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On some fourth-order semilinear elliptic problems in $\mathbb{R}^N$ ☆

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1. Introduction

The main purpose of this paper is to establish the existence of two solutions for the following fourth-order problem involving critical growth from the viewpoint of Sobolev embedding:

$$\begin{aligned} &\Delta^2 u - \lambda g(x)u = f(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \\ &u \in D^{2,2}(\mathbb{R}^N) - \{0\}, \end{aligned} \tag{1}$$

where $\lambda > 0$, $p = 2N/(N-4)$ is the critical Sobolev exponent, $N \geqslant 5$, the coefficient $f(x)$ is a continuous bounded function varying in sign and $g \in L^{N/4}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) - \{0\}$ is a nonnegative locally Hölder continuous function.

The critical growth in second-order semilinear problems, on bounded domains, has been extensively investigated in recent years, starting with the work of Brézis and Nirenberg [7]. For fourth-order equations on bounded domains involving critical Sobolev exponent, we refer to [5,16] and references therein.

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We note that problems on unbounded domains involving critical Sobolev exponent have attracted a lot of attention [4,11,15,22,23]. In these papers various existence results have been obtained for the Laplace and biharmonic operators and p-Laplacian.

Here we extend the results of the Drábek–Huang paper [15] to problem (1). The basic tools employed in our paper are the mountain-pass theorem, constrained minimization and concentration-compactness principle [21] and its variant at infinity [9,10]. Unlike the paper of Drábek–Huang, we use the concentration-compactness principle at infinity to investigate the loss of mass at infinity of weakly convergent sequences in $D^{2,2}(\mathbb{R}^N)$. Also, to overcome the difficulty that has arisen from the lack of compactness of the associated energy functional of problem (1), we exploit the fact that the best constant for the Sobolev embedding $D^{2,2}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$, $N \geqslant 5$, is attained.

To discuss the existence of solutions for the perturbed problem (1), it is first necessary to investigate the existence of positive principal eigenvalues for the corresponding linear elliptic problem

$$\begin{aligned} &\Delta^2 u = \lambda g(x) u, \quad \text{in } \mathbb{R}^N, \\ &u \in D^{2,2}(\mathbb{R}^N) - \{0\}. \end{aligned} \tag{2}$$

By a principal eigenvalue of (2) we mean a value of λ corresponding to which there is a positive solution u of (2). The existence, uniqueness and simplicity of positive principal eigenvalues for problem (2) are proved here.

Principal eigenvalues for second-order elliptic equations with weight on unbounded domains, have been discussed by various authors, see for example [2,3,6,8,9,17,20,26], and references therein.

We would like to recall that we can prove, in many second-order eigenvalue problems, that the principal eigenfunction can be taken to be positive, using the classical Stampacchia result, namely, if $w \in W^{1,q}$ then $w^+ = \max\{w, 0\}$ and $w^- = \max\{0, -w\}$ are also in $W^{1,q}$ (cf. [13]).

The situation is more complicated in the case of eigenvalue problems involving equations of order greater than two. It is not possible to use this method to prove that (2) admits a principal eigenfunction, since a similar Stampacchia result on Sobolev spaces involving derivatives of order greater than one, is not valid.

Here, to overcome this nontrivial technical difficulty, we prove, by a fixed point theorem in cones, the existence of principal eigenfunction for the biharmonic operator in bounded domains with Navier boundary conditions. Then, to obtain the principal eigenfunction of problem (2), we use an approximation method; that is, we take the limit of a sequence of principal eigenfunctions associated to eigenvalue problems in balls.

The method used here gives a straightforward proof of the fact that the first eigenvalue is simple and is the unique positive eigenvalue having a positive eigenfunction. This idea can also be applied to problems considered in [9,17], where similar results, to the Laplacian and p-Laplacian operators, respectively, were obtained, by using some technical lemmas involving asymptotic behavior of the solutions and their derivatives at infinity.

Our paper is organized in two parts. Part I is dedicated to proving some properties of principal eigenvalues for problem (2). This study is also fundamental for our existence and multiplicity results of problem (1), which is stated in Part II.

Part I of this paper is organized as follows. In Section 2, we present some basic definitions and establish some preliminary results. In Section 3, we prove the existence, uniqueness and simplicity of positive principal eigenvalues for problem (2), using a theorem of the Krein–Rutman type for the biharmonic operator in bounded domains with Navier boundary conditions, which is proved in Section 4.

Part II of this paper is organized as follows. In Section 5 we examine Palais–Smale sequences to find the range of a variational functional associated to problem (1) for which the Palais–Smale condition holds. We apply in Section 6 the mountain-pass principle to obtain the first existence result for problem (1). Section 7 is devoted to the proof of the existence of a second solution for λ in a small right neighborhood of the first eigenvalue of the associated weighted eigenvalue problem (2). The essential assumption here is that $f(x)$ changes sign which is expressed by $\int_{\mathbb{R}^N} f(x)\phi_1^p \,\mathrm{d}x < 0$, where ϕ_1 is the first eigenfunction for problem (2). This assumption is quite natural and has already appeared for the papers [1,15] for the case of the Laplacian and p-Laplacian, respectively.

Part I. Principal eigenvalues for problem (2)

Our main intention here is to extend some of the results, contained in the references [2,3,8,9,17,20], to the biharmonic operator. The main result in this part may be stated as follows:

Theorem 1. *Assume that $g \in L^{N/4}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) - \{0\}$, $N \geqslant 5$, is a nonnegative and locally Hölder continuous function. Then (2) admits a positive eigenvalue $\lambda_1(g)$ characterized by being the unique positive eigenvalue having positive eigenfunctions. Moreover, $\lambda_1(g)$ is simple and if $\tilde{g}(x) \geqslant g(x)$ for all $x \in \mathbb{R}^N$ we have $\lambda_1(\tilde{g}) \leqslant \lambda_1(g)$.*

Using standard bootstrap argument we also prove regularity and determine the asymptotic behavior of solutions of (2).

2. Preliminary results

We emphasize that throughout this section the weight g can change sign.

We shall start by recalling some basic definitions. In a given Banach space X, we denote weak convergence by "$\rightharpoonup$" and strong convergence by "$\to$". We recall that $D^{2,2}(\mathbb{R}^N)$ is the closure of $C_0^\infty(\mathbb{R}^N)$ functions with respect to the norm $\|u\| = \sqrt{\langle u, u\rangle}$ associated with inner-product given by $\langle u, v\rangle = \int_{\mathbb{R}^N} \Delta u\, \Delta v \,\mathrm{d}x$. By the Sobolev embedding $D^{2,2}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$, $N \geqslant 5$, we see that

$$D^{2,2}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N):\ \Delta u \in L^2(\mathbb{R}^N)\}.$$

The norm $\|u\| = \|\Delta u\|_{L^2}$ is equivalent to the norm $\|D^2 u\|_{L^2}$ (for more details see [21, p. 164]).

We consider here the following variational formulation of problem (2): we say that $u \in D^{2,2}(\mathbb{R}^N) - \{0\}$ is an eigenfunction if

$$\int_{\mathbb{R}^N} \Delta u \, \Delta v \, \mathrm{d}x = \lambda \int_{\mathbb{R}^N} guv \, \mathrm{d}x, \ \forall v \in D^{2,2}(\mathbb{R}^N). \tag{3}$$

The corresponding real number λ is called an eigenvalue.

Using the hypotheses on $g(x)$, we see by Hölder and Sobolev inequalities, for fixed $u \in D^{2,2}(\mathbb{R}^N)$, that the map $v \mapsto \int_{\mathbb{R}^N} guv \, \mathrm{d}x$ is well defined and bounded linear functional in $D^{2,2}(\mathbb{R}^N)$, since

$$\int_{\mathbb{R}^N} |guv| \, \mathrm{d}x \leqslant C\|g\|_{L^{N/4}}\|u\| \, \|v\|.$$

So by the Riesz–Fréchet representation theorem, there is an element Tu in $D^{2,2}(\mathbb{R}^N)$, such that

$$\langle Tu, v \rangle = \int_{\mathbb{R}^N} guv \, \mathrm{d}x.$$

Clearly $T : D^{2,2}(\mathbb{R}^N) \to D^{2,2}(\mathbb{R}^N)$ is linear, symmetric and bounded. Furthermore, we have the following result.

Lemma 2. *The linear operator T is compact.*

The proof of Lemma 2, based on Sobolev embedding theorems, is standard and is omitted. Some related results in weight Sobolev spaces for second-order elliptic equations can be found in the paper by Hanouzet [19].

Notice that for $\lambda \neq 0$, the eigenvalue problem (2) may be rewritten as $Tu = \lambda^{-1}u$. Now, we can apply the spectral analysis of compact symmetric operators, to describe the eigenvalues and eigenvectors of problem (2) (cf. [13]).

Theorem 3. *If $g^+ = \max(g, 0)$ is nontrivial, then problem (2) has a sequence of eigenvalues*

$$0 < \lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_k \to +\infty,$$

whose variational characterizations are

$$\lambda_k^{-1} = \sup_{F_k} \inf \left\{ \int_{\mathbb{R}^N} gu^2 \, \mathrm{d}x \Big/ \int_{\mathbb{R}^N} |\Delta u|^2 \, \mathrm{d}x : u \in F_k - \{0\} \right\}, \tag{4}$$

where F_k varies over all k-dimensional subspaces of $D^{2,2}(\mathbb{R}^N)$.

Remark 1. (i) A similar statement holds when g^- is not trivial.

(ii) It follows readily from the variational characterization (4) that if g, $\tilde{g} \in L^{N/4}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $N \geqslant 5$, are functions such that $\tilde{g}(x) \geqslant g(x)$ for all $x \in \mathbb{R}^N$. Then we have $\lambda_k(\tilde{g}) \leqslant \lambda_k(g)$, provided that the eigenvalues $\lambda_k(\tilde{g})$ and $\lambda_k(g)$ exist, for $k = 1, 2, \ldots.$

(iii) In particular if $g(x)$ is a nonnegative and nontrivial function, we find the following characterization of the first eigenvalue,

$$\lambda_1 = \inf\left\{\int_{\mathbb{R}^N} |\Delta u|^2\,\mathrm{d}x \Big/ \int_{\mathbb{R}^N} g(x)u^2\,\mathrm{d}x\colon u \in D^{2,2} - \{0\}\right\} > 0.$$

3. Proof of Theorem 1

Theorem 4. *Assume that $g \in L^{N/4}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) - \{0\}$, $N \geqslant 5$, is nonnegative and locally Hölder continuous function. Then problem* (2) *has a principal eigenvalue, that is, there is a positive eigenvalue $\lambda_1(g)$ with an associated positive eigenfunction ϕ_1.*

Proof. In order to prove Theorem 4, without loss of generality, we may assume that $g(0) > 0$ and we consider the following family of eigenvalue problems:

$$\begin{aligned} &\Delta^2 u = \lambda g(x) u \quad \text{in } B_n,\\ &u = \Delta u = 0 \qquad \text{on } \partial B_n, \end{aligned} \tag{5}$$

where $B_n = B(0,n)$ is the ball centered at origin with radius n. As a consequence of Theorem 7 (to be proved in the next section) we have a solution $(\lambda_{1,n}, \phi_{1,n})$ of (5) with $\phi_{1,n} \in W_0^{1,2}(B_n) \cap W^{2,2}(B_n)$, $\phi_{1,n} > 0$ in $B(0,n)$ and $(\lambda_{1,n})$ is a nonincreasing sequence of positive numbers with the following characterization:

$$\lambda_{1,n} = \inf\left\{\int_{B(0,n)} |\Delta u|^2\,\mathrm{d}x \Big/ \int_{B(0,n)} gu^2\,\mathrm{d}x\colon u \in W_0^{1,2}(B_n) \cap W^{2,2}(B_n) - \{0\}\right\}.$$

Since $W_0^{1,2}(B_n) \cap W^{2,2}(B_n) \hookrightarrow D^{2,2}(\mathbb{R}^N)$ we have that $\lambda_{1,n} \geqslant \lambda_1$. Moreover, $\lambda_{1,n} \searrow \lambda_1$. Indeed, given $\delta > 0$, there exist $u \in D^{2,2}(\mathbb{R}^N) - \{0\}$ such that

$$\frac{\int_{\mathbb{R}^N} |\Delta u|^2\,\mathrm{d}x}{\int_{\mathbb{R}^N} g(x)u^2\,\mathrm{d}x} < \lambda_1 + \delta.$$

Let $\psi \in C_0^\infty(\mathbb{R}^N, [0,1])$, such that $\psi \equiv 1$ on $|x| \leqslant 1$, $\psi \equiv 0$ on $|x| \geqslant 2$, $|\nabla\psi| \leqslant 2$ and $|\Delta\psi| \leqslant 2$ and we set $u_\varepsilon(x) = \psi(\varepsilon x)u$. Notice that $u_\varepsilon \to u$ in $D^{2,2}(\mathbb{R}^N)$ as $\varepsilon \to 0$ and $u_\varepsilon \in W_0^{1,2}(B_n) \cap W^{2,2}(B_n)$ for some n. Thus, we can choose $\varepsilon > 0$ such that

$$\lambda_{1,n} \leqslant \frac{\int_{\mathbb{R}^N} |\Delta u_\varepsilon|^2\,\mathrm{d}x}{\int_{\mathbb{R}^N} g(x)u_\varepsilon^2\,\mathrm{d}x} < \lambda_1 + \delta.$$

Now, normalizing $\phi_{1,n}$ by $\|\Delta\phi_{1,n}\|_{L^2(B_n)}$, up to a subsequence, we have $\phi_{1,n} \rightharpoonup \phi_1$ weakly in $D^{2,2}(\mathbb{R}^N)$ and,

$$\int_{\mathbb{R}^N} g(x)\phi_{1,n}^2\,\mathrm{d}x \to \int_{\mathbb{R}^N} g(x)\phi_1^2\,\mathrm{d}x,$$

since T is compact. Thus, using that

$$1 = \int_{\mathbb{R}^N} |\Delta\phi_{1,n}|^2\,\mathrm{d}x = \lambda_{1,n}\int_{\mathbb{R}^N} g(x)\phi_{1,n}^2\,\mathrm{d}x,$$

we conclude that $\phi_{1,n} \to \phi_1$ in $D^{2,2}(\mathbb{R}^N)$ and $\|\Delta\phi_1\|_{L^2} = 1$. Furthermore, ϕ_1 is a nontrivial weak solution of

$$\begin{aligned} &\Delta^2\phi_1 = \lambda_1 g(x)\phi_1 \quad \text{in } \mathbb{R}^N,\\ &\phi_1 \geqslant 0 \qquad\qquad\quad \text{in } \mathbb{R}^N. \end{aligned}$$

We set $\psi_1 = -\Delta\phi_1$. Then ψ_1 is a weak nontrivial solution of the problem

$$-\Delta\psi_1 = \lambda_1 g(x)\phi_1 \quad \text{in } \mathbb{R}^N.$$

Using a standard bootstrap argument, see Lemma 5 below, we show that $u \in C^{4,\alpha}_{loc}(\mathbb{R}^N)$ and

$$\lim_{|x|\to+\infty} D^i\phi_1 = 0, \quad i = 0,1,2,3.$$

Hence, by the maximum principle we have $\psi_1 > 0$ in $\mathbb{R}^N$. Consequently, ϕ_1 is a solution of the Dirichlet problem

$$-\Delta\phi_1 = \psi_1 \quad \text{in } \mathbb{R}^N,$$

and again by the maximum principle $\phi_1 > 0$ in $\mathbb{R}^N$. □

Lemma 5. *Suppose that $u \in D^{2,2}(\mathbb{R}^N)$ is a solution of* (2). *Then u is a classical solution, that is, $u \in C^{4,\alpha}_{loc}(\mathbb{R}^N)$ and*

$$\lim_{|x|\to+\infty} D^i u = 0, \quad i = 0,1,2,3.$$

Proof. Let $u \in D^{2,2}(\mathbb{R}^N)$ be a solution of (2). Thus,

$$\Delta^2 u = \lambda g(x)u \quad \text{in } \mathbb{R}^N. \tag{6}$$

We set $v = -\Delta u$ and (6) is equivalent to the system

$$\begin{aligned} &-\Delta v = \lambda g(x)u \quad \text{in } \mathbb{R}^N,\\ &-\Delta u = v \qquad\qquad \text{in } \mathbb{R}^N. \end{aligned} \tag{7}$$

Using a standard bootstrap argument (cf. [18]), we see that $v \in L^q(\mathbb{R}^N)$ for $q \geqslant 1$ large, and so also $u \in L^q(\mathbb{R}^N)$. Now, using the embedding of $W^{4,q}(B_1(x) \hookrightarrow C^{3,\alpha}(B_1(x))$, and system (7) we find that $u \in C^{4,\alpha}_{loc}(\mathbb{R}^N)$ and

$$\|u\|_{C^{3,\alpha}(\overline{B_1(x)})} \leqslant C\|u\|_{L^q(B_2(x))}, \tag{8}$$

where the constant C is independent of x. Finally, from this estimate we complete the proof. □

Theorem 6. *The eigenspace corresponding to the eigenvalue λ_1 is one dimensional and λ_1 is the only eigenvalue of* (2) *to which there corresponds an eigenfunction which does not change sign.*

Proof. Suppose that $\phi \in D^{2,2}(\mathbb{R}^N)$ is an eigenfunction of (2) corresponding to the principal eigenvalue λ_1 and that $u \in D^{2,2}(\mathbb{R}^N)$ is a positive eigenfunction of (2) corresponding to the eigenvalue λ. Of course we have $\lambda \geqslant \lambda_1$. Since

$$\int_{\mathbb{R}^N} \Delta\phi\,\Delta v\,\mathrm{d}x = \lambda_1 \int_{\mathbb{R}^N} g\phi v\,\mathrm{d}x, \quad \forall v \in D^{2,2}(\mathbb{R}^N), \tag{9}$$

taking $v = \phi$, we have

$$\int_{\mathbb{R}^N} |\Delta\phi|^2\,\mathrm{d}x = \lambda_1 \int_{\mathbb{R}^N} g(x)\phi^2\,\mathrm{d}x. \tag{10}$$

Let $\psi \in C_0^\infty(\mathbb{R}^N, [0,1])$, such that $\psi \equiv 1$ on $|x| \leqslant 1$, $\psi \equiv 0$ on $|x| \geqslant 2$, $|\nabla\psi| \leqslant 2$ and $|\Delta\psi| \leqslant 2$ and we set $\phi_R(x) = \psi(x/R)\phi$. Notice that $\phi_R \to \phi$ in $D^{2,2}(\mathbb{R}^N)$ as $R \to +\infty$ and $\phi_R \in W_0^{1,2}(B_{3R}) \cap W^{2,2}(B_{3R})$. Multiplying by ϕ_R^2/u the equation

$$\Delta^2 u = \lambda g(x) u$$

and integrating by parts over B_{3R}, we find

$$\begin{aligned}
&2\int_{B_{3R}} \frac{\phi_R \Delta\phi_R \Delta u}{u}\,\mathrm{d}x + 2\int_{B_{3R}} \frac{|\nabla\phi_R|^2 \Delta u}{u}\,\mathrm{d}x,\\
&-4\int_{B_{3R}} \frac{\phi_R \nabla\phi_R \nabla u \Delta u}{u^2}\,\mathrm{d}x - \int_{B_{3R}} \frac{\phi_R^2 |\Delta u|^2}{u^2}\,\mathrm{d}x,\\
&2\int_{B_{3R}} \frac{\phi_R^2 |\nabla u|^2 \Delta u}{u^3}\,\mathrm{d}x = \lambda \int_{B_{3R}} g\phi_R^2\,\mathrm{d}x.
\end{aligned} \tag{11}$$

Subtracting Eq. (11) from (10), we get

$$\Psi(R) + \int_{|x| \geqslant 3R} |\Delta\phi|^2\,\mathrm{d}x = \lambda_1 \int_{\mathbb{R}^N} g\phi^2\,\mathrm{d}x - \lambda \int_{B_{3R}} g\phi_R^2\,\mathrm{d}x, \tag{12}$$

where

$$\Psi(R) = \int_{B_{3R}} \left[\left| \Delta\phi - \frac{\phi\Delta u}{u} \right|^2 - 2\frac{\Delta u}{u} \left| \nabla\phi - \frac{\phi\nabla u}{u} \right|^2 \right] \mathrm{d}x.$$

Notice that, since $\Delta^2 u = \lambda g(x) u > 0$, in $\mathbb{R}^N$, using Lemma 5, by the maximum principle, we have that $-\Delta u > 0$, in $\mathbb{R}^N$. Thus $\Psi(R)$ is a nonnegative and nondecreasing function of R. Thus, $\Psi(R)$ converges as $R \to \infty$. From (12), taking the limit as $R \to \infty$ and using Fatou's lemma, we find

$$\int_{\mathbb{R}^N} \left\{ \left| \Delta\phi - \frac{\phi\Delta u}{u} \right|^2 - \frac{2\Delta u}{u} \left| \nabla\phi - \frac{\phi\nabla u}{u} \right|^2 \right\} \mathrm{d}x = (\lambda_1 - \lambda) \int_{\mathbb{R}^N} g\phi^2 \mathrm{d}x.$$

Since $\lambda \geqslant \lambda_1$, we get $u\nabla\phi - \phi\nabla u = 0$. Therefore, $\phi = cu$, in $\mathbb{R}^N$. From this we conclude that $\lambda = \lambda_1$. $\square$

4. A theorem of the Krein–Rutman type

In this section we investigate the following linear eigenvalue problem:

$$\begin{aligned} &\Delta^2 u = \lambda g(x)u \quad &&\text{in } \Omega,\\ &\Delta u = 0, &&\text{on } \partial\Omega,\\ &u = 0, &&\text{on } \partial\Omega \end{aligned} \tag{13}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and $g : \bar{\Omega} \to \mathbb{R}$ is a continuous nonnegative and nontrivial function.

Our main result in this section is the following.

Theorem 7. *Problem* (13) *admits a positive eigenvalue $\lambda_1(g)$ characterized by being the unique positive eigenvalue having a positive eigenfunction. Moreover, $\lambda_1(g)$ is simple and if $\tilde{g}(x) \geqslant g(x)$ for all $x \in \bar{\Omega}$ we have $\lambda_1(\tilde{g}) \leqslant \lambda_1(g)$.*

We shall use the fixed point method to cones in order to prove Theorem 7. Actually we shall apply Theorem 19.3 in [14], which for the sake of completeness we state now.

Theorem 8. *Let X be a Banach space and $K \subset X$ a cone with $Int(K) \neq \emptyset$. Let $T : X \to X$ a compact and strongly positive linear operator, i.e. such that $T(K - \{0\}) \subset Int(K)$, and we denote by $r(T)$ its spectral radius. Then we have*

(i) *$r(T)$ is positive and $r(T)$ is a simple eigenvalue with an eigenvector $v \in Int(K)$ and there is no other eigenvalue with a positive eigenvector.*
(ii) *$|\lambda| < r(T)$ for all eigenvalues $\lambda \neq r(T)$.*
(iii) *If $T_1 : X \to X$ and $T_1 x \geqslant Tx$ on K, then $r(T_1) \geqslant r(T)$.*

We work in the real Banach space

$$E = \{v \in C(\bar{\Omega}) : v = 0 \text{ on } \partial\Omega\},$$

provided with the natural ordering given by the cone

$$C_E = \{v \in E\colon v(x) \geqslant 0, \quad \forall x \in \Omega\}.$$

We use also the real Banach space

$$X = \{v \in C^1(\bar{\Omega})\colon v = 0 \text{ on } \partial\Omega\},$$

with the natural ordering given by the cone

$$C_X = \{v \in X : v(x) \geqslant 0, \ \forall x \in \Omega\}$$

Consider the problem

$$\begin{aligned} &\Delta^2 u = h \quad &&\text{in } \Omega,\\ &\Delta u = 0 &&\text{on } \partial\Omega,\\ &u = 0 &&\text{on } \partial\Omega, \end{aligned} \tag{14}$$

and the solution operator $S : C(\bar{\Omega}) \to C(\bar{\Omega})$.

Lemma 9. *S is well-defined and compact.*

Proof. It is well known that if $h \in L^2(\Omega)$, then problem (14) has a unique weak solution, that is, there is a unique $u \in H = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ such that

$$\int_\Omega \Delta u \, \Delta v \, \mathrm{d}x - \int_\Omega hv \, \mathrm{d}x = 0, \quad \forall v \in H. \tag{15}$$

Notice that the left side of (15) is the directional derivative of the C^1-functional

$$J(u) = \frac{1}{2} \int_\Omega |\Delta u|^2 \, \mathrm{d}x - \int_\Omega hu \, \mathrm{d}x$$

defined on the Hilbert space H, in the direction of the vector v. Moreover, J is coercive:

$$J(u) \geqslant \tfrac{1}{2}\|u\|_H^2 - C\|h\|_{L^2}\|u\|_H.$$

Finally, J is sequentially weakly lower semi-continuous in H, since the functional $u \to \int_\Omega hu \, \mathrm{d}x$ belongs to H^{-1} as a consequence of Sobolev imbedding (cf. [12]).

We know that problem (14) is equivalent to the system

$$\begin{aligned} -\Delta u &= v \quad \text{in } \Omega,\\ -\Delta v &= h \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega,\\ v &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{16}$$

Thus, applying Agmon–Douglas–Nirenberg estimates to the equations of this system, we see that if $h \in L^q(\Omega)$, $q \geqslant 2$, the solution $S(h) = u$ of (15) satisfies

$$\|u\|_{W^{4,q}} \leqslant C\|h\|_{L^q}$$

where $C = C(\Omega, N, q)$. Thus, using Rellich–Kondrachov Theorem, we have that the solution operator $S\colon C(\bar{\Omega}) \to C(\bar{\Omega})$ is well defined and compact, since:

$$C(\bar{\Omega}) \hookrightarrow L^q(\Omega) \overset{S}{\hookrightarrow} W^{4,q}(\Omega) \cap W_0^{1,q}(\Omega) \hookrightarrow C(\bar{\Omega}). \qquad \square$$

Let $M : E \to E$ denote the multiplication operator induced by the function $g(x)$. We see that for $\lambda \neq 0$, problem (14) is equivalent to the problem:

$$SMu = \lambda^{-1} u$$

in E. Let us consider the compact operator $K : E \to E$ given by $K = SM$.

Lemma 10. *K is a strongly positive operator, that is, $K(C_X) \subset Int(C_X)$.*

Proof. First, we observe that if $u \in K(C_X)$, we see by Lemma 3.4 in [18] that $u > 0$ in Ω and the normal derivative on the boundary is negative, that is, $\partial u/\partial \nu(x) < 0$ for every $x \in \partial\Omega$. Thus, we have

$$K(C_X) \subset A = \left\{ u \in X\colon u > 0 \text{ in } \Omega \text{ and } \frac{\partial u}{\partial \nu} < 0 \text{ on } \partial\Omega \right\}.$$

Next we are going to prove that $A \subset int(C_X)$. Let $u \in K(C_X)$. Since $\partial\Omega$ is compact we can find positive constants δ_1 and ε such that for all $v \in X$ and $\|u-v\|_X \leqslant \delta_1$ we have $(\partial v/\partial v)(x) < -\varepsilon < 0$ for all $x \in \partial\Omega$. Thus, we can choose $\rho > 0$ such that if $v \in X$ and $\|u-v\|_X \leqslant \delta_1$, we have $v(x) > 0$ for all $x \in \Omega$ such that $d(x, \partial\Omega) \leqslant \rho$, since $v = 0$ on $\partial\Omega$. On the other hand, since $u > 0$ in the compact set $\Omega_\rho = \{x \in \Omega\colon d(x, \partial\Omega) \geqslant \rho\}$ we can choose $\delta \leqslant \delta_1$ such that for all $v \in X$ with $\|u - v\|_X \leqslant \delta_1$ we have $v(x) > 0$ for all $x \in \Omega_\rho$. $\square$

Finally, to prove Theorem 7, we use Lemmas 9 and 10 and apply Theorem 8.

Part II. Existence and multiplicity of solutions for problem (1)

We study the existence of solutions of problem (1), understood as critical points of the functional

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\Delta u|^2 \,\mathrm{d}x - \frac{\lambda}{2}\int_{\mathbb{R}^N} g(x)u^2 \,\mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^N} f(x)|u|^p \,\mathrm{d}x, \tag{17}$$

defined on the space $D^{2,2}(\mathbb{R}^N)$ and with Fréchet derivative

$$\langle I'_\lambda(u), v\rangle = \int_{\mathbb{R}^N} \Delta u\, \Delta v \,\mathrm{d}x - \lambda\int_{\mathbb{R}^N} g(x)uv \,\mathrm{d}x - \int_{\mathbb{R}^N} f(x)|u|^{p-2}uv \,\mathrm{d}x.$$

In this work we are denoting by K_0 the best Sobolev constant to the Sobolev embedding, $D^{2,2}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$, that is,

$$K_0 = \inf\{\|\Delta u\|_{L^2}^2\colon u \in D^{2,2}(\mathbb{R}^N), \quad \|u\|_{L^p} = 1\}. \tag{18}$$

According to Theorem 2.1 in [16] (see also [23]) this infimum K_0 is attained by the functions u_ε given by

$$u_\varepsilon(x) = \frac{C_N \varepsilon^{(N-4)/2}}{(|x-x_0|^2 + \varepsilon^2)^{(N-4)/2}}, \quad C_N = [(N-4)(N-2)N(N+2)]^{(N-4)/8}, \tag{19}$$

for any $x_0 \in \mathbb{R}^N$ and $\varepsilon > 0$. The functions u_ε, with $x_0 = 0$, are the only positive, spherically symmetric solutions of the equation

$$\Delta^2 u = u^{(N+4)/(N-4)} \quad \text{in } \mathbb{R}^N.$$

As usual, we say that a C^1-functional $\Phi\colon X \to \mathbb{R}$ satisfies the Palais–Smale condition at level c (the $(PS)_c$ condition for short) if every Palais–Smale sequence of Φ at level c, that is, $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$ in a dual space X^*, is relatively compact.

We frequently use the concentration-compactness principle due to P. L. Lions [21]. We state below a version adequate for our purposes (For the proof we refer to [11,21,25]).

Proposition 11. *Let* $(u_n) \subset D^{2,2}(\mathbb{R}^N)$ *be a sequence such that*

$$
\begin{array}{ll}
u_n \rightharpoonup u & \text{in } D^{2,2}(\mathbb{R}^N),\\
u_n \to u & \text{a.e. on } \mathbb{R}^N,\\
|\nabla u_n| \to |\nabla u| & \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^N),\\
|u_n|^p \overset{*}{\rightharpoonup} \nu & \text{in the sense of measure},\\
|\Delta u_n|^2 \overset{*}{\rightharpoonup} \mu & \text{in the sense of measure}
\end{array}
$$

and define

$$\alpha_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geqslant R} |u_n|^p \, \mathrm{d}x,$$

$$\beta_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geqslant R} |\Delta u_n|^2 \, \mathrm{d}x.$$

Then there exist an at most countable index set J, *distinct points* $(x_k) \subset \mathbb{R}^N$ *and nonnegative weights* $\{\nu_k, \mu_k\}$ *such that*

$$\nu = |u|^p + \sum_{k\in J} \nu_k \delta_k,$$

$$\mu \geqslant |\Delta u|^2 + \sum_{k\in J} \mu_k \delta_k,$$

$$K_0 \nu_k^{(N-4)/N} \leqslant \mu_k \text{ for each } k,$$

where δ_{x_k} *is Dirac mass at* $x_k \in \mathbb{R}^N$. *Furthermore, we have*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^p \, \mathrm{d}x = \int_{\mathbb{R}^N} |u|^p \, \mathrm{d}x + \sum_{k\in J} \nu_k + \alpha_\infty,$$

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |\Delta u_n|^2 \, \mathrm{d}x \geqslant \int_{\mathbb{R}^N} |\Delta u|^2 \, \mathrm{d}x + \sum_{k\in J} \mu_k + \beta_\infty,$$

$$K_0 \alpha_\infty^{(N-4)/N} \leqslant \beta_\infty.$$

5. Palais–Smale condition

We establish the existence of solutions for $\lambda \in (0, \lambda_1 + \delta)$ where $\delta > 0$ is a small positive number.

We commence by establishing the Palais–Smale condition. This will allow us to obtain the first existence result by straightforward application of the mountain-pass theorem.

Proposition 12. *Suppose that* $\lambda \in (0, \lambda_1)$ *and function* f *satisfies*
(f_∞) $\lim_{|x|\to\infty} f(x) = f(\infty)$.
Then the Palais–Smale condition holds for all

$$c < \frac{2}{N} K_0^{N/4} \|f\|_{L^\infty}^{1-N/4}.$$

Proof. Let $(u_n) \subset D^{2,2}(\mathbb{R}^N)$ be a $(PS)_c$ sequence for I_λ, defined by (17), that is,

$$I_\lambda(u_n) \to c \quad \text{and } I_\lambda(u_n) \to 0.$$

Since $\lambda \in (0, \lambda_1)$ it is easy to check that (u_n) is bounded in $D^{2,2}(\mathbb{R}^N)$. Thus, up to a subsequence, we have $u_n \rightharpoonup u$ in $D^{2,2}(\mathbb{R}^N)$ and according to Proposition 11, we may assume $|u_n|^p \overset{*}{\rightharpoonup} v$ and $|\Delta u_n|^2 \overset{*}{\rightharpoonup} \mu$ in the sense of measures, with

$$v = |u|^p + \sum_{k\in J} v_k \delta_k,$$

$$\mu \geqslant |\Delta u|^2 + \sum_{k\in J} \mu_k \delta_k,$$

where J is at most countable set and $K_0 v_k^{(N-4)/N} \leqslant \mu_k$ for each k. Let $x_k \in \mathbb{R}^N$ be in the support of the singular part of $\mathrm{d}v$ and $\mathrm{d}\mu$. We now take as a test function $u_n\psi$, where $\psi \in C_0^\infty(\mathbb{R}^N, [0, 1])$, such that $\psi \equiv 1$ on $B(x_k, \varepsilon)$, $\psi \equiv 0$ on $\mathbb{R}^N - B(x_k, 2\varepsilon)$, $|\nabla\psi| \leqslant 2/\varepsilon$ and $|\Delta\psi| \leqslant 2/\varepsilon^2$. Hence

$$\int_{\mathbb{R}^N} \Delta u_n \Delta(u_n\psi)\,\mathrm{d}x - \lambda \int_{\mathbb{R}^N} g(x) u_n^2 \psi\,\mathrm{d}x = \int_{\mathbb{R}^N} f(x)|u_n|^p \psi\,\mathrm{d}x + \mathrm{o}_n(1). \tag{20}$$

We observe that

$$\int_{\mathbb{R}^N} \Delta u_n \Delta(u_n\psi)\,\mathrm{d}x$$
$$= \int_{\mathbb{R}^N} |\Delta u_n|^2 \psi\,\mathrm{d}x + 2\int_{\mathbb{R}^N} \Delta u_n \nabla u_n \nabla\psi\,\mathrm{d}x + \int_{\mathbb{R}^N} u_n \Delta u_n\, \Delta\psi\,\mathrm{d}x.$$

Since

$$\lim_{n\to\infty} \left| \int_{\mathbb{R}^N} \Delta u_n \nabla u_n \nabla\psi\,\mathrm{d}x \right|$$
$$\leqslant \lim_{n\to\infty} \left\{ \left(\int_{\mathbb{R}^N} |\Delta u_n|^2\,\mathrm{d}x \right)^{1/2} \left(\int_{\mathbb{R}^N} |\nabla u_n|^2 |\nabla\psi|^2\,\mathrm{d}x \right)^{1/2} \right\}$$
$$\leqslant C \left(\int_{B(x_k,\,2\varepsilon)} |\nabla u|^{2N/(N-2)}\,\mathrm{d}x \right)^{(N-2)/2N} \left(\int_{B(x_k,\,2\varepsilon)} |\nabla\psi|^N\,\mathrm{d}x \right)^{1/N}$$
$$\leqslant C_1 \left(\int_{B(x_k,\,2\varepsilon)} |\nabla u|^{2N/(N-2)}\,\mathrm{d}x \right)^{(N-2)/2N},$$

where $C > 0$ and $C_1 > 0$ are constants independent of n, we see that

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\Delta u_n\nabla u_n\nabla\psi\,\mathrm{d}x = 0.$$

Similarly, since $\Delta\psi_\varepsilon \sim \varepsilon^{-2}$ and $u_n \to u$ in $L^2_{loc}(\mathbb{R}^N)$, we have

$$\begin{aligned}\lim_{n\to\infty}\left|\int_{\mathbb{R}^N} u_n\,\Delta u_n\,\Delta\psi\,\mathrm{d}x\right| &\leqslant C\lim_{n\to\infty}\left(\int_{B(x_k,2\varepsilon)} u_n^2|\Delta\psi|^2\,\mathrm{d}x\right)^{1/2}\\ &\leqslant C\left(\int_{B(x_k,2\varepsilon)} u^2|\Delta\psi|^2\,\mathrm{d}x\right)^{1/2}\\ &\leqslant C\left(\int_{B(x_k,2\varepsilon)}|u|^p\right)^{1/p}\left(\int_{B(x_k,2\varepsilon)}|\Delta\psi|^{N/2}\right)^{2/N}\\ &\leqslant C_1\left(\int_{B(x_k,2\varepsilon)}|u|^p\right)^{1/p},\end{aligned}$$

for some positive constants C and C_1 independent of ε. Consequently,

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\left|\int_{\mathbb{R}^N} u_n\,\Delta u_n\,\Delta\psi\,\mathrm{d}x\right| = 0.$$

Letting $n\to\infty$ and then $\varepsilon\to 0$ in (20) we obtain

$$\mu_k = f(x_k)\nu_k.$$

Since the numbers μ_k and ν_k are positive the concentration occurs only at points where $f(x_k) > 0$. If $\lim_{|x|\to\infty} f(x) \leqslant 0$, there is no concentration at infinity. Hence we may assume that $0 < f(\infty) = \lim_{|x|\to\infty} f(x)$. To examine a possible concentration of the sequence (u_n) at infinity let $\phi_R \in C_0^\infty(\mathbb{R}^N,[0,1])$ be such that $\phi_R(x) = 0$ for $|x| \leqslant R$ and $\phi_R(x) = 1$ for $|x| \geqslant R+1$. Taking as a test function $u_n\phi_R$, we obtain

$$\int_{\mathbb{R}^N}\Delta u_n\,\Delta(u_n\phi_R)\,\mathrm{d}x - \lambda\int_{\mathbb{R}^N} g(x)u_n^2\phi_R\,\mathrm{d}x = \int_{\mathbb{R}^N} f(x)|u_n|^p\phi_R\,\mathrm{d}x + \mathrm{o}(1). \tag{21}$$

Letting $n\to\infty$ and then $R\to\infty$ in (21), using notation from Proposition 11 we obtain

$$\beta_\infty = f(\infty)\alpha_\infty.$$

If $f(x_k) > 0$ and $f(\infty) > 0$, then by Proposition 11 we have

$$\left(\frac{K_0}{f(x_k)}\right)^{N/4} \leqslant \nu_k \quad\text{and}\quad \left(\frac{K_0}{f(\infty)}\right)^{N/4} \leqslant \alpha_\infty.$$

To complete the proof, we observe that

$$I_\lambda(u_n) - \frac{1}{2}\langle I_\lambda'(u_n),u_n\rangle = \frac{2}{N}\int_{\mathbb{R}^N} f(x)|u_n|^p\,\mathrm{d}x + \mathrm{o}(1)$$

and letting $n \to \infty$ we get

$$c = \frac{2}{N}\left[\sum_k \nu_k f(x_k) + f(\infty)\nu_\infty\right].$$

If a concentration occurs either at x_k or at ∞, we derive from the last identity that

$$c \geqslant \frac{2}{N} K_0^{N/4} \|f\|_{L^\infty}^{1-N/4},$$

which is impossible. Consequently, $\nu_k = 0$ for all $k \in J$ and $\alpha_\infty = 0$, which yields $u_n \to u$ in $L^p(\mathbb{R}^N)$. It is now routine to show that $u_n \to u$ in $D^{2,2}(\mathbb{R}^N)$. □

Remark 2. (i) Inspection of the proof of Proposition 12 shows that this proposition remains true with assumption (f_∞) replaced by $f(x) \leqslant 0$, for $|x| \geqslant R$, for some $R > 0$.
(ii) If $\lambda = \lambda_1$, then the following modification of Proposition 12 holds: every bounded sequence in $D^{2,2}(\mathbb{R}^N)$ satisfying

$$I_{\lambda_1}(u_n) \to c < (2/N) K_0^{N/4} \|f\|_{L^\infty}^{1-N/4} \quad \text{and } I'_{\lambda_1}(u_n) \to 0 \quad \text{in } D^{2,2}(\mathbb{R}^N)$$

is relatively compact in $D^{2,2}(\mathbb{R}^N)$.

6. Mountain-pass solution

Our first existence result is a consequence of the mountain-pass theorem. It is easy to check that I_λ has the mountain-pass geometry, that is,

Lemma 13. *Assume that $0 < \lambda < \lambda_1$. Then I_λ satisfies the following conditions*:
(i) *there exist ρ, $\delta > 0$, such that $I_\lambda(u) \geqslant \rho$ for $\|u\| = \delta$ and*
(ii) *for all $u \in D^{2,2}(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} f(x)|u|^p \,\mathrm{d}x > 0$ we have*

$$\lim_{t \to +\infty} I_\lambda(tu) = -\infty.$$

To proceed further, we introduce the following assumptions (f_1), (f_2), (g_1):
(f_1) $f(x_0) = \sup_{x \in \mathbb{R}^N} f(x)$;
(f_2) for x close to x_0 we have

$$f(x) = \begin{cases} f(x_0) + \mathrm{o}(|x - x_0|^4) & \text{if } N \geqslant 7, \\ f(x_0) + \mathrm{o}(|x - x_0|^2) & \text{if } N = 6, \\ f(x_0) + \mathrm{o}(|x - x_0|) & \text{if } N = 5; \end{cases}$$

(g_1) there exist positive constants g_0 and R such that

$$g(x) \geqslant g_0 \quad \text{for all } |x - x_0| \leqslant R.$$

We are using in the next lemma, the extremal functions u_ε given by (19), in order to get a suitable estimate on the minimax mountain-pass level associated to energy functional I_λ. Let $\varphi \in C_0^\infty(\mathbb{R}^N,[0,1])$, such that $\varphi \equiv 1$ on $B(x_0,R)$, $\varphi \equiv 0$ on $\mathbb{R}^N - B(x_0,2R)$, $|\nabla\varphi| \leqslant 2/R$ and $|\Delta\varphi| \leqslant 2/R^2$ and set $v_\varepsilon(x)=\varphi(x)u_\varepsilon(x)$, $\varepsilon>0$.

Lemma 14. *Suppose* (f_1) *and* (f_2) *hold. Then for* $\varepsilon>0$ *sufficiently small, we have*

$$\sup_{t\geqslant 0} I_\lambda(tv_\varepsilon) < \frac{2}{N}K_0^{N/4}\|f\|_{L^\infty}^{1-N/4}.$$

Proof. For simplicity, we assume that $x_0=0$. It follows from [5] or [23]

$$\int_{\mathbb{R}^N} |\Delta v_\varepsilon|^2\,\mathrm{d}x = \int_{\mathbb{R}^N} |\Delta u_1|^2\,\mathrm{d}x + \mathrm{O}(\varepsilon^{N-4}), \tag{22}$$

$$\int_{\mathbb{R}^N} v_\varepsilon^p\,\mathrm{d}x = \int_{\mathbb{R}^N} u_1^p\,\mathrm{d}x + \mathrm{O}(\varepsilon^N) \tag{23}$$

and

$$\int_{\mathbb{R}^N} v_\varepsilon^2\,\mathrm{d}x = \begin{cases} \mathrm{O}(\varepsilon^l) & \text{if } N \geqslant 5,\ N\neq 8,\\ \mathrm{O}(\varepsilon^l|\log\varepsilon|) & \text{if } N=8, \end{cases} \tag{24}$$

where $l=\min\{4,N-4\}$. We also have

$$\begin{aligned}\int_{\mathbb{R}^N} |v_\varepsilon|^2\,\mathrm{d}x &\geqslant C_N^2\omega_N\varepsilon^{N-4}\int_0^R \frac{r^{N-1}}{(\varepsilon^2+r^2)^{N-4}}\,\mathrm{d}r\\ &= C_N^2\omega_N\varepsilon^4\int_0^{R/\varepsilon}\frac{r^{N-1}}{(1+r^2)^{N-4}}\,\mathrm{d}r\\ &\geqslant C_0\varepsilon^4 \quad \text{if } N\geqslant 7,\end{aligned} \tag{25}$$

$$\begin{aligned}\int_{\mathbb{R}^6} v_\varepsilon^2\,\mathrm{d}x &\geqslant C_6^2\omega_6\varepsilon^4\int_0^{R/\varepsilon}\frac{r^5}{(1+r^2)^2}\,\mathrm{d}r\\ &\geqslant C_6^2\omega_6\varepsilon^4\int_1^{R/\varepsilon}\frac{r^5}{(1+r^2)^2}\,\mathrm{d}r\\ &\geqslant \frac{C_6^2\omega_6\varepsilon^4}{4}\int_1^{R/\varepsilon} r\,\mathrm{d}r\\ &= K_1\varepsilon^2(R^2-\varepsilon^2) \quad \text{if } N=6,\end{aligned} \tag{26}$$

$$\begin{aligned}\int_{\mathbb{R}^5} v_\varepsilon^2\,\mathrm{d}x &\geqslant C_5^2\omega_5\varepsilon^4\int_0^{R/\varepsilon}\frac{r^4}{(1+r^2)}\,\mathrm{d}r\\ &\geqslant C_5^2\omega_5\varepsilon^4\int_1^{R/\varepsilon}\frac{r^4}{(1+r^2)}\,\mathrm{d}r\\ &\geqslant K_2\varepsilon(R^3-\varepsilon^3), \quad \text{if } N=5.\end{aligned} \tag{27}$$

for all $0<\varepsilon<\varepsilon_0$ and some positive constants C_0, K_1 and K_2.

Since λ_1 is the first eigenvalue, we have for all $\lambda \in (0, \lambda_1]$,

$$\int_{\mathbb{R}^N} \left(|\Delta v_\varepsilon|^2 - \lambda g v_\varepsilon^2\right) \, dx > 0. \tag{28}$$

Thus, in view of Lemma 13, for each $\varepsilon > 0$, suitably small, there exists $t_\varepsilon > 0$ such that

$$I_\lambda(t_\varepsilon v_\varepsilon) = \max_{t \geqslant 0} I_\lambda(t v_\varepsilon).$$

Moreover, we have

$$t_\varepsilon \int_{\mathbb{R}^N} \left(|\Delta v_\varepsilon|^2 - \lambda g v_\varepsilon^2\right) \, dx - t_\varepsilon^{p-1} \int_{\mathbb{R}^N} f(x) v_\varepsilon^p \, dx = 0$$

and

$$t_\varepsilon^{p-2} = \frac{\int_{\mathbb{R}^N} (|\Delta v_\varepsilon|^2 - \lambda g v_\varepsilon^2) \, dx}{\int_{\mathbb{R}^N} f(x) v_\varepsilon^p \, dx} \geqslant \frac{\int_{\mathbb{R}^N} (|\Delta v_\varepsilon|^2 - \lambda v_\varepsilon^2) \, dx}{\int_{\mathbb{R}^N} f(0) v_\varepsilon^p \, dx}.$$

From this and estimates (22)–(24), it follows that $t_\varepsilon \geqslant a_2$, for all $0 < \varepsilon < \varepsilon_0$ where a_2 is a positive constant independent of ε.

We now write $I_\lambda(t_\varepsilon v_\varepsilon)$ as $I_\lambda(t_\varepsilon v_\varepsilon) = E(\varepsilon) - F(\varepsilon)$, where

$$E(\varepsilon) = \frac{t_\varepsilon^2}{2} \int_{\mathbb{R}^N} |\Delta v_\varepsilon|^2 \, dx - \frac{f(0)}{p} t_\varepsilon^p \int_{\mathbb{R}^N} v_\varepsilon^p \, dx,$$

$$F(\varepsilon) = \frac{\lambda t_\varepsilon^2}{2} \int_{\mathbb{R}^N} \lambda g(x) v_\varepsilon^2 \, dx - \frac{t_\varepsilon^p}{p} \int_{\mathbb{R}^N} (f(0) - f(x)) v_\varepsilon^p \, dx.$$

The function

$$\kappa(t) = \frac{a}{2} t^2 - \frac{b}{p} t^p, \quad t > 0,$$

with $a > 0$ and $b > 0$, attains its maximum at

$$t_{\max} = \left(\frac{a}{b}\right)^{1/(p-2)} \quad \text{and} \quad \kappa(t_{\max}) = \frac{2}{N} \frac{a^{p/(p-2)}}{b^{2/(p-2)}}.$$

Applying this to $E(\varepsilon)$ and using (22)–(23), we obtain the following estimate:

$$\begin{aligned} E(\varepsilon) &\leqslant \frac{2}{N} \frac{(\int_{\mathbb{R}^N} |\Delta v_\varepsilon|^2 \, dx)^{N/4}}{(f(0) \int_{\mathbb{R}^N} v_\varepsilon^p \, dx)^{(N-4)/4}} \\ &= \frac{2}{N} \|f\|_{L^\infty}^{1-N/4} \left\{ \frac{\int_{\mathbb{R}^N} |\Delta v_\varepsilon|^2 \, dx}{[\int_{\mathbb{R}^N} v_\varepsilon^p \, dx]^{2/p}} \right\}^{N/4} \\ &\leqslant \frac{2}{N} K_0^{N/4} \|f\|_{L^\infty}^{1-N/4} + O(\varepsilon^{N(N-4)/4}). \end{aligned} \tag{29}$$

Using estimates (25)–(27) and assumptions (f_2) and (g_1), we derive the following estimates and assumptions:

$$F(\varepsilon) \geqslant \begin{cases} c_1\varepsilon^4 + o(\varepsilon^4) & \text{if } N \geqslant 7, \\ c_1\varepsilon^2(R^2-\varepsilon^2) + o(\varepsilon^2) & \text{if } N = 6, \\ c_1\varepsilon(R^3-\varepsilon^3) + o(\varepsilon) & \text{if } N = 5. \end{cases}$$

Combining the last three estimates with (29) the result follows. □

Let

$$\Gamma = \{\gamma \in C([0,1], D^{2,2}(\mathbb{R}^N)\colon h(0)=0,\ h(1)=t_0v_\varepsilon\},$$

where $t_0>0$ is such that $I_\lambda(t_0v_\varepsilon)<0$ and $\|t_0v_\varepsilon\| \geqslant \delta$, and set

$$c_1 = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\lambda(\gamma(t)).$$

It follows from Lemma 14 that $c_1 < (2/N)K_0^{N/4}\|f\|_{L^\infty}^{1-N/4}$ and in view of Proposition 12 and Lemma 13, applying the mountain-pass theorem (cf. [24]) we obtain the first existence result.

Theorem 15. *Suppose that* (f_∞), (f_1), (f_2) *and* (g_1) *hold and let* $\lambda\in(0,\lambda_1)$. *Then problem* (1) *has a solution.*

7. Existence of the second solution

Throughout this section we make the additional assumption:

(f_3) $$\int_{\mathbb{R}^N} f(x)\phi_1^p\,\mathrm{d}x < 0,$$

where $\phi_1>0$ is the first eigenfunction associated with weighted eigenvalue problem (2). We also assume that $\|\phi_1\|=1$.

We closely follow the approach from the paper of Drábek–Huang [15]. By Λ_λ we denote the Nehari manifold,

$$\begin{aligned}\Lambda_\lambda &= \{u\in D^{2,2}(\mathbb{R}^N)\colon \langle I'(u),u\rangle = 0\}\\ &= \{u\in D^{2,2}(\mathbb{R}^N)\colon \int_{\mathbb{R}^N}(|\Delta u|^2-\lambda g(x)u^2)\,\mathrm{d}x = \int_{\mathbb{R}^N} f(x)|u|^p\,\mathrm{d}x\}.\end{aligned}$$

We now set

$$\Psi_\lambda(u) = \langle I'_\lambda(u),u\rangle.$$

Using Ψ_λ we decompose Λ_λ into three disjoint sets:

$$\Lambda_\lambda = \Lambda_\lambda^+ \cup \Lambda_\lambda^0 \cup \Lambda_\lambda^-,$$

where

$$\Lambda_\lambda^+ = \{u \in \Lambda_\lambda \colon \langle \Psi_\lambda'(u), u\rangle > 0\},$$
$$\Lambda_\lambda^- = \{u \in \Lambda_\lambda \colon \langle \Psi_\lambda'(u), u\rangle < 0\},$$
$$\Lambda_\lambda^0 = \{u \in \Lambda_\lambda \colon \langle \Psi_\lambda'(u), u\rangle = 0\}.$$

Equivalently, we can write

$$\Lambda_\lambda^+ = \{u \in \Lambda_\lambda \colon \int_{\mathbb{R}^N} f(x)|u|^p \,\mathrm{d}x < 0\},$$
$$\Lambda_\lambda^- = \{u \in \Lambda_\lambda \colon \int_{\mathbb{R}^N} f(x)|u|^p \,\mathrm{d}x > 0\},$$
$$\Lambda_\lambda^0 = \{u \in \Lambda_\lambda \colon \int_{\mathbb{R}^N} f(x)|u|^p \,\mathrm{d}x = 0\}.$$

Remark 3. (i) $\Lambda_\lambda^+ = \emptyset$ for all $\lambda \in (0, \lambda_1]$ and $\Lambda_\lambda^+ \neq \emptyset$ for all $\lambda > \lambda_1$ if (f_3) holds. Obviously, we have $t > 0$ such that $t\phi_1 \in \Lambda_\lambda$, thus $t\phi_1 \in \Lambda_\lambda^+$, since $\int_{\mathbb{R}^N} f(x)\phi_1^p \,\mathrm{d}x < 0$.
(ii) $\Lambda_\lambda^- \neq \emptyset$. Indeed, since we are assuming that f is a continuous function varying in sign we have for small $\varepsilon > 0$ that $\int_{\mathbb{R}^N} f(x)v_\varepsilon^p > 0$ and using (28) we have that $tv_\varepsilon \in \Lambda_\lambda$ for some $t > 0$.
(iii) It is clear that $0 \notin \Lambda_\lambda^+$ and $0 \notin \Lambda_\lambda^-$.

In the next lemma below we show that the first eigenfunction ϕ_1 is at a positive distance from Λ_λ^-.

Lemma 16. *There exists a constant $\tau > 0$ such that for $\lambda > 0$,*

$$\left\| \frac{u}{\|u\|} - \phi_1 \right\| \geqslant \tau, \quad \forall u \in \Lambda_\lambda^-.$$

Proof. Arguing indirectly we can find sequences $\lambda_n > 0$ and $u_n \in \Lambda_{\lambda_n}^-$ such that

$$\frac{u_n}{\|u_n\|} \to \phi_1 \quad \text{in } D^{2,2}(\mathbb{R}^N).$$

Since $I_{\lambda_n}(u) > 0$ in $\Lambda_{\lambda_n}^-$, we have

$$0 < \frac{1}{\|u_n\|^p} \int_{\mathbb{R}^N} (|\Delta u_n|^2 - \lambda_n g(x) u_n^2) \,\mathrm{d}x < \frac{p}{2} \int_{\mathbb{R}^N} f(x)|v_n|^p \,\mathrm{d}x,$$

where $v_n = u_n/\|u_n\|$. We now observe that

$$0 \leqslant \lim_{n\to\infty} \int_{\mathbb{R}^N} f(x)|v_n|^p \,\mathrm{d}x = \int_{\mathbb{R}^N} f(x)\phi_1^p \,\mathrm{d}x,$$

and we get a contradiction with our assumption (f_3). $\square$

Lemma 17. *For every $\rho > 0$, there exists $\tilde{\lambda} = \tilde{\lambda}(\rho) > \lambda_1$ such that for all u satisfying*

$$\|u\| = 1 \quad \text{and} \quad \| \pm u - \phi_1\| \geqslant \rho,$$

we have

$$\int_{\mathbb{R}^N} |\Delta u|^2 \,\mathrm{d}x \geqslant \tilde{\lambda} \int_{\mathbb{R}^N} g(x) u^2 \,\mathrm{d}x.$$

Proof. In the contrary case there exist $\rho > 0$ and sequences (λ_n) and $(u_n) \subset D^{2,2}(\mathbb{R}^N)$ satisfying

$$\|u_n\| = 1 \quad \text{and} \quad \| \pm u_n - \phi_1\| \geqslant \rho,$$

and $\lambda_n \to \lambda_1$ such that

$$\int_{\mathbb{R}^N} |\Delta u_n|^2 \,\mathrm{d}x = \lambda_n \int_{\mathbb{R}^N} g(x) u_n^2 \,\mathrm{d}x.$$

We may assume that $u_n \rightharpoonup u_0$ in $D^{2,2}(\mathbb{R}^N)$. Then

$$\int_{\mathbb{R}^N} g(x) u_n^2 \,\mathrm{d}x \to \int_{\mathbb{R}^N} g(x) u_0^2 \,\mathrm{d}x$$

and

$$0 \leqslant \int_{\mathbb{R}^N} (|\Delta u_0|^2 - \lambda_1 g(x) u_0^2) \,\mathrm{d}x \leqslant \liminf_{n\to\infty} \int_{\mathbb{R}^N} (|\Delta u_n|^2 - \lambda_n g(x) u_n^2) \,\mathrm{d}x = 0,$$

which means that $u_n \to u_0$ in $D^{2,2}(\mathbb{R}^N)$. Since $\|u_n\| = 1$, we must have that $\|u_0\| = 1$. It follows form the variational characterization of the first eigenvalue that $u_0 = \phi_1$ or $u_0 = -\phi_1$. However, this is impossible because

$$\| \pm u_n - \phi_1\| \geqslant \rho. \qquad \square$$

Lemma 18. *There exists a constant $\tilde{\lambda} > \lambda_1$ such that for each $\lambda \in (\lambda_1, \tilde{\lambda})$ the set Λ_λ^- is closed in $D^{2,2}(\mathbb{R}^N)$ and open in Λ_λ.*

Proof. The second part of this lemma is obvious and the first is a consequence of Lemmas 16 and 17. $\square$

Lemma 19. *There exists $\lambda^* > \lambda_1$ such that for $\lambda \in (\lambda_1, \lambda^*)$, the set Λ_λ^+ is bounded.*

Proof. Arguing indirectly assume that there exist sequences $\lambda_n > \lambda_1$ and $(u_n) \subset \Lambda_{\lambda_n}^+$ such that $\lambda_n \to \lambda_1$ and $\|u_n\| \to \infty$. Since $(u_n) \subset \Lambda_{\lambda_n}^+$, we have

$$\begin{aligned} 0 &> \int_{\mathbb{R}^N} (|\Delta u_n|^2 - \lambda_n g(x) u_n^2) \,\mathrm{d}x > \frac{p}{2} \int_{\mathbb{R}^N} f(x) |u_n|^p \,\mathrm{d}x \\ &= \frac{p}{2} \int_{\mathbb{R}^N} (|\Delta u_n|^2 - \lambda_n g(x) u_n^2) \,\mathrm{d}x. \end{aligned} \tag{30}$$

Dividing by $\|u_n\|^p$ and taking the limit we find

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} f(x)|v_n|^p\,\mathrm{d}x = 0, \tag{31}$$

where $v_n = u_n/\|u_n\|$. Up to a subsequence, we can assume that $v_n \rightharpoonup v_0$ in $D^{2,2}(\mathbb{R}^N)$. Thus,

$$\int_{\mathbb{R}^N} g(x)v_n^2\,\mathrm{d}x \to \int_{\mathbb{R}^N} g(x)v_0^2\,\mathrm{d}x,$$

which implies

$$0 \leqslant \int_{\mathbb{R}^N}(|\Delta v_0|^2 - \lambda_1 g(x)v_0^2)\,\mathrm{d}x \leqslant \liminf_{n\to\infty}\int_{\mathbb{R}^N}(|\Delta v_n|^2 - \lambda_n g(x)v_n^2)\,\mathrm{d}x.$$

It then follows from (30) that

$$\int_{\mathbb{R}^N}(|\Delta v_0|^2 - \lambda_1 g(x)v_0^2)\,\mathrm{d}x = 0 \quad \text{and} \quad v_n \to v_0 \quad \text{in } D^{2,2}(\mathbb{R}^N).$$

Hence $v_0 = \pm\phi_1$ since $\|v_n\| = 1$. Therefore,

$$\int_{\mathbb{R}^N} f(x)|v_n|^p\,\mathrm{d}x \to \int_{\mathbb{R}^N} f(x)\phi_1^p\,\mathrm{d}x,$$

which, using (31), we reach a contradiction with our assumption (f_3). □

According to Lemma 19 and the fact that $I_\lambda > 0$ in Λ_λ^- we see that I_λ is bounded from below on Λ_λ for $\lambda \in (\lambda_1, \lambda^*)$ and we set

$$c_1 = \inf_{w\in\Lambda_\lambda} I_\lambda(w).$$

Clearly, we have

$$c_1 = \inf_{w\in\Lambda_\lambda^+} I_\lambda(w) < 0 \quad \text{for } \lambda > \lambda_1.$$

Proposition 20. (i) *For $\lambda \in [\lambda_1, \tilde{\lambda})$ every minimizing sequence (u_n) for I_λ on Λ_λ^- such that*

$$c_1 \leqslant I(u_n) \leqslant c_1 + \frac{2}{N}\|f\|_{L^\infty}^{1-N/4}K_0^{N/4}$$

is relatively compact in $D^{2,2}(\mathbb{R}^N)$.

(ii) *Let $\lambda \in (\lambda_1, \lambda^*)$. Then every minimizing sequence of I_λ on Λ_λ^+ is relatively compact.*

Proof. (i) We distinguish two cases: (a) $\lambda \in (\lambda_1, \tilde{\lambda})$ and (b) $\lambda = \lambda_1$.

Case (a). Arguing indirectly we can show that (u_n) is bounded. Indeed, assume that $\|u_n\| \to \infty$ and let $v_n = u_n/\|u_n\|$. Now, arguing as in the proof of Lemma 19 and using Lemma 17 we find

$$0 \leqslant (\tilde{\lambda} - \lambda)\int_{\mathbb{R}^N} g(x)v_n^2\,\mathrm{d}x \leqslant \int_{\mathbb{R}^N}(|\Delta v_n|^2 - \lambda g(x)v_n^2)\,\mathrm{d}x.$$

Since $I_\lambda(u_n)$ is bounded and

$$\int_{\mathbb{R}^N} (|\Delta v_n|^2 - \lambda g(x)v_n^2)\,\mathrm{d}x = \frac{2}{N}\frac{I_\lambda(u_n)}{\|u_n\|^2},$$

we obtain

$$\int_{\mathbb{R}^N} g(x)v_n^2\,\mathrm{d}x \to 0,$$

which implies that $\|v_n\| \to 0$, contradicting the fact that $\|v_n\| = 1$.

We may assume that $u_n \rightharpoonup u$ in $D^{2,2}(\mathbb{R}^N)$.

According to the Lagrange multiplier method, we have

$$I_\lambda'(u_n) - a_n\Psi_\lambda'(u_n) \to 0,$$

where $a_n \in \mathbb{R}$ satisfies

$$\langle I_\lambda'(u_n), u_n\rangle = a_n\langle \Psi_\lambda'(u_n), u_n\rangle.$$

Since $u_n \in \Lambda_\lambda^-$, we get $a_n = 0$. This yields that $I_\lambda'(u_n) \to 0$ in $D^{2,2}(\mathbb{R}^N)$. It is obvious that

$$\Delta^2 u - \lambda g u = f|u|^{p-2}u,$$

hence $u \in \Lambda_\lambda$ and $I_\lambda(u) \geqslant c_1$.

We now repeat the argument from the proof of Proposition 12 to obtain

$$\frac{2}{N}\|f\|_{L^\infty}^{1-N/4}K_0^{N/4} + c_1 \geqslant \frac{2}{N}\int_{\mathbb{R}^N} f(x)|u|^p\,\mathrm{d}x + \frac{2}{N}\sum_{j\in J} f(v_j)v_j + \frac{2}{N}f(\infty)v_\infty.$$

If a concentration occurs either at a finite point x_j or at infinity, we deduce a contradiction from the last inequality. Therefore, $v_j = v_\infty = 0$ for all $j \in J$ and $u_n \to u$ in $L^p(\mathbb{R}^N)$. Since (u_n) is a (PS) sequence, we see that $u_n \to u$ in $D^{2,2}(\mathbb{R}^N)$. Obviously, $u \in \Lambda_\lambda^-$, as Λ_λ^- is closed in $D^{2,2}(\mathbb{R}^N)$, thus $c_1 = I_\lambda(u) > 0$ and u is a nontrivial solution.

Case (b). Let (u_n) be a minimizing sequence for I_{λ_1} on Λ_{λ_1} satisfying conditions stated in part (1) of our Proposition. First, we show that the sequence (u_n) is bounded in $D^{2,2}(\mathbb{R}^N)$. Assume by contradiction that $\|u_n\| \to \infty$. First, we observe that

$$I_{\lambda_1}(u_n) = \frac{2}{N}\int_{\mathbb{R}^N} f(x)|u_n|^p\,\mathrm{d}x$$

and

$$\int_{\mathbb{R}^N} (|\Delta u_n|^2 - \lambda_1 g(x)u_n^2)\,\mathrm{d}x < \int_{\mathbb{R}^N} f(x)|u_n|^p\,\mathrm{d}x. \qquad (32)$$

Hence $\int_{\mathbb{R}^N} f(x)|u_n|^p\,\mathrm{d}x$ is bounded. We now set $v_n = u_n/\|u_n\|$. It then follows from (32) that

$$0 \leqslant \int_{\mathbb{R}^N} (|\Delta v_n|^2 - \lambda_1 g(x)v_n^2)\,\mathrm{d}x \leqslant \frac{1}{\|u_n\|^2}\int_{\mathbb{R}^N} f(x)|u_n|^p\,\mathrm{d}x \to 0. \qquad (33)$$

We now apply Lemmas 16 and 17 to get

$$(\tilde{\lambda} - \lambda_1) \int_{\mathbb{R}^N} g(x) v_n^2 \, dx \to 0.$$

Hence, by virtue of (33), we obtain that $v_n \to 0$ in $D^{2,2}(\mathbb{R}^N)$, which is impossible as $\|v_n\| = 1$ for each n. Thus (u_n) is bounded in $D^{2,2}(\mathbb{R}^N)$. As in the Case (a), using the Lagrange multiplier method in $\Lambda_{\lambda_1}^-$ we show that $I'_{\lambda_1}(u_n) \to 0$ in $D^{-2,2}(\mathbb{R}^N)$. Considering a possible concentration at finite points, or at infinity, we show that $u_n \to u$ in $D^{2,2}(\mathbb{R}^N)$. Repeating the argument from Case (a) the conclusion readily follows.

(ii) According to Lemma 19 (u_n) is bounded in $D^{2,2}(\mathbb{R}^N)$. As in the first part we show that the sequence (u_n) is relatively compact in $D^{2,2}(\mathbb{R}^N)$ and up to a subsequence, $u_n \to u_0$ in $D^{2,2}(\mathbb{R}^N)$. To show that $u_0 \in \Lambda_\lambda^+$ it is sufficient to observe that $I_\lambda(u_0) \leqslant I_\lambda(u_n) < 0$ and this completes the proof. □

We are now in a position to formulate the existence results for $\lambda \geqslant \lambda_1$.

Theorem 21. *Suppose that assumptions* (f_1), (f_2), (f_3) *and* (g_1) *hold. Then problem* (1) *with* $\lambda = \lambda_1$ *has at least one solution.*

Proof. Since $\Lambda_{\lambda_1}^+ = \emptyset$, we see that $I_{\lambda_1}(u) > 0$ for $u \in \Lambda_{\lambda_1}^-$. On the other hand by Lemma 14, $tv_\varepsilon \in \Lambda_{\lambda_1}^-$ with suitable $t > 0$, we therefore conclude that assumptions of Proposition 20 are fulfilled. Thus, up to a subsequence, $u_n \to u_0$ in $D^{2,2}(\mathbb{R}^N)$. Since $\Lambda_{\lambda_1}^-$ is closed, $u_0 \in \Lambda_{\lambda_1}^-$ and $I_{\lambda_1}(u_0) > 0$. □

The second part of Proposition 20 yields the existence of a minimizer of I_λ on Λ_λ^+ for $\lambda \in (\lambda_1, \lambda^*)$.

Theorem 22. *Suppose that the assumptions* (f_1), (f_2) *and* (f_3) *hold. Then problem* (1) *with* $\lambda_1 < \lambda < \lambda^*$ *has at least one solution.*

Let $z \in \Lambda_\lambda^+$, with $\lambda_1 < \lambda < \lambda^*$, be a solution obtained in Theorem 22. We obviously have $I_\lambda(z) = c_1$. To obtain a second solution we need the following result.

Lemma 23. *Suppose that assumptions* (f_1), (f_2), (f_3) *and* (g_1) *hold. Then for* $\varepsilon > 0$ *sufficiently small we have*

$$\sup I_\lambda(z + tv_\varepsilon) < I_\lambda(z) + \frac{2}{N} \|f\|_{L^\infty}^{1-N/4} K_0^{N/4}.$$

The proof is identical to that of Proposition 5 in the paper [10]. Lemma 23 yields that $z + tv_\varepsilon \in \Lambda_\lambda^-$ for t sufficiently large. Combining this observation with the first part of Proposition 20, we easily derive the following existence result.

Theorem 24. *Suppose that assumptions* (f_1), (f_2), (f_3) *and* (g_1) *hold. Let* $\bar{\lambda} = \min\{\lambda^*, \tilde{\lambda}\}$. *Then for* $\lambda \in (\lambda_1, \bar{\lambda})$ *problem* (1) *has at least two solutions.*

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