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On a class of singular biharmonic problems involving critical exponents ☆

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Abstract

This paper deals with the following class of singular biharmonic problems

$$\text{(P)}\qquad \begin{cases} \Delta^2 u + V(x)|u|^{q-1}u = |u|^{2^*-2}u, & \text{in } \Omega \subset \mathbb{R}^N, \\ u \in D_o^{2,2}(\Omega), & N \geqslant 5, \end{cases}$$

where $1 \leqslant q < 2^* - 1$, $2^* = 2N/(N-4)$ is the critical Sobolev exponent, Δ^2 denotes the biharmonic operator, Ω is open domain (not necessarily bounded, it may be equal to $\mathbb{R}^N$) and V is a potential that changes sign in Ω with some points of singularities in Ω. Some results on the existence of solutions are obtained by combining the Mountain Pass Theorem and Hardy inequality with some arguments used by Brézis and Nirenberg.

Keywords: Biharmonic operator; Critical Sobolev exponents; Hardy inequality; Singular and indefinite potential

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1. Introduction

In this paper we are concerned with a class of singular biharmonic problems involving critical Sobolev exponents of the type

$$(\mathrm{P})\quad \begin{cases} \Delta^2 u + V(x)|u|^{q-1}u = |u|^{2^*-2}u, & \text{in } \Omega \subset \mathbb{R}^N, \\ u \in D_o^{2,2}(\Omega), & N \geqslant 5, \end{cases}$$

where $1 \leqslant q < 2^* - 1$, $2^* = 2N/(N-4)$ is the critical Sobolev exponent, Δ^2 denotes the biharmonic operator, Ω is open domain (not necessarily bounded, it may be equal to $\mathbb{R}^N$) and $V : \Omega \to \mathbb{R}$ is a potential that changes sign with some points of singularities in Ω.

When the potential V is a regular function, this kind of the problem has been studied by several authors. We would like to mention the paper by Bernis et al. [4] and references therein for problem (P) in a bounded domain with V equal to a constant. For the other references about this problem (P) in whole space with regular potential, we cite a recent paper by Chabrowski and do Ó [9]. Some results of existence involving the p-Laplacian operator with potential V, still being a regular function, were considered by several authors, more precisely, for the following quasilinear elliptic problem

$$(\mathrm{P}_1)\quad \begin{cases} -\Delta_p u + V(x)|u|^{q-1}u = |u|^{s-2}u, & \text{in } \mathbb{R}^N, \\ u \in D^{1,p}\big(\mathbb{R}^N\big), \end{cases}$$

where Δ_p denotes p-Laplacian operator, $N > p > 1$, $p < q + 1 < p^* = Np/(N-p)$, $D^{1,p}(\mathbb{R}^N)$ is the completion of $C_o^\infty(\mathbb{R}^N)$ in the L^p-norm of ∇u and V is a continuous function in $\mathbb{R}^N$. Related to the problem above with $s = p^*$, among others, we would like to cite the papers of Benci and Cerami in [2] which treated the case $q = 1$, $p = 2$ and $V = V^- \in L^{N/2}(\mathbb{R}^N)$, while Pan in [20] studied the case $q > 1$, $p = 2$ and $V = V^+ \in L^{N/2}(\mathbb{R}^N)$. In [3] Ben-Naoum et al. considered $V = V^+ - V^-$, $V^\pm \neq 0$, $V \in L^{p^*/(p^*-p)}(\mathbb{R}^N)$ satisfying

$$|V|_{p^*/(p^*-p)} = \left(\int_{\mathbb{R}^N} |V|^{p^*/(p^*-p)}\, dx \right)^{(p^*-p)/p} < S,$$

where S is the best constant of the Sobolev embedding $W^{1,p}(\mathbb{R}^N)$ in to $L^{p^*}(\mathbb{R}^N)$, and they proved that problem (P_1) has a positive solution provided

$$q > \frac{(N+1)p^2 - 2Np}{(N-p)(p-1)}, \quad p \geqslant \sqrt{N}.$$

In the case when the potential V has some singularity, for instance $V(x) = \lambda/|x|^\beta$, we observe that by Pohozaev identity [21] (see also [13]) the problem (P_1) has no solution when $\beta = p = q$, $\lambda < 0$, $s = p^* := pN/(N-p)$ and Ω is a bounded starshaped with respect to the origin. Still in a bounded domain, Garcia Azorero and Peral Alonso in [13] have shown that problem above has at least one solution provided that one of the conditions below hold

(i) $\beta = p = q$, $s < p^*$, $0 < -\lambda < \lambda_N := ((N-p)/p)^2$,

(ii) $\beta < p = q$, $s = p^*$, $0 < -\lambda < \lambda_1$, $N \geqslant p^2 - (p-1)\beta$,

where λ_N and λ_1 denote the best constant of the Hardy inequality and the first eigenvalue of $(-\Delta_p, W_o^{1,p}(\Omega))$, respectively.

In [14], Ghoussoub and Yuan not only extended the results of existence and nonexistence mentioned above when V has a singularity at the origin but also get multiplicity results for problem (P_1) on a bounded domain, considering a large class of possibilities among the numbers p, q, β, s, p^* and $\beta^* = (N-\beta)p/(N-p)$ (Hardy's critical exponent). Now when $\Omega = \mathbb{R}^N$, Terracini in [24], among others results, applying Pohozaev type inequality concluded that (P_1) has no solution when $\lambda \neq 0$, $\beta \neq 2$, $p = 2$ and $s = 2^*$. While, Lions in [16] proved the existence of positive solutions for the case $\lambda < 0$, $\beta = 2 = p$ and $s = 2^*$, and Jannelli in [15] presented a explicit positive solution for this situation.

With respect to the problems involving the biharmonic operators with potential V having singularities, more exactly, the problem of the type (P), we remark firstly that with some changes the nonexistence result obtained in [24] also holds for problem (P) with $\beta \neq 4$. Still related to problem (P), Noussair et al. in [19], applying a compactness result due to Egnell [11, Lemma 10], treated the situation when the potential V is nonpositive and verifies the condition:

$$(\mathrm{H}_1)\quad V(x) = \begin{cases} O(|x|^{\nu}) & \text{as } |x| \to \infty, \\ O(|x|^{\mu}) & \text{as } |x| \to 0, \end{cases}$$

where

$$-4 < \nu < \mu \quad \text{and} \quad \frac{2(N+\nu)}{N-4} < q+1 < \frac{2(N+\mu)}{N-4}.$$

Notice that this condition implies that V should cross the critical hyperbole $|x|^{-4}$.

In this work, motivated by the papers mentioned above, we study problem (P), mainly in a domain not necessarily bounded,when the potential $V = V^+ - V^-$ changes sign and the positive part V^+ either cross the hyperbole $|x|^{-4}$ likes above or it remains below of the critical hyperbole near of the origin and at the infinity, that is, $\nu < -4 < \mu$. More exactly, we will impose the following condition:

$$(\mathrm{H}_2)\quad \begin{cases} V \in L^1_{\mathrm{loc}}(\Omega), \quad V = V^+ - V^- = V_1 + V_2 - V^-, \quad V^{\pm} \neq 0, \\ V_1, V^- \in L^{\alpha_o}(\Omega) \cap L^{\infty}(\Omega), \quad \alpha_o = \frac{2^*}{2^* - (q+1)}, \\ \forall y \in \overline{\Omega}, \quad \lim_{x \to y,\ x \in \Omega} |x-y|^{\alpha} V_2(x) = 0 \\ \text{and} \quad \lim_{|x| \to \infty} |x|^{\alpha} V_2(x) = 0, \end{cases}$$

where

$$\alpha := \begin{cases} 4 & \text{if } q = 1, \\ N - \frac{(N-4)(q+1)}{2} & \text{if } q > 1, \end{cases}$$

and

$$V^{\pm}(x) := \max\{\pm V(x), 0\}.$$

Example 1.1. Consider the potential $V = V_1 + V_2 - V^- := V_1 + \epsilon|x|^{-\alpha} - V^-$ on $\mathbb{R}^N$ where $V_1, V^- \in L^{\alpha_o}(\Omega) \cap L^{\infty}(\Omega)$, $V^{\pm} \neq 0$ and ϵ is sufficiently small.

Example 1.2. Consider the potential $V = V_1 + V_2 - V^-$, $V^\pm \neq 0$, $V_1, V^- \in L^{\alpha_o}(\Omega) \cap L^\infty(\Omega)$ and V_2 behaves like $|x|^{-\alpha+\epsilon}$ and $|x|^{-\alpha-\epsilon}$ at the origin and the infinity, respectively.

Remark 1.1. The number α defined above, it exactly the exponent where the Hardy type inequality hold (see, e.g., Lions [17]), that is,

$$\eta \int_{\mathbb{R}^N} \frac{u^{q+1}}{|x|^\alpha}\, dx \leqslant \left(\int_{\mathbb{R}^N} |\Delta u|^2\, dx \right)^{(q+1)/2}, \quad q \geqslant 1,\ u \in D^{2,2}\big(\mathbb{R}^N\big), \tag{1.1}$$

where $\alpha := N - (N-4)(q+1)/2$, $N \geqslant 5$, and when $\alpha = 4$ and $q = 1$ then $\eta \leqslant \lambda_N := ((N-4)/4)^2$ is the optimal constant, whose the proof can be found in [10]. The p-Laplacian version of this inequality is proved in [7].

We shall state our first result.

Theorem 1.1. *Assume either* (H$_2$) *or that* $V^\pm \neq 0$ *with* V^+ *verifying* (H$_1$). *Then, problem* (P) *has at least one nontrivial solution provided that*

(i) $1 \leqslant q < 2^* - 1$, *if* $N \geqslant 8$,
(ii) $\frac{4}{N-4} < q < 2^* - 1$, *if* $N = 5, 6$ *and* $N = 7$,
(iii) $1 \leqslant q \leqslant \frac{4}{N-4}$, *if* $|V^-|_\infty$ *is sufficiently large.*

Here, we also study the same kind of results when the potential V is below of the critical hyperbole $|x|^{-4}$ at the origin and Ω a bounded domain, that is, we consider the following problem:

$$(\mathrm{P}_2) \qquad \begin{cases} \Delta^2 u + V(x)u = \mu |u|^{q-1}u + |u|^{2^*-2}u, & \text{in } \Omega \subset \mathbb{R}^N, \\ u \in D_o^{2,2}(\Omega), \quad N \geqslant 5, \quad \mu > 0, \quad 1 \leqslant q < p^* - 1. \end{cases}$$

In this case we shall state the following result.

Theorem 1.2. *Suppose that* Ω *is a bounded smooth domain. Assume that either* $V = \lambda/|x|^4$, $\lambda < \lambda_N$ *or* $V := \lambda/|x|^\beta$, $\lambda < 0$, $\beta < 4$. *Then, the problem* (P$_2$) *has at least one nontrivial solution provided*

(i) $1 \leqslant q < 2^* - 1$, *if* $N \geqslant 8$,
(ii) $\frac{4}{N-4} < q < 2^* - 1$, *if* $N = 5, 6$ *and* $N = 7$,
(iii) $1 \leqslant q \leqslant \frac{4}{N-4}$, *if* $|V^-|_\infty$ *is sufficiently large.*

In order to conclude, we like to say that our main theorems extend or complement the results obtained in [3] for the fourth order operator Δ^2 as well as results get in [19], considering the potential V more general having some singularities. In addition, when the domain is bounded, we also complement some results proved in [14].

This paper is organized as follows. Section 2 contains the statements and the proofs of two crucial lemmas related with the proof of Theorem 1.1. Section 3 and Section 4 deal with the proof of Theorems 1.1 and 1.2, respectively. Finally in Section 5 we have some generalization and concluding remarks.

Notation. In this paper we make use of the following notation.

$C, C_1, C_2, \ldots$ denote (possibly different) positive constants;

$B[p,R]$ and $B(p,R)$ denote the closed and open ball with the radius R centered at point p of $\mathbb{R}^N$, respectively;

$L^p(\Omega)$, $1 \leqslant p \leqslant \infty$, denote Lebesgue spaces; the norm in L^p is denoted by $|u|_p$;

$D_o^{2,2}(\Omega)$ denotes the completion of the space $C_o^\infty(\Omega)$ in the norm $\|u\| := (\int_\Omega |\Delta u|^2\,dx)^{1/2}$.

In this work we are denoting by S the best constant to the Sobolev embedding, $D_o^{2,2}(\Omega) \hookrightarrow L^{2^*}(\Omega)$, that is,

$$S := \inf\bigl\{|\Delta u|_2^2\colon\ u \in D_o^{2,2}(\Omega) \text{ and } |u|_{2^*} = 1\bigr\}.$$

It is known (see [19]) that for $\Omega = \mathbb{R}^N$ the best constant is attained by the functions

$$s_{\epsilon,x_o}(x) := \frac{C_N \epsilon^{(N-4)/2}}{(\epsilon^2 + |x - x_o|^2)^{(N-4)/2}},$$
$$C_N := \bigl(N(N-4)\bigl(N^2-4\bigr)\bigr)^{(N-4)/8}. \tag{1.2}$$

2. Preliminary results

We begin this section stating the following crucial result:

Lemma 2.1. *Suppose either* (H$_2$) *or that* $V^\pm \neq 0$ *with* V^+ *verifying* (H$_1$). *Then the function* $\psi : D_o^{2,2}(\Omega) \to \mathbb{R}$, *given by*

$$\psi(u) := \int_\Omega V^+ |u|^{q+1}\,dx, \quad 1 \leqslant q < 2^* - 1,$$

is weakly continuous.

Proof. If $V^\pm \neq 0$ with V^+ satisfying (H$_1$), this case is exactly one of results in [11, Lemma 10]. If (H$_2$) holds, the case $q = 1$ was proved in [22, Lemma 2.1] (see also [23,25] for related results). For the case $q > 1$, the proof is done adapting arguments by [22], thus, we are going to give only a sketch of the proof. Since $V_1 \in L^{\alpha_o}(\Omega)$, from a result by Brezis and Lieb [5] it follows that

$$u \mapsto \int_\Omega V_1 |u|^{q+1}\,dx$$

is weakly continuous. Hence it remains to show that

$$u \mapsto \int_\Omega V_2 |u|^{q+1}\,dx$$

is weakly continuous. Let $u_n \rightharpoonup u$ (weakly) and $\epsilon > 0$. By (H_2), there exists $R > 0$ such that

$$|x|^\alpha V_2(x) \leqslant \epsilon, \quad x \in \Omega,\ |x| \geqslant R. \tag{2.1}$$

Define

$$\Omega_1 := \Omega \setminus B[0,R], \quad \Omega_2 := \Omega \cap B(0,R).$$

From (2.1) and by using the Hardy type inequality (see [17]) we infer that

$$\int_{\Omega_1} V_2|u_n|^{q+1}\,dx \leqslant \epsilon \int_{\Omega_1} \frac{|u_n|^{q+1}}{|x|^\alpha}\,dx \leqslant \epsilon C\left(\int_{\Omega_1} |\Delta u_n|^2\,dx\right)^{(q+1)/2},$$

so that

$$\int_{\Omega_1} V_2|u_n|^{q+1}\,dx \leqslant C\epsilon, \quad \forall n,\ C > 0. \tag{2.2}$$

By compactness of $\overline{\Omega}_2$, there is a finite covering of $\overline{\Omega}_2$ by closed ball $B[x_k, r_k]$, $k = 1, 2, \dots, m$, such that

$$|x - x_k|^\alpha V_2(x) \leqslant \epsilon, \quad |x - x_k| < r_k, \quad k = 1, 2, \dots, m. \tag{2.3}$$

Taking $r := \min\{r_k, k = 1, 2, \dots, m\}$ we obtain

$$|x - x_k|^\alpha V_2(x) \leqslant \epsilon, \quad |x - x_k| < r, \quad k = 1, 2, \dots, m. \tag{2.4}$$

Define

$$A := \bigcup_{k=1}^{m} B[x_k, r],$$

then invoking again the Hardy inequality we get

$$\int_A V_2|u_n|^{q+1}\,dx \leqslant \epsilon \int_A \frac{|u_n|^{q+1}}{|x - x_k|^\alpha}\,dx \leqslant \epsilon C\left(\int_A |\Delta u_n|^2\,dx\right)^{(q+1)/2},$$

thus

$$\int_A V_2|u_n|^{q+1}\,dx \leqslant C\epsilon, \quad \forall n,\ C > 0. \tag{2.5}$$

On the other hand, since from (2.2) we have $V_2 \in L^\infty(\Omega_2 \setminus A)$, and since $\Omega_2 \setminus A$ is bounded, we can assume that $V_2 \in L^{\alpha_o}(\Omega_2 \setminus A)$. Then, by a result by Brezis and Lieb [5], we infer that

$$\int_{\Omega_2\setminus A} V_2|u_n|^{q+1}\,dx \to \int_{\Omega_2\setminus A} V_2|u|^{q+1}\,dx, \quad \text{as } n \to \infty. \tag{2.6}$$

This inequality together with (2.2), (2.4) and (2.5) we conclude

$$\int_\Omega V_2|u_n|^{q+1}\,dx \to \int_\Omega V_2|u|^{q+1}\,dx, \quad \text{as } n \to \infty.$$

This completes the proof of Lemma 2.1. □

Next result will be used in order to get a weak solution.

Lemma 2.2. *Suppose either that* (H_2) *or that* $V^{\pm} \neq 0$ *with* V^+ *satisfying* (H_1). *Then the function* $\phi : D_o^{2,2}(\Omega) \to \mathbb{R}$, *given by*

$$\phi(u) := \int_{\Omega} V^+ |u|^{q-1} u v \, dx, \quad 1 \leqslant q < 2^* - 1, \ v \in D_o^{2,2}(\Omega) \text{ fixed},$$

is weakly continuous.

Proof. If $V^{\pm} \neq 0$ with V^+ satisfying (H_1), the proof is similar to those given above (see [11]). Suppose that (H_2) holds and let (u_n) be a sequence such that $u_n \rightharpoonup u$ (weakly), $u, u_n \in D_o^{2,2}(\Omega)$ and $\epsilon > 0$. We are using the same notations of the proof of Lemma 2.1. Since $V_1 \in L^{\alpha_o}(\Omega)$ by a result due to Brezis and Lieb [5], we have

$$\int_{\Omega} V_1 |u_n|^{q-1} u_n v \, dx \to \int_{\Omega} V_1 |u|^{q-1} u v \, dx, \quad \text{as } n \to \infty.$$

Hence it remains to show that

$$\int_{\Omega} V_2 |u_n|^{q-1} u_n v \, dx \to \int_{\Omega} V_2 |u|^{q-1} u v \, dx, \quad \text{as } n \to \infty.$$

Once more, by the Hardy type inequality and (2.1), we have

$$\begin{aligned} \int_{\Omega_1} V_2 |u_n|^{q-1} u_n v \, dx &\leqslant \left(\int_{\Omega_1} v^{2^*} \, dx \right)^{1/2^*} \left(\int_{\Omega_1} V^{p'} u_n^{qp'} \, dx \right)^{1/p'} \\ &\leqslant \epsilon \left(\int_{\Omega_1} v^{2^*} \, dx \right)^{1/2^*} \left(\int_{\Omega_1} |x|^{-\alpha p'} u_n^{qp'} \, dx \right)^{1/p'} \\ &\leqslant \epsilon C \left(\int_{\Omega_1} v^{2^*} \, dx \right)^{1/2^*} \left(\int_{\Omega_1} |\Delta u|^2 \, dx \right)^{qp'/(2p')} \end{aligned}$$

where in the last inequality we used the Hardy inequality, since $\alpha p' = N - (qp'(N-4)/2$ because $\alpha = N - (q+1)(n-4)/2$ and $p' = 2N/(N+4)$. Then,

$$\int_{\Omega_1} V_2 |u_n|^{q-1} u_n v \, dx \leqslant C\epsilon. \tag{2.7}$$

Similarly, we have the following estimates

$$\int_{\Omega_1} V_2|u|^{q-1}uv\,dx \leqslant C\epsilon, \qquad \int_A V_2|u_n|^{q-1}u_n v\,dx \leqslant C\epsilon,$$
$$\int_A V_2|u|^{q-1}uv\,dx \leqslant C\epsilon,$$

where C does not depend on n. Combining these inequalities with (2.7) we obtain

$$\int_{\Omega_2\setminus A} V_2|u_n|^{q-1}u_n v\,dx \to \int_{\Omega_2\setminus A} V_2|u|^{q-1}uv\,dx, \quad \text{as } n\to\infty,$$

thus, the Lemma 2.2 is proved. □

3. Proof of Theorem 1.1

The proof of the theorem is done adapting some ideas from [3] and arguments by [6]. The associated energy functional to problem (P) is $I : D_o^{2,2}(\Omega) \to \mathbb{R}$, defined by

$$I(u) := \int_\Omega \left(\frac{1}{2}|\Delta u|^2 + \frac{1}{q+1}V|u|^{q+1} - \frac{1}{2^*}|u|^{2^*}\right)dx,$$

which is a C^1 functional and its Fréchet derivative is given by

$$I'(u)v := \int_\Omega \left(\Delta u\Delta v + V|u|^{q-1}uv - |v|^{2^*-2}uv\right)dx.$$

We shall prove that the functional I verifies the mountain pass geometry conditions, namely

Lemma 3.1. *Suppose either* (H$_2$) *or that* $V^\pm \neq 0$ *with* V^+ *verifying* (H$_1$). *Then* I *verifies the following conditions*:

(a) *There exist constants* $\rho, \beta > 0$, *such that* $I(u) \geqslant \beta$, $\|u\| = \rho$.
(b) *There exists* $e \in D_o^{2,2}(\Omega)$ *with* $\|e\| > \rho$, *such that* $I(e) \leqslant 0$.

Proof. (a) Since $V_2, u_+^{q+1} \geqslant 0$ and $V_1, V^- \in L^{\alpha_o}(\Omega)\cap L^\infty(\Omega)$, we have

$$I(u) \geqslant \frac{1}{2}\|u\|^2 - C\|u\|^{q+1} - C\|u\|^{2^*} \geqslant \beta, \quad \|u\| = \rho,\ C > 0.$$

(b) Let $w \in D_o^{2,2}(\Omega) - \{0\}$; thus

$$I(tw) := \frac{t^2}{2}\int_\Omega |\Delta w|^2\,dx + \frac{t^{q+1}}{q+1}\int_\Omega V|w|^{q+1}\,dx - \frac{t^{2^*}}{2^*}\int_\Omega |w|^{2^*}\,dx.$$

Since $2 \leqslant q+1 < 2^*$ we have that

$$I(tw) \to -\infty, \quad \text{as } t \to +\infty.$$

This proves Lemma 3.1. $\square$

From Lemma 3.1, applying the Mountain Pass Theorem [1], there exists a sequence $\{u_n\} \subset D_o^{2,2}(\Omega)$ such that

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0, \quad \text{as } n \to \infty,$$

where

$$(*) \quad c := \inf_{h \in \Gamma} \sup_{t \in [0,1]} I\big(h(t)\big),$$

$$\Gamma := \big\{h \in C\big([0,1], D_o^{2,2}(\Omega)\big):\ h(0) = 0,\ h(1) = e\big\}.$$

It is standard to prove that the sequence above is bounded in $D_o^{2,2}(\Omega)$, so that passing to the subsequence if necessary, we can assume that

$$u_n \rightharpoonup u, \quad \text{weakly in } D_o^{2,2}(\Omega),\ \text{as } n \to \infty.$$

By using Lemmas 2.1, 2.2 and a result by Brezis and Lieb, we conclude that u is a weak solution. Finally, by virtue of next result, Lemma 3.2, and arguing as in Brezis and Nirenberg [6] we reach that u is nontrivial.

Lemma 3.2. *The mountain pass level c given in $(*)$ verifies the inequality*

$$0 < \beta \leqslant c < \frac{2}{N} S^{N/4}.$$

Proof. Let $\chi \in C_o^\infty(\Omega)$ be a positive function satisfying

$$\Lambda := \operatorname{int}\big\{x \in \Omega:\ \chi(x) = 1\big\} \neq \emptyset \quad \text{and} \quad \operatorname{supp} V^- \cap \Lambda \neq \emptyset.$$

Notice that the functions s_{ϵ,x_o} defined in (1.2) satisfies

$$s_{\epsilon,x_o} \rightharpoonup 0, \quad \text{weakly in } D_o^{2,2}(\Omega),\ \text{as } \epsilon \to 0$$

and then

$$\chi s_{\epsilon,x_o} \rightharpoonup 0, \quad \text{weakly in } D_o^{2,2}(\Omega),\ \text{as } \epsilon \to 0.$$

From Lemma 2.1, we infer that

$$\int_\Omega V^+ (\chi s_{\epsilon,x_o})^{q+1}\, dx \to 0, \quad \text{as } \epsilon \to 0. \tag{3.1}$$

We now claim that

$$s_{\epsilon,0}^{q+1} \in L^1\big(\mathbb{R}^N\big), \quad \text{if } q > \frac{4}{N-4}.$$

Indeed,

$$\begin{aligned}
\left|s_{\epsilon,0}^{q+1}\right|_1 &= C_N^{q+1}\epsilon^{(N-4)(q+1)/2}\int\limits_{\mathbb{R}^N}\frac{dx}{(\epsilon^2+|x|^2)^{(N-4)(q+1)/2}} \\
&= C_N^{q+1}\epsilon^{-(N-4)(q+1)/2+N}\int\limits_0^\infty\frac{r^{N-1}dr}{(1+|r|^2)^{(N-4)(q+1)/2}} \\
&< \infty, \quad \text{if } q > \frac{4}{N-4}.
\end{aligned}$$

Next, define

$$u_{\epsilon,x_0}(x) := \chi(x)s_{\epsilon,x_o} = \chi(x)s_{\epsilon,0}(x-x_0),$$

then (see [4]) we have

$$|\Delta u_{\epsilon,s}|_2^2 = |\Delta u_{\epsilon,0}|_2^2 + o\big(\epsilon^{N-4}\big), \qquad |u_{\epsilon,s}|_{2^*}^{2^*} = |u_{\epsilon,0}|_{2^*}^{2^*} + o\big(\epsilon^N\big),$$

$$|u_{\epsilon,x_0}|_r^r = \begin{cases} K\epsilon^{(N-4)r/2} + o\big(\epsilon^{(N-4)r/2}\big) & \text{if } r < \frac{N}{N-4}, \\ K\epsilon^{N-(N-4)r/2}|\log\epsilon| + o\big(\epsilon^{N-(N-4)r/2}\big)|\log\epsilon| & \text{if } r = \frac{N}{N-4}, \\ K\epsilon^{N-(N-4)r/2} + o\big(\epsilon^{N-(N-4)r/2}\big) & \text{if } r > \frac{N}{N-4}. \end{cases}$$

Now we consider $f_\epsilon := s_{\epsilon,0}^{q+1}/|s_{\epsilon,0}^{q+1}|_1$ as an approximation of identity, then

$$\begin{aligned}
\int\limits_\Omega V u_{\epsilon,x_0}^{q+1}\,dx &= \int\limits_\Omega V(x)\chi^{q+1}(x)s_{\epsilon,0}^{q+1}(x-s)\,dx \\
&= \int\limits_\Omega\left(\big(V\chi^{q+1}\big)(x) * \frac{s_{\epsilon,0}^{q+1}(x)}{|s_{\epsilon,0}^{q+1}|_1}\right)\left|s_{\epsilon,0}^{q+1}\right|_1 dx \\
&= \left|s_{\epsilon,0}^{q+1}\right|_1\big(V(x_0)\chi^{q+1}(x_0)+o(1)\big) \\
&= C_N^{q+1}\epsilon^{-(N-4)(q+1)/2+N} \\
&\quad\times\int\limits_0^\infty\frac{r^{N-1}\,dr}{(1+|r|^2)^{(N-4)(q+1)/2}}\big(V(x_0)\chi^{q+1}(x_0)+o(1)\big) \\
&= C_N^{q+1}\epsilon^{-(N-4)(q+1)/2+N}C\big(V(x_0)\chi^{q+1}(x_0)+o(1)\big),
\end{aligned}$$

$$C > 0, \text{ a.e. } x_0 \in \mathbb{R}^N.$$

From the assumption on the function χ ,that is, choosing x_0 such that

$$\chi(x_0) = 1, \quad V(x_0) = -V^-(x_0) < 0$$

and that the convergence holds, we obtain

$$\int\limits_\Omega V u_{\epsilon,x_0}^{q+1}\,dx = \epsilon^{-(N-4)(q+1)/2+N}C_1\big(-V^-(x_0)+o(1)\big). \tag{3.2}$$

Define

$$v_\epsilon := \frac{u_{\epsilon,x_0}}{|u_{\epsilon,x_0}|_{2^*}},$$

then (as in [6]) we have

$$X_\epsilon := |\Delta v_\epsilon|_2^2 \leqslant S + O(\epsilon^L), \quad L := \min\{N-4, 4\}. \tag{3.3}$$

On the other hand, notice that

$$I(tv_\epsilon) \to -\infty, \quad \text{as } t \to \infty,$$

where

$$I(tv_\epsilon) = \frac{t^2}{2}X_\epsilon - \frac{t^{2^*}}{2^*} + \frac{t^{q+1}}{q+1}\int_\Omega V v_\epsilon^{q+1}\,dx,$$

thus there exists $t_\epsilon > 0$ such that

$$Y_\epsilon := \sup_{t\geqslant 0} I(tv_\epsilon) = I(t_\epsilon v_\epsilon).$$

In addition,

$$g(t) := \frac{t^2}{2}X_\epsilon - \frac{t^{2^*}}{2^*}$$

achieves its unique positive maximum at $X_\epsilon^{1/(2^*-2)}$, so that

$$Y_\epsilon \leqslant \frac{2}{N}X_\epsilon^{2^*/(2^*-2)} + \frac{t_\epsilon^{q+1}}{q+1}\int_\Omega V v_\epsilon^{q+1}\,dx.$$

From (3.3), we have

$$Y_\epsilon \leqslant \frac{2}{N}S^{N/4} + O(\epsilon^L) + \frac{t_\epsilon^{q+1}}{q+1}\int_\Omega V v_\epsilon^{q+1}\,dx.$$

Now since $|u_{\epsilon,x_0}|_{2^*}$ is bounded away from zero by a constant C independent of ϵ, we obtain

$$Y_\epsilon \leqslant \frac{2}{N}S^{N/4} + O(\epsilon^L) + \frac{Ct_\epsilon^{q+1}}{q+1}\int_\Omega V u_{\epsilon,x_0}^{q+1}\,dx.$$

Observing that $t_\epsilon \to S^{1/(2^*-2)}$, as $\epsilon \to 0$ (see [8]), inserting (3.2) in to the inequality above we have

$$\begin{aligned} Y_\epsilon &\leqslant \frac{2}{N}S^{N/4} + O(\epsilon^L) + C\frac{S^{1/(2^*-2)}}{q+1}\epsilon^{-(N-4)(q+1)/2+N}\big(-V^-(x_0) + o(1)\big) \\ &\leqslant \frac{2}{N}S^{N/4} + \epsilon^{N-4}\big(C - |V^-|_\infty \epsilon^{-(N-4)(q+1)/2+N-(N-4)} \\ &\qquad + o(1)\epsilon^{-(N-4)(q+1)/2+4}\big). \end{aligned}$$

Since $q > \frac{4}{N-4}$, we get

$$Y_\epsilon \leqslant \frac{2}{N} S^{N/4} + \epsilon^{N-4}\big(C - \big(|V^-|_\infty - o(1)\big)\epsilon^{-(N-4)(q+1)/2+4}\big)$$
$$< \frac{2}{N} S^{N/4}, \quad \text{as } \epsilon \to 0.$$

This completes the proof of Lemma 3.2 as well as the proof of Theorem 1.1. □

4. Proof of Theorem 1.2

For that matter we start by proving the following crucial result.

Lemma 4.1. *Suppose $\beta < 4$ and that Ω is a bounded domain. Then*

$$f : D_o^{2,2}(\Omega) \mapsto \mathbb{R}$$

given by

$$f(u) := \int_\Omega \frac{u^2}{|x|^\beta}\, dx$$

is weakly continuous.

Proof. Notice that

$$\int_{B(0,\epsilon)} \frac{1}{|x|^{\beta N/4}}\, dx = C \int_0^\epsilon \frac{r^{N-1}}{r^{\beta N/4}}\, dr < \infty;$$

then $1/|x|^\beta \in L^{N/4}(\Omega)$ and since $u^2 \in L^{N/(N-4)}(\Omega)$ using a result by Brezis and Lieb we conclude the proof of Lemma 4.1. □

Now, from the Hardy inequality it is easy to see that

$$\|u\|_* = \left(\int_\Omega \left(|\Delta u|^2 + \frac{\lambda}{|x|^4} u^2\right) dx\right)^{1/2}, \quad \lambda < \lambda_N,$$

define an equivalent norm in $D_o^{2,2}(\Omega)$. Thus, in both cases, arguing as in [6] (or [8,19]) and combining the remarks above with the arguments used in the proof of Theorem 1.1 we complete the proof of Theorem 1.2. □

5. Further results and concluding remarks

Finally, we point out that the same argument done to prove the Theorem 1.1 can also be used without difficulty in order to get similar results involving the p-Laplacian operator. In this case, the main tool is the Hardy type inequality (see, e.g., [17] or [7]) given by

$$\eta \int_{\mathbb{R}^N} \frac{u^{q+1}}{|x|^\alpha}\,dx \leqslant \left(\int_{\mathbb{R}^N} |\nabla u|^p\,dx \right)^{(q+1)/p}, \quad q \geqslant p-1,\ u \in D^{1,p}(\mathbb{R}^N), \tag{5.1}$$

where $\alpha := N - (N-p)(q+1)/p$, $N \geqslant p$.

Thus assume that either the conditions (H3) or (H4) below hold, where $V(x)$ is a nonpositive function which is locally bounded in $\mathbb{R}^N \setminus \{0\}$ and

$$(\mathrm{H}_3) \quad V(x) = \begin{cases} O(|x|^\nu) & \text{as } |x| \to \infty, \\ O(|x|^\mu) & \text{as } |x| \to 0, \end{cases}$$

where $-p < \nu < \mu$ and $p(N+\nu)/(N-p) < q+1 < p(N+\mu)/(N-p)$ and

$$(\mathrm{H}_4) \quad \begin{cases} V \in L^1_{\mathrm{loc}}(\Omega), \quad V = V^+ - V^- = V_1 + V_2 - V^-, \quad V^\pm \neq 0, \\ V_1, V^- \in L^{\alpha_o}(\Omega) \cap L^\infty(\Omega), \quad \alpha_o = \frac{p^*}{p^*-(q+1)}, \\ \forall y \in \overline{\Omega}, \quad \lim_{x\to y,\ x\in\Omega} |x-y|^\alpha V_2(x) = 0 \\ \text{and} \quad \lim_{|x|\to\infty} |x|^\alpha V_2(x) = 0, \end{cases}$$

where

$$\alpha := \begin{cases} p & \text{if } q = p-1, \\ N - \frac{(N-p)(q+1)}{p} & \text{if } q > p-1. \end{cases}$$

Theorem 5.1. *Suppose either* (H$_4$) *or* $V^\pm \neq 0$ *and* V^+ *verifying* (H$_3$). *Then problem* (P$_1$) *has at least one nontrivial solution provided*

(i) $N \geqslant p^2$, $p < q+1 < s = p^*$ *and* $\mu > 0$,
(ii) $p < N < p^2$, $p^* - \frac{p}{p+1} < q+1 < s = p^*$ *and* $\mu > 0$.

We remark that the proof of Theorem 5.1 follows the same line to those made in the proof of Theorem 1.1, because the condition (H$_3$) for p-Laplacian, which was used by [18], is equivalent to the condition (H$_1$) for the biharmonic operator. In addition, this theorem complement the results in [3] for problem (P$_1$) in the sense that it is true for a class of potentials changing sign with singularities.

Finally we shall state the similar results to Theorem 1.2 for the p-Laplacian operator, whose the proof follows as in the proof of Theorem 1.2, that is, when the potential V is below of the critical hyperbole $|x|^{-p}$ at the origin and Ω a bounded domain for the problem below

$$(\mathrm{P}_3) \quad \begin{cases} -\Delta_p u + V(x)u^{p-1} = \mu|u|^{q-1}u + |u|^{p^*-2}u, & \text{in } \Omega \subset \mathbb{R}^N, \\ u \in D_o^{1,p}(\Omega), \quad \mu > 0, \quad p-1 < q < p^*-1. \end{cases}$$

Theorem 5.2. *Suppose that Ω is a bounded smooth domain. Assume that either* $V = \lambda/|x|^p$, $-\lambda < \lambda_N$ *or* $V := \lambda/|x|^\beta$, $\lambda < 0$, $\beta < p$. *Then, the problem* (P$_3$) *has at least one nontrivial solution provided*

(i) $N \geqslant p^2$, $p < q+1 < p^*$ *and* $\mu > 0$,

(ii) $p < N < p^2$, $p^* - \frac{p}{p+1} < q + 1 < p^*$ *and* $\mu > 0$.

This theorem is related to some results found in [13] and [14], because in our case we studied situation not considered by them, for example, in [13] the authors worked with the case $\mu = 0$ and $V = -V^-$, while in [14] is analyzed the situation where $\mu = 0$ and $\lambda \in (-\lambda_N, 0)$.

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