
&\leqslant \int_{Q_k} \rho(x) u_k^{p-1} |z_k| \, \mathrm{d}x \\
&\leqslant c |u_k|_{L^p}^{p-1} |z_k|_{L^p} \to 0,
\end{aligned}
$$

it follows $\|u_k\|_k \to 0$ and (i) holds. □

3. Proof of the main result

As a consequence of the results of the previous section, we have a bounded sequence of solutions u_k of $(1.1)_{a,k}$ which verifies

$$
|u_k|^2_{L^2(Q_s(\xi_k))} \geqslant \eta > 0,
$$

for all k and for some $s \in (0,1)$.

Now, we denote by ξ^i the ith component of vector ξ_k^i, the center of cube $Q_s(\xi_k^i)$ given in Proposition 2.7 and $b_k^i = [\xi_k^i]$, $i = 1, \ldots, N$ is the greatest integer equal or less than ξ_k^i. Next, defining a new sequence $\hat{u}_k$ as

$$
\hat{u}_k(x) = u_k(x + b_k)
$$

we find that

$$
|\hat{u}_k|^2_{L^2(Q_{s+1}(0))} \geqslant \eta. \tag{3.1}
$$

On the other hand, since K, V and ρ are 1-periodics we get, by taking Q_k centered at the origin,

$$
\begin{aligned}
J_{a,k}(u_k) &= \frac{1}{2} \int_{Q_k} (|\nabla u_k(x)|^2 + V(x) u_k(x)^2) \, \mathrm{d}x \\
&\quad - \frac{1}{2^*} \int_{Q_k} K(x) |u(x)|^{2^*} \, \mathrm{d}x - \int_{Q_k} \rho(x) H(u(x) - a)(u^p(x) - a^p) \, \mathrm{d}x \\
&= \frac{1}{2} \int_{\hat{Q}_k} (|\nabla \hat{u}_k(x)|^2 + V(x) \hat{u}_k(x)^2) \, \mathrm{d}x - \frac{1}{2^*} \int_{\hat{Q}_k} K(x) |\hat{u}_k(x)|^{2^*} \, \mathrm{d}x \\
&\quad - \int_{\hat{Q}_k} \rho(x) H(\hat{u}_k(x) - a)(\hat{u}_k^p(x) - a^p) \, \mathrm{d}x \\
&\equiv \hat{J}_{a,k}(\hat{u}_k),
\end{aligned}
$$

where $\hat{Q}_k$ is the cube in $\mathbb{R}^N$ with length k and centered in $-b_k$.

Now using that u_k is a critical point of $J_{a,k}$, we have $\sigma_k \in \partial\Phi_{a,k}(u_k)$ verifying

$$\int_{Q_k} f(x, u_k(x) - 0)\phi(x)\,\mathrm{d}x \leqslant \int_{Q_k} \sigma_k(x)\phi(x)\,\mathrm{d}x \leqslant \int_{Q_k} f(x, u(x) + 0)\phi(x)\,\mathrm{d}x$$

for $\phi \in C_0^\infty(Q_k)$, $\phi \geqslant 0$. Then

$$\begin{aligned} 0 &= \int_{Q_k} (\nabla u_k(x)\nabla\phi(x) + V(x)u_k(x)\phi(x))\,\mathrm{d}x \\ &\quad - \int_{\hat{Q}_k} K(x)|u_k^{2^*-1}|u(x)\phi(x)\,\mathrm{d}x - \int_{Q_k} \sigma_k(x)\phi(x)\,\mathrm{d}x \\ &= \int_{\hat{Q}_k} (\nabla \hat{u}_k(x)\nabla\hat{\phi}(x) + V(x)\hat{u}_k(x)\hat{\phi}(x))\,\mathrm{d}x \\ &\quad - \int_{\hat{Q}_k} K(x)|\hat{u}_k^{2^*-1}|\hat{u}(x)\hat{\phi}(x)\,\mathrm{d}x - \int_{\hat{Q}_k} \hat{\sigma}_k(x)\hat{\phi}(x)\,\mathrm{d}x \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \int_{\hat{Q}_k} f(x, \hat{u}_k(x) - 0)\hat{\phi}(x)\,\mathrm{d}x &\leqslant \int_{\hat{Q}_k} \hat{\sigma}_k(x)\hat{\phi}(x)\,\mathrm{d}x \\ &\leqslant \int_{\hat{Q}_k} f(x, \hat{u}(x) + 0)\hat{\phi}(x)\,\mathrm{d}x, \end{aligned} \tag{3.3}$$

where here $\hat{\sigma}_k(x) = \sigma_k(x + b_k)$ and $\hat{\phi}(x) = \phi(x + b_k)$.

Thus from (3.2) and (3.3), we have $\hat{u}_k$ as a critical point of $\hat{J}_{a,k}$. Now, by using the same arguments as before, we can conclude that $\hat{u}_k$ is bounded in $H^1_{\mathrm{loc}}(\mathbb{R}^N)$ and, taking subsequence if necessary, we obtain $u \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ such that $\hat{u}_k \rightharpoonup u$.

Furthermore, from the assumption on the growth of the function f it follows that $\|\hat{\sigma}_k\|_k \leqslant c$, where c is independent of k. Hence, taking a subsequence we have $\hat{\sigma}_k \rightharpoonup \sigma_0$ and $\hat{\sigma}_k(x) \to \sigma_0(x)$ almost everywhere $x \in \mathbb{R}^N$ for some $\sigma_0 \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$. Therefore, by taking limit in (3.3), we get

$$\sigma_0(x) \in [f(u(x) - 0), f(u(x) + 0)],$$

almost everywhere in $\mathbb{R}^N$. Then passing to the limit in (3.2) and from the interior elliptic estimates one gets $u \in W^{2,2^*}_{\mathrm{loc}}(\mathbb{R}^N)$ and

$$-\Delta u(x) + V(x)u(x) + K(x)u(x)^{2^*-1} \in [f(u(x) - 0), f(u(x) + 0)],$$

almost everywhere in $\mathbb{R}^N$, which proved that u is a solution of (1.1). Finally we observe that, by (3.1), $u \neq 0$. □

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