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On Semilinear Elliptic Equations Involving Concave and Convex Nonlinearities

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Abstract. We prove some existence results for the problem $(1_{\lambda,\mu})$ with $0 < q < 1 < p$ depending on the range of parameters λ and μ. To establish the existence of solutions we use the method of successive approximations and the monotone method of sub and supersolutions. The cases where (i) $a(x)$ is bounded from below by a positive constant and (ii) $a(x)$ is bounded below by a positive constant outside a ball are considered. We also discuss the case where $\mu = 0$ and λ is replaced by a positive or negative function. In this situation we use the variational method based on a constrained minimization combined with concentration–compactness principle at infinity.

1. Introduction

The main purpose of this article is to construct solutions to the problem

$$(1_{\lambda,\mu}) \qquad \begin{cases} -\Delta u + a(x)u = \lambda u^q + \mu u^p & \text{in} \quad \mathbb{R}^N, \\ u(x) > 0 & \text{on} \quad \mathbb{R}^N, \end{cases}$$

where $0 < q < 1 < p$ and $\lambda > 0$, $\mu > 0$ are parameters. The coefficient $a(x)$ is positive, locally Hölder continuous and bounded on $\mathbb{R}^N$.

In the case of the Dirichlet problem in a bounded domain $\Omega \subset \mathbb{R}^N$

$$(D) \qquad \begin{cases} -\Delta u = \lambda\, |u|^{q-2}u + \mu\, |u|^{p-2}u & \text{in} \quad \Omega, \\ u(x) = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

with $1 < q < 2 < p$, there are a number of existence results. In particular, problem (D) admits infinitely many solutions for some values of parameters λ and μ [ABC], [BW] and at least two positive solutions for $\lambda > 0$ small and $\mu = 1$ [RU] in the case where Ω is a ball and exactly two solutions if $2 < p < \frac{2N}{N-2}$ [APY]. Some existence results

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(infinitely many) for problem $\left(1_{\lambda,\mu}\right)$ can be found in the papers [CH3], [GM] and [ST]. However, in these papers the nonlinearity is replaced by $h(x)\,|u|^{q-2}u + k(x)\,|u|^{p-2}u$, with k and h satisfying some integrability conditions.

In Sections 2 and 3 we present some existence results based on the method of successive approximations with no restrictions on p.

Section 4 is devoted to a purely concave nonlinearity. Solutions obtained by the method of successive approximations in Sections 3 and 4 are bounded from below by positive constants. Section 5 contains some existence results for problem $\left(1_{\lambda,\mu}\right)$ obtained by the use of the method of sub and supersolutions under less restrictive conditions on λ and μ and we derive some estimates from below. In general solutions of problem $\left(1_{\lambda,\mu}\right)$, whose existence have been established in Theorems 3.1 and 5.1 exhibit the following behaviour when λ or μ tend to 0: (i) if $\lambda \to 0$, then solutions converge uniformly to 0 on $\mathbb{R}^N$, (ii) however, if $\mu \to 0$, then there exists a nonzero limit which is a solution of problem $(1_{\lambda,0})$. In Section 6 we relax some assumptions on the coefficient $a(x)$ and we allow the nonlinearity to interfere with the first eigenvalue $\lambda_1(a)$ of the operator $-\Delta u + a(x)u$ on $\mathbb{R}^N$. We give some existence results in two cases: $\lambda_1(a) > 0$ and $\lambda_1(a) < 0$. In both cases we use the method of sub– and supersolutions. Finally, in Section 6 we again consider the purely concave nonlinearity. Here we give some existence results for $\lambda = 1, -1$ and $\mu = 0$. Problems $\left(1_{\lambda,\mu}\right)$ with $\lambda = 1, -1$ and $\mu = 0$ stem from the problems in scalar curvature of warped products of semiriemannian manifolds [DL]. In the case $\lambda = -1$, $\mu = 0$ we offer a very simple proof of the existence of a solution based on a constrained minimization. To obtain relative compactness of a minimizing sequence we apply the concentration–compactness principle at infinity [CH2], which generalizes slightly the result from the paper [LI].

2. Preliminaries

Let $a_0 = \inf_{x\in\mathbb{R}^N} a(x)$ and $A = \sup_{x\in\mathbb{R}^N} a(x)$. We shall always assume, except in Sections 6 and 7, that $a_0 > 0$. Let $0 < a_1 < a_0$ and set

$$H(x,\delta) \;=\; \prod_{j=1}^{N} \cosh \delta x_j$$

for $x \in \mathbb{R}^N$, where $\delta > 0$ is a small parameter. By straightforward calculations we check that there exists a number $\delta_0 > 0$ such that

$$-\Delta H + a(x)H \;\geq\; a_1 H \quad \text{on} \quad \mathbb{R}^N \tag{2.1}$$

for all $0 \leq \delta \leq \delta_0$.

Using the function H it is easy to show the following version of the maximum principle in $\mathbb{R}^N$ for the operator $-\Delta u + a(x)u$ [CH1].

Proposition 2.1. *Suppose that* $u(x) \leq Ce^{\delta \sum_{i=1}^{N} |x_i|}$ *on* $\mathbb{R}^N$ *for some constants* $C > 0$ *and* $0 \leq \delta \leq \delta_0$ *and that*

$$-\Delta u + a(x)u \;\leq\; 0 \quad \textit{in} \quad \mathbb{R}^N\,.$$

Then $u(x) \leq 0$ *on* $\mathbb{R}^N$.

From this we easily derive the following lower and upper estimates:

Corollary 2.2. *Suppose that* $|u(x)| \leq Ce^{\delta \sum_{i=1}^N |x_i|}$ *on* $\mathbb{R}^N$ *for some constants* $C > 0$ *and* $0 \leq \delta \leq \delta_0$ *and that*

$$-\Delta u + a(x)u \;=\; f \quad \text{on} \quad \mathbb{R}^N,$$

where f *is a bounded function. Then*

$$\frac{\inf_{x\in\mathbb{R}^N} f(x)}{A} \;\leq\; u(x) \;\leq\; \frac{\sup_{x\in\mathbb{R}^N} f(x)}{a_0} \quad \text{on} \quad \mathbb{R}^N.$$

This result will be frequently used in Sections 3 and 4 to construct a solution to problem $\left(1_{\lambda,\mu}\right)$ by a method of successive approximations.

3. Successive approximations

In this section we construct a sequence of successive approximations to problem $\left(1_{\lambda,\mu}\right)$.

Let $u_0(x) = M$, where $0 < M \leq 1$ is a constant. We define u_j for $j \geq 1$ by

$$\left(2_j\right) \qquad -\Delta u_j + a(x)u_j \;=\; \lambda u_{j-1}^q + \mu u_{j-1}^p \quad \text{in} \quad \mathbb{R}^N.$$

Equations $\left(2_j\right)$ have unique bounded solutions on $\mathbb{R}^N$.

Theorem 3.1. *Suppose that* $\lambda + \mu \leq a_0$. *Then problem* $\left(1_{\lambda,\mu}\right)$ *admits a solution satisfying*

$$(3.1) \qquad \left(\frac{\lambda}{A}\right)^{\frac{1}{1-q}} \;\leq\; u(x) \;\leq\; \left(\frac{\lambda+\mu}{a_0}\right)^{\frac{1}{1-q}} \quad \text{on} \quad \mathbb{R}^N.$$

Proof. Since $0 < M \leq 1$, we have

$$-\Delta u_1 + a(x)u_1 \;\leq\; (\lambda+\mu)M^q \quad \text{in} \quad \mathbb{R}^N$$

and by Corollary 2.2 we have

$$u_1(x) \;\leq\; \frac{(\lambda+\mu)M^q}{a_0} \quad \text{on} \quad \mathbb{R}^N.$$

Similarly, we have

$$-\Delta u_2 + au_2 \;\leq\; \frac{\lambda(\lambda+\mu)^q M^{q^2}}{a_0^q} + \frac{\mu(\lambda+\mu)^p M^{pq}}{a_0^p} \;\leq\; \frac{(\lambda+\mu)^{1+q} M^{q^2}}{a_0^q} \quad \text{in} \quad \mathbb{R}^N$$

and by Corollary 2.2

$$u_2(x) \;\leq\; \frac{(\lambda+\mu)^{1+q} M^{q^2}}{a_0^{1+q}} \quad \text{on} \quad \mathbb{R}^N.$$

Using mathematical induction we show that

$$u_n(x) \;\le\; \frac{(\lambda+\mu)^{1+q+\ldots+q^{n-1}}}{a_0^{1+q+\ldots+q^{n-1}}}\, M^{q^n} \quad \text{on} \quad \mathbb{R}^N . \tag{3.2}$$

We now derive lower bounds for the sequence $\{u_n\}$. First, we observe that

$$-\Delta u_1 + a u_1 \;\ge\; \lambda M^q \quad \text{in} \quad \mathbb{R}^N .$$

Hence by Corollary 2.2 we have

$$u_1(x) \;\ge\; \frac{\lambda M^q}{A} \quad \text{on} \quad \mathbb{R}^N .$$

We easily show using mathematical induction that

$$u_n(x) \;\ge\; \frac{\lambda^{1+q+\ldots+q^{n-1}}}{A^{1+q+\ldots+q^{n-1}}}\, M^{q^n} \quad \text{on} \quad \mathbb{R}^N . \tag{3.3}$$

Let $\Omega_1 \subset \Omega_2 \subset \Omega_3$ be bounded domains in $\mathbb{R}^N$, with $\overline{\Omega}_1 \subset \Omega_2$ and $\overline{\Omega}_2 \subset \Omega_3$. It follows from Theorem 8.24 in [GT] that

$$\|u_n\|_{C^\alpha(\overline{\Omega}_2)} \;\le\; K, \quad n \;=\; 1, 2, \ldots$$

where $K = K\big(\overline{\Omega}_3, A, a_0, \lambda, \mu, M\big)$. Using this estimate we deduce from Theorem 6.2 in [GT] the following estimate

$$\|Du_n\|_{C^\alpha(\overline{\Omega}_1)}, \; \big\|D^2 u_n\big\|_{C^\alpha(\overline{\Omega}_1)} \;\le\; K_1$$

for some constant $K_1 > 0$ independent of n. By standard arguments involving the Ascoli–Arzela theorem, we can choose a subsequence of $\{u_n\}$, denoted again by $\{u_n\}$, such that

$$u_n \longrightarrow u, \quad Du_n \longrightarrow Du \quad \text{and} \quad D^2 u_n \longrightarrow D^2 u$$

uniformly on each bounded domain of $\mathbb{R}^N$. Obviously, $u(x)$ satisfies $\big(1_{\lambda,\mu}\big)$ and the estimate (3.1). □

As an immediate consequence of Theorem 3.1 we obtain the following existence results:

Corollary 3.2. *Suppose that $a_0 \ge 2$. Then problem*

$$\begin{cases} -\Delta u + a(x) u \;=\; u^q + u^p & \text{in} \quad \mathbb{R}^N, \\ u(x) \;>\; 0 & \text{on} \quad \mathbb{R}^N \end{cases} \tag{$1_{1,1}$}$$

admits a solution $u(x)$ satisfying

$$\frac{1}{A^{\frac{1}{1-q}}} \;\le\; u(x) \;\le\; \left(\frac{2}{a_0}\right)^{\frac{1}{1-q}} \quad \text{on} \quad \mathbb{R}^N .$$

Corollary 3.3. *Suppose that* $1 < a_0 < 2$. *Then for each* $0 < \lambda \le a_0 - 1$ *problem*

$$(1_{\lambda,1}) \qquad \begin{cases} -\Delta u + a(x)u \ = \ \lambda u^q + u^p & in \quad \mathbb{R}^N, \\ u(x) \ > \ 0 \quad on \quad \mathbb{R}^N \end{cases}$$

admits a solution $u(x)$ *satisfying*

$$\left(\frac{\lambda}{A}\right)^{\frac{1}{1-q}} \ \le \ u(x) \ \le \ \left(\frac{\lambda+1}{a_0}\right)^{\frac{1}{1-q}} \quad on \quad \mathbb{R}^N .$$

Corollary 3.4. *Suppose that* $1 < a_0 < 2$. *Then for each* $0 < \mu \le a_0 - 1$ *problem*

$$(1_{1,\mu}) \qquad \begin{cases} -\Delta u + a(x)u \ = \ u^q + \mu u^p & in \quad \mathbb{R}^N, \\ u(x) \ > \ 0 \quad on \quad \mathbb{R}^N \end{cases}$$

admits a solution u *satisfying*

$$\left(\frac{1}{A}\right)^{\frac{1}{1-q}} \ \le \ u(x) \ \le \ \left(\frac{1+\mu}{a_0}\right)^{\frac{1}{1-q}} .$$

If u is a bounded solution of problem $(1_{\lambda,\mu})$ with $K = \inf_{x\in\mathbb{R}^N} u(x) > 0$, then by Corollary 2.2

$$u(x) \ \ge \ \frac{\lambda K^q + \mu K^p}{A} \quad \text{on} \quad \mathbb{R}^N$$

and hence

$$K \ \ge \ \frac{\lambda K^q + \mu K^p}{A} .$$

This inequality will be used to derive some nonexistence results.

Proposition 3.5. (a) *If* $\lambda \ge A$ *and* $\mu \ge A$, *then problem* $(1_{\lambda,\mu})$ *does not have a bounded solution which is bounded from below by a positive constant on* $\mathbb{R}^N$.

(b) *Let* $\mu > 0$ *be fixed. Then for* $\lambda > \mu^{-\frac{1-q}{p-1}} A^{\frac{p-q}{p-1}}$ *problem* $(1_{\lambda,\mu})$ *does not have a bounded solution which is bounded from below by a positive constant. Similarly, for* $\lambda > 0$ *fixed and* $\mu > \lambda^{-\frac{p-1}{1-q}} A^{\frac{p-q}{1-q}}$ *problem* $(1_{\lambda,\mu})$ *does not have a bounded solution which is bounded from below by a positive constant.*

(c) *Let* $a_0 = A$. *If the equation* $At = \lambda t^q + \mu t^p$ *does not have a positive solution in* t, *then problem* $(1_{\lambda,\mu})$ *does not have a bounded solution which is bounded from below by a positive constant.*

Proof. (a) If $K \ge 1$, then

$$K \ \ge \ \frac{\lambda K^q + \mu K^p}{A} \ > \ \frac{\mu K^p}{A}$$

which implies that $\mu < AK^{1-p} \le A$. On the other hand, if $0 < K < 1$, then $K > \frac{\lambda K^q}{A}$ and hence $\lambda < A$.

(b) Assuming that u is a bounded solution which is bounded from below by a positive constant on $\mathbb{R}^N$ we have $K > \left(\frac{\lambda}{A}\right)^{\frac{1}{1-q}}$. Then

$$-\Delta u + a(x)u \;=\; \lambda u^q + \mu u^p \;>\; \mu u^p \;>\; \mu \left(\frac{\lambda}{A}\right)^{\frac{p}{1-q}} \quad \text{in} \quad \mathbb{R}^N$$

and by Corollary 2.2 we have

$$u(x) \;\geq\; \frac{\mu}{A}\left(\frac{\lambda}{A}\right)^{\frac{p}{1-q}} \;=\; \frac{\mu \lambda^{\frac{p}{1-q}}}{A^{\frac{1-q+p}{1-q}}} \quad \text{on} \quad \mathbb{R}^N .$$

These inequalities can now be iterated in the following way. Since

$$-\Delta u + a(x)u \;\geq\; \mu u^p \;\geq\; \frac{\mu^{p+1} A^{\frac{p^2}{1-q}}}{A^{\frac{(1-q)p+p^2}{1-q}}} \quad \text{in} \quad \mathbb{R}^N ,$$

we deduce from Corollary 2.2 that

$$u(x) \;\geq\; \frac{\mu^{1+p} \lambda^{\frac{p^2}{1-q}}}{A^{\frac{1-q+(1-q)p+p^2}{1-q}}} \quad \text{on} \quad \mathbb{R}^N .$$

By induction we check that

$$u(x) \;\geq\; \frac{\mu^{1+p+\ldots+p^{n-1}} \lambda^{\frac{p^n}{1-q}}}{A^{\frac{(1-q)(1+p+\ldots+p^{n-1})+p^n}{1-q}}} \;=\; \frac{\mu^{-\frac{1}{p-1}} \left(\mu^{\frac{1}{p-1}} \lambda^{\frac{1}{1-q}}\right)^{p^n}}{A^{-\frac{1}{p-1}} A^{p^n\left(\frac{1}{1-q}+\frac{1}{p-1}\right)}}$$

on $\mathbb{R}^N$ for each $n \geq 1$. This shows that if $\mu^{\frac{1}{p-1}} \lambda^{\frac{1}{1-q}} > A^{\frac{1}{p-1}+\frac{1}{1-q}}$, then $u(x) = \infty$ on $\mathbb{R}^N$, which is impossible. The last inequality justifies the assertion (b).

(c) This is a direct consequence of the inequality $AK \geq \lambda K^q + \mu K^p$. □

The choice of M satisfying $0 < M \leq 1$ in the proof of Theorem 3.1 ensures that the sequences of upper and lower bounds are bounded. If $M > 1$ and $\lambda + \mu \leq a_0$, then using the above method we can show that

$$u_n(x) \;\leq\; \frac{(\lambda+\mu)^{1+q+\ldots+q^{n-1}}}{a_0^{1+q+\ldots+q^{n-1}}} M^{p^n} \quad \text{on} \quad \mathbb{R}^N .$$

Similarly, if $a_0 < \lambda + \mu$ and $0 < M \leq 1$, then

$$u_n(x) \;\leq\; \left(\frac{\lambda+\mu}{a_0}\right)^{1+p+\ldots+p^{n-1}} M^{q^n} \quad \text{on} \quad \mathbb{R}^N .$$

Both sequences are unbounded.

It is worth pointing out that if $a_0 = A$, that is, a is a constant function, by Theorem 3.1 each successive approximation must be a constant function. Therefore in this case this method leads to a constant solution. We can also start successive approximations from $u_0 = f(x)$, where f is a locally Hölder continuous function on $\mathbb{R}^N$ such that $\delta \leq f(x) \leq 1$ on $\mathbb{R}^N$ for some constant $0 < \delta$. Inspection of the proof of Theorem 3.1

shows that we also obtain a solution satisfying the estimate (3.1). However, it is not clear whether different first approximations lead to distinct solutions. It is easy to see that if $f_1(x) \le f_2(x)$ on $\mathbb{R}^N$ and u^1 and u^2 are solutions corresponding to f_1 and f_2, respectively, then $u^1(x) \le u^2(x)$ on $\mathbb{R}^N$.

Theorem 3.1 can be easily extended to the problem

$$(1_{\Lambda,M}) \qquad \begin{cases} -\Delta u + a(x)u \ = \ \sum_{j=1}^k \lambda_j u^{q_j} + \sum_{j=1}^l \mu_j u^{p_j} \ \text{ in } \mathbb{R}^N\,, \\ u(x) \ > \ 0 \quad \text{on} \quad \mathbb{R}^N\,, \end{cases}$$

where $0 < q_1 < q_2 < \ldots < q_k < 1 < p_1 < p_2 < \ldots < p_l$ and λ_j and μ_j are positive parameters.

Proposition 3.6. *Suppose that* $\sum_{j=1}^k \lambda_j + \sum_{j=1}^l \mu_j \le a_0$. *Then problem* $(1_{\Lambda,M})$ *admits a positive solution* u *satisfying*

$$\left(\frac{\lambda_1}{A}\right)^{\frac{1}{1-q_1}} \ \le \ u(x) \ \le \ \left(\frac{\sum_{j=1}^k \lambda_j + \sum_{j=1}^l \mu_j}{a_0}\right)^{\frac{1}{1-q_1}} \qquad on \quad \mathbb{R}^N\,.$$

4. Purely concave nonlinearity

In the case where $\mu = 0$ the successive approximations

$$\begin{aligned} w_0(x) \ &= \ M\,, \quad M \ > \ 0\,, \\ -\Delta w_j + a(x)w_j \ &= \ \lambda w_{j-1}^q \quad \text{in} \quad \mathbb{R}^N\,, \quad j \ = \ 1\,,2\,,\ldots \end{aligned}$$

satisfy the estimates

$$w_n(x) \ \le \ \frac{\lambda^{1+q+\ldots+q^{n-1}} M^{q^n}}{a_0^{1+q+\ldots+q^{n-1}}} \quad \text{on} \quad \mathbb{R}^N$$

and

$$w_n(x) \ \ge \ \frac{\lambda^{1+q+\ldots+q^{n-1}} M^{q^n}}{A^{1+q+\ldots+q^{n-1}}} \quad \text{on} \quad \mathbb{R}^N\,.$$

Therefore we can formulate the following result

Theorem 4.1. *Suppose that* $0 < a_0 = \inf_{x\in\mathbb{R}^N} a(x)$ *and* $\sup_{x\in\mathbb{R}^N} a(x) = A < \infty$. *Then for each* $\lambda > 0$ *problem* $(1_{\lambda,0})$ *has a solution* $w(x)$ *satisfying*

$$\left(\frac{\lambda}{A}\right)^{\frac{1}{1-q}} \ \le \ w(x) \ \le \ \left(\frac{\lambda}{a_0}\right)^{\frac{1}{1-q}}\,.$$

In Proposition 4.2 below we show that a solution of problem $(1_{\lambda,0})$ provides a lower estimate for a solution of problem $\left(1_{\lambda,\mu}\right)$.

Proposition 4.2. *Suppose that* $\lambda + \mu \le a_0$. *Let* u *be a solution of problem* $\left(1_{\lambda,\mu}\right)$ *from Theorem* 3.1 *and* w *a solution of problem* $(1_{\lambda,0})$ *from Theorem* 4.1. *Then* $w(x) \le u(x)$ *on* $\mathbb{R}^N$.

Proof. For a fixed $0 < M \leq 1$ we compare successive approximations for problems $(1_{\lambda,\mu})$ and $(1_{\lambda,0})$. As in the proof of Theorem 3.1 we can establish the inequality

$$u_n(x) \;\geq\; \left(\frac{\mu}{A}\right)^{1+p+\ldots+p^{n-1}} M^{p^n} \quad \text{on} \quad \mathbb{R}^N$$

for each n. This inequality will be used to estimate $u_j - w_j$. Since

$$-\Delta(u_1 - w_1) + a(u_1 - w_1) \;=\; \lambda M^q + \mu M^p - \lambda M^q \;=\; \mu M^p \quad \text{in} \quad \mathbb{R}^N\,,$$

by Corollary 2.2 we have $u_1 - w_1 \geq \frac{\mu M^p}{A}$ on $\mathbb{R}^N$. Similarly,

$$-\Delta(u_2 - w_2) + a(u_2 - w_2) \;=\; \lambda u_1^q + \mu u_1^p - \lambda w_1^q \;\geq\; \mu u_1^p \;\geq\; \frac{\mu^{1+p} M^{p^2}}{A^p} \quad \text{in} \quad \mathbb{R}^N$$

and by Corollary 2.2 we get

$$u_2 - w_2 \;\geq\; \left(\frac{\mu}{A}\right)^{1+p} M^{p^2} \quad \text{on} \quad \mathbb{R}^N\,.$$

It easy to see that

$$u_n - w_n \;\geq\; \left(\frac{\mu}{A}\right)^{1+p+\ldots+p^{n-1}} M^{p^n} \quad \text{on} \quad \mathbb{R}^N$$

for each n. Letting $n \to \infty$ the result readily follows. □

For a fixed $\lambda > 0$ and $\mu > 0$ satisfying $\lambda + \mu \leq a_0$ we denote by $u_{\lambda,\mu}$ a solution of $(1_{\lambda,\mu})$ from Theorem 3.1. As an immediate consequence of the estimate (3.1) and the interior Schauder estimates we get

Proposition 4.3. *Suppose that $\lambda + \mu \leq a_0$. Then for a fixed $\lambda > 0$*

$$\lim_{\mu\to 0} u_{\lambda,\mu}(x) \;=\; w_\lambda(x) \quad \text{on} \quad \mathbb{R}^N\,,$$

where w_λ is a solution of problem $(1_{\lambda,0})$ satisfying

$$\left(\frac{\lambda}{A}\right)^{\frac{1}{1-q}} \;\leq\; w_\lambda(x) \;\leq\; \left(\frac{\lambda}{a_0}\right)^{\frac{1}{1-q}} \quad \text{on} \quad \mathbb{R}^N\,.$$

Proposition 4.4. *Suppose that $\lambda + \mu \leq a_0$. Then for each fixed $0 < \mu < a_0$ we have $\lim_{\lambda\to 0} u_{\lambda,\mu}(x) = 0$ uniformly on $\mathbb{R}^N$.*

Proof. Let $S_{\lambda,\mu} = \sup_{x\in\mathbb{R}^N} u_{\lambda,\mu}(x)$. Since $\lambda + \mu \leq a_0$, it follows from the estimate (3.1) that $S_{\lambda,\mu} \leq 1$. By virtue of Corollary 2.2 we have

$$u_{\lambda,\mu}(x) \;\leq\; \frac{\lambda S_{\lambda,\mu}^q + \mu S_{\lambda,\mu}^p}{a_0} \quad \text{on} \quad \mathbb{R}^N\,.$$

Hence

$$a_0 S_{\lambda,\mu} \;\leq\; \lambda S_{\lambda,\mu}^q + \mu S_{\lambda,\mu}^p\,.$$

Letting $\lambda \to 0$, we get

$$a_0 \limsup_{\lambda\to 0} S_{\lambda,\mu} \;\le\; \mu \left(\limsup_{\lambda\to 0} S_{\lambda,\mu}\right)^p .$$

If $\limsup_{\lambda\to 0} S_{\lambda,\mu} > 0$, then we must have $\limsup_{\lambda\to 0} S_{\lambda,\mu} > 1$, which is impossible. □

It should be emphasized that problem $\left(1_{\lambda,\mu}\right)$ may have a solution $u_{\lambda,\mu}$ for which $\lim_{\lambda\to 0} u_{\lambda,\mu}(x) \not\equiv 0$. Here we give an example of problem $\left(1_{\lambda,\mu}\right)$ having a solution $u_{\lambda,\mu}$ with $\lim_{\lambda\to 0} u_{\lambda,\mu}(x) = u_\mu(x)$, where u_μ is a solution of problem $\left(1_{0,\mu}\right)$. However, we were unable to establish this fact in a general case. As an example we consider the problem

$$\begin{cases} -\Delta u + u \;=\; \lambda u^{\frac{1}{2}} + \dfrac{1}{2}\, u^{\frac{3}{2}} \quad \text{in} \quad \mathbb{R}^N , \\ u(x) \;>\; 0 \quad \text{on} \quad \mathbb{R}^N , \end{cases}$$

with $0 < \lambda < \frac{1}{2}$. Successive approximations are given by $u_0 \equiv 1$ and

$$-\Delta u_j + u_j \;=\; \lambda u_{j-1}^{\frac{1}{2}} + \frac{1}{2}\, u_{j-1}^{\frac{3}{2}} \quad \text{in} \quad \mathbb{R}^N .$$

By the uniqueness of bounded solutions for the corresponding equations on $\mathbb{R}^N$ we get the following relation

$$u_j \;=\; \lambda u_{j-1}^{\frac{1}{2}} + \frac{1}{2}\, u_{j-1}^{\frac{3}{2}}$$

and all successive approximations are constant functions. It is easy to check that $\{u_j\}$ is a decreasing sequence for $\lambda > 0$ small enough and $\lim_{j\to\infty} u_j = \left(1 - \sqrt{1-2\lambda}\right)^2 = u_{\lambda,\frac{1}{2}}$. The constant function $\left(1 - \sqrt{1-2\lambda}\right)^2$ is a solution of our problem satisfying estimate (3.1) of Theorem 3.1. We now notice that this problem has a second solution $u_{\lambda,\frac{1}{2}}(x) = \left(1 + \sqrt{1-2\lambda}\right)^2$, which has a property $\lim_{\lambda\to 0} u_{\lambda,\frac{1}{2}} = 4$ and a constant function 4 is a solution of the limit equation

$$-\Delta u + u \;=\; \frac{1}{2}\, u^{\frac{3}{2}} \quad \text{in} \quad \mathbb{R}^N .$$

We observe here that a solution $\left(1 + \sqrt{1-2\lambda}\right)^2$ does not satisfy the corresponding estimate (3.1), that is, it is not determined by the method of successive approximations from Section 3.

Theorem 4.1 continues to hold if λ is replaced by a function $b(x)$ which is locally Hölder continuous and satisfies

$$0 \;<\; b_0 \;\le\; b(x) \;\le\; B \quad \text{on} \quad \mathbb{R}^N$$

for some constants b_0 and B. In this case we can show the existence of a solution $u(x)$ satisfying

$$\left(\frac{b_0}{A}\right)^{\frac{1}{1-q}} \;\le\; u(x) \;\le\; \left(\frac{B}{a_0}\right)^{\frac{1}{1-q}} .$$

The maximum principle given by Proposition 2.1 is also valid for the operator $-\Delta u + \sum_{i=1}^N b_i(x)u_{x_i} + a(x)u$, where b_i are bounded functions on $\mathbb{R}^N$. Therefore Theorems 3.1 and 4.1 can be extended to this more general operator. Obviously in order to obtain C^2–solutions we need to assume that b_i are bounded and locally Hölder continuous. These observations will be used to show the existence of a solution of the problem

$$(1_b)\qquad \begin{cases} -\Delta u + a(x)u \ = \ b(x)u^q \quad \text{in} \quad \mathbb{R}^N\,, \\ u(x) \ > \ 0 \quad \text{on} \quad \mathbb{R}^N\,, \end{cases}$$

with $b(x)$ satisfying suitable growth or decay conditions for large $|x|$.

Let $H(x,\delta)$ be a function defined in the Introduction, which for a given $a_1 < a_0$ satisfies (2.1) for $0 \le \delta \le \delta_0$. Similarly, given $0 < a_1 < a_0$ there exists $\delta_1 > 0$ such that

$$(4.1)\qquad a(x) + \frac{\Delta H(x,\delta)}{H(x,\delta)} - 2\frac{|\nabla H(x,\delta)|^2}{H(x,\delta)^2} \ \ge \ a_1 \quad \text{in} \quad \mathbb{R}^N$$

for each $0 \le \delta \le \delta_1$.

Theorem 4.5. *Suppose that $b(x)$ is locally Hölder continuous on $\mathbb{R}^N$ and such that*

$$(4.2)\qquad b_0 e^{-(1-q)\delta_1 \sum_{i=1}^N |x_i|} \ \le \ b(x) \ \le \ B e^{-(1-q)\delta_1 \sum_{i=1}^N |x_i|} \quad \textit{on} \quad \mathbb{R}^N$$

for some constants $b_0 > 0$ and $B > 0$. Then problem (1_b) admits a solution $u(x)$ satisfying

$$c_1 H(x,\delta_1)^{-1} \ \le \ u(x) \ \le \ C_1 H(x,\delta_1)^{-1} \quad \textit{on} \quad \mathbb{R}^N$$

for some constants $c_1 > 0$ and $C_1 > 0$.

Proof. We introduce a new unknown function w by

$$u(x) \ = \ \frac{w(x)}{H(x,\delta_1)}\,.$$

Then w satisfies the equation

$$-\Delta w + \frac{2}{H}\nabla H \nabla w + \left(a + \frac{\Delta H}{H} - 2\,\frac{|\nabla H|^2}{H^2}\right) w \ = \ b(x) H^{1-q} w^q \quad \text{in} \quad \mathbb{R}^N\,.$$

It follows from (4.2) that

$$\frac{b_0}{2^{N(1-q)}} \ \le \ b(x) H(x,\delta_1)^{1-q} \ \le \ B \quad \text{on} \quad \mathbb{R}^N\,.$$

The assertion follows from Theorem 4.1. □

In a similar manner we can establish the following existence result

Theorem 4.6. *Suppose that $b(x)$ is locally Hölder continuous on $\mathbb{R}^N$ and such that*

$$b_0 e^{(1-q)\delta_0 \sum_{i=1}^N |x_i|} \ \le \ b(x) \ \le \ B e^{(1-q)\delta_0 \sum_{i=1}^N |x_i|} \quad \textit{on} \quad \mathbb{R}^N$$

for some constants $b_0 > 0$ *and* $B > 0$. *Then problem* (1_b) *admits a solution satisfying*

$$c_1 H(x, \delta_0) \ \leq \ u(x) \ \leq \ C_1 H(x, \delta_0) \quad on \quad \mathbb{R}^N$$

for some constant $c_1 > 0$ *and* $C_1 > 0$.

5. Method of sub and supersolutions

In this section we construct a solution of the problem $(1_{\lambda,\mu})$ by a monotone method based on the use of sub and supersolutions. Applying this method we obtain the existence of a solution in the case where $\lambda + \mu > a_0$ may not be satisfied.

We recall that a positive C^2–function U on $\mathbb{R}^N$ is a supersolution for $(1_{\lambda,\mu})$ if

$$-\Delta U + a(x)U \ \geq \ \lambda U^q + \mu U^p \quad \text{in} \quad \mathbb{R}^N .$$

A positive subsolution is obtained by reversing this inquality. In the next section we shall use the definition of sub– and supersolution in a weak sense. Throughout this section we assume that $a_0 = \inf_{x\in\mathbb{R}^N} a(x) > 0$ and $A = \sup_{x\in\mathbb{R}^N} a(x) > 0$.

Theorem 5.1. (i) *For each* $\mu_0 > 0$ *there exist* $\lambda_0 > 0$ *and* $M > 0$ *such that for each* $0 < \lambda \leq \lambda_0$ *and* $0 < \mu \leq \mu_0$ *problem* $(1_{\lambda,\mu})$ *admits a minimal solution* u *and a maximal solution* v *satisfying*

$$C_1 e^{-\delta_1 \sum_{i=1}^N |x_i|} \ \leq \ u(x) \ \leq \ v(x) \ \leq \ M \quad on \quad \mathbb{R}^N$$

for some constants $\delta_1 > 0$ *and* $C_1 > 0$.

(ii) *For each* $\lambda_0 > 0$ *there exist* $\mu_0 > 0$ *and* $M > 0$ *such that for each* $0 < \lambda \leq \lambda_0$ *and* $0 < \mu \leq \mu_0$ *the assertion from part* (i) *remains valid.*

Proof. (i) Given $\mu_0 > 0$ we can find $M = M(\mu_0, a_0) > 0$ such that $Ma(x) \geq Ma_0 > \mu_0 M^p$ on $\mathbb{R}^N$. Then we choose $\lambda_0 = \lambda_0(M, \mu_0)$ such that

$$Ma(x) \ \geq \ Ma_0 \ \geq \ \mu_0 M^p + \lambda_0 M^q \ \geq \ \mu M^p + \lambda M^q \quad \text{on} \quad \mathbb{R}^N$$

for all $0 < \lambda \leq \lambda_0$ and $0 < \mu \leq \mu_0$. This means that $U(x) \equiv M$ is a supersolution. We now proceed to the construction of a subsolution V such that $V(x) \leq U(x)$ on $\mathbb{R}^N$. We set $V(x) = \frac{1}{H(x,\delta)+K}$, and $H(x, \delta)$ is a function defined in Section 2, where $\delta > 0$ and $K > 0$ are constants to be determined. Let $A_1 > A$. We choose $\delta_1 > 0$ so that

$$-\Delta V + aV \ = \ V\left(a - 2\delta^2 H^2 V^2 \sum_{i=1}^N \tanh^2 \delta x_i + \delta^2 HV\right) \ \leq \ A_1 V \quad \text{on} \quad \mathbb{R}^N$$

for all $0 < \delta \leq \delta_1$ and $K > 0$. We now select $K > 0$ large enough so that $A_1 V(x) \leq \lambda V(x)^q$ on $\mathbb{R}^N$, which means that V is a subsolution for $(1_{\lambda,\mu})$. Taking $K > 0$ larger, if necessary, we may assume that $V(x) \leq M$ on $\mathbb{R}^N$ for all $0 < \delta \leq \delta_1$. We now follow a standard method [DL] to construct minimal and maximal solutions. Let $u_0 = V$ and define u_j, $j \geq 1$, to be a unique bounded positive solution of the equation

$$-\Delta u_j + a(x)u_j \ = \ \lambda u_{j-1}^q + \mu u_{j-1}^p \quad \text{in} \quad \mathbb{R}^N .$$

Since

$$-\Delta\big(u_1 - V\big) + a\big(u_1 - V\big) \;\geq\; 0 \quad \text{in} \quad \mathbb{R}^N\,,$$

we derive from Proposition 2.1 that $u_1(x) \geq V(x)$ on $\mathbb{R}^N$. In a similar manner we check that $u_1(x) \leq U(x)$ on $\mathbb{R}^N$ and that

$$V(x) \;\leq\; u_{j-1}(x) \;\leq\; u_j(x) \;\leq\; U(x) \quad \text{on} \quad \mathbb{R}^N$$

for each $j \geq 1$. Using the interior Schauder estimates we show that

$$\lim_{j\to\infty} u_j(x) \;=\; u(x)$$

on $\mathbb{R}^N$ and u is a solution of $\big(1_{\lambda,\mu}\big)$. It remains to prove that u is minimal among all solutions in the order interval $[V, U]$. This follows from the fact that if w is any such solution then, repeating the above argument, we get $u_j(x) \leq w(x)$ on $\mathbb{R}^N$ for each j. Letting $j \to \infty$, the claim readily follows. To construct a maximal solution in the order interval $[V, U]$ we set $v_0(x) = U(x)$ on $\mathbb{R}^N$ and define v_j for $j \geq 1$ to be a unique positive and bounded solution of the equation

$$-\Delta v_j + a v_j \;=\; \lambda v_{j-1}^q + \mu v_{j-1}^p \quad \text{in} \quad \mathbb{R}^N\,.$$

Applying Proposition 2.1 we demonstrate that

$$V(x) \;\leq\; u_j(x) \;\leq\; v_j(x) \;\leq\; v_{j-1}(x) \;\leq\; U(x) \quad \text{on} \quad \mathbb{R}^N$$

for each j and that $v(x) = \lim_{j\to\infty} v_j(x)$ is a maximal solution of $\big(1_{\lambda,\mu}\big)$ in the order interval $[V, U]$. Obviously the maximal and minimal solutions may well coincide. We point out here that since we can take $\mu_0 > 0$ large the condition $\lambda + \mu \leq a_0$, used in the proof of Theorem 3.1, may not be satisfied for all $0 < \lambda \leq \lambda_0$ and $0 < \mu \leq \mu_0$.

(ii) We construct a supersolution by starting with $\lambda_0 > 0$ and selecting $M = M(a_0, \lambda_0) > 0$ so that

$$a(x)M^{1-q} \;\geq\; a_0 M^{1-q} \;>\; \lambda_0 \quad \text{on} \quad \mathbb{R}^N\,.$$

In the next step we choose $\mu_0 = \mu_0(M, \lambda_0)$ such that $a_0 M^{1-q} \geq \lambda_0 + \mu_0 M^{p-q}$. This implies that $U(x) \equiv M$ is a supersolution for $\big(1_{\lambda,\mu}\big)$ for all $0 < \lambda \leq \lambda_0$ and $0 < \mu \leq \mu_0$. The construction of a subsolution and the remaining part of the proof are the same as in the case (i). □

Let us denote for each $0 < \lambda \leq \lambda_0$ and $0 < \mu \leq \mu_0$ by $u_{\lambda,\mu}$ and $v_{\lambda,\mu}$ minimal and maximal solutions, respectively. The next proposition contains asymptotic properties and estimates of minimal and maximal solutions which are similar to those obtained in Section 4.

Proposition 5.2. (i) *Let $\mu_0 \geq a_0$ and λ_0 be chosen as in the part* (i) *of Theorem* 5.1. *Then for each fixed $0 < \mu \leq \mu_0$*

$$\lim_{\lambda\to 0} u_{\lambda,\mu}(x) \;=\; \lim_{\lambda\to 0} v_{\lambda,\mu}(x) \;=\; 0$$

uniformly on $\mathbb{R}^N$.

(ii) *Let* $\lambda_0 > 0$ *and* $\mu_0 > 0$ *be chosen as in the part* (ii) *of Theorem* 5.1. *Then for each* $0 < \lambda \le \lambda_0$ *and* $0 < \mu \le \mu_0$ *the maximal solution* $v_{\lambda,\mu}$ *satisfies*

$$v_{\lambda,\mu}(x) \;\ge\; \left(\frac{\lambda}{A}\right)^{\frac{1}{1-q}} \quad on \quad \mathbb{R}^N .$$

If $\lambda_0 > A$ *and* μ_0 *be chosen as in the part* (ii) *of Theorem* 5.1, *then for each* $0 < \lambda \le \lambda_0$ *and* $0 < \mu \le \mu_0$ *satisfying* $\lambda + \mu > A$ *the maximal solution* $v_{\lambda,\mu}$ *satisfies the estimate*

$$v_{\lambda,\mu}(x) \;\ge\; \left(\frac{\lambda+\mu}{A}\right)^{\frac{1}{1-q}} \quad on \quad \mathbb{R}^N .$$

Moreover if $\lambda > A$, *then*

$$\lim_{\mu\to 0} v_{\lambda,\mu}(x) \;=\; v_\lambda(x) \quad on \quad \mathbb{R}^N ,$$

where v_λ *is a solution of problem* $(1_{\lambda,0})$. *For each fixed* $0 < \mu < \mu_0$ *we also have*

$$\lim_{\lambda\to 0} u_{\lambda,\mu}(x) \;=\; \lim_{\lambda\to 0} v_{\lambda,\mu}(x) \;=\; 0$$

uniformly on $\mathbb{R}^N$.

Proof. (i) Since $\mu_0 \ge a_0$, then a supersolution $U(x) \equiv M$ from the part (i) of Theorem 5.1 satisfies $Ma(x) \ge Ma_0 > \mu_0 M^p$, that is, $M < \left(\frac{a_0}{\mu_0}\right)^{\frac{1}{p-1}} \le 1$. Hence $u_{\lambda,\mu}(x) \le v_{\lambda,\mu}(x) < 1$ on $\mathbb{R}^N$. Repeating the argument from the proof of Proposition 4.4, we obtain the assertion.

(ii) The first estimate can be established as in the proof of Theorem 3.1. To establish the second estimate, let $\{v_j\}$ be a decreasing sequence converging to $v_{\lambda,\mu}$ from the proof of part (ii) of Theorem 5.1. It also follows from the proof of Theorem 5.1 that

$$M \;>\; \left(\frac{\lambda_0}{a_0}\right)^{\frac{1}{1-q}} \;\ge\; \left(\frac{\lambda_0}{A}\right)^{\frac{1}{1-q}} \;>\; 1 .$$

Therefore

$$-\Delta v_1 + a v_1 \;=\; \lambda M^q + \mu M^p \;>\; (\lambda+\mu) M^q \quad \text{in} \quad \mathbb{R}^N$$

and from Corollary 2.2 we have

$$v_1(x) \;\ge\; \frac{(\lambda+\mu)M^q}{A} \quad \text{on} \quad \mathbb{R}^N .$$

If $\lambda + \mu > A$, then by induction we show that

$$v_n(x) \;\ge\; \left(\frac{\lambda+\mu}{A}\right)^{1+q+\ldots+q^{n-1}} M^{q^n} \quad \text{on} \quad \mathbb{R}^N .$$

Letting $n \to \infty$ the estimate readily follows. According to the choice of λ_0 and μ_0 we have $a(x) \geq a_0 M > \mu M^p$ on $\mathbb{R}^N$. Hence $M < \left(\frac{a_0}{\mu}\right)^{\frac{1}{p-1}}$. If $S_{\lambda,\mu} = \sup_{x\in\mathbb{R}^N} v_{\lambda,\mu}(x)$, then

$$a_0 S_{\lambda,\mu} \;\leq\; \lambda S^q_{\lambda,\mu} + \mu S^p_{\lambda,\mu}\,.$$

Letting $\lambda \to 0$ we get

$$a_0 \limsup_{\lambda\to 0} S_{\lambda,\mu} \;\leq\; \mu\Bigl(\limsup_{\lambda\to 0} S_{\lambda,\mu}\Bigr)^p .$$

If $\limsup_{\lambda\to 0} S_{\lambda,\mu} > 0$, then

$$\left(\frac{a_0}{\mu}\right)^{\frac{1}{p-1}} \;\leq\; \limsup_{\lambda\to 0} S_{\lambda,\mu} \;\leq\; M \;<\; \left(\frac{a_0}{\mu}\right)^{\frac{1}{p-1}},$$

which is impossible. □

6. Resonance case

In this section we relax the assumptions on the coefficient $a(x)$. We assume that a is bounded and locally Hölder continuous on $\mathbb{R}^N$ and that

(a) $\;a(x) \geq \delta > 0$ for all $x \in \mathbb{R}^N - B(0,R)$

for some constants δ and $R > 0$.

The first eigenvalue of the operator $-\Delta u + a(x)u$ in $\mathbb{R}^N$ is defined by (see [BD])

$$\lambda_1(a) \;=\; \inf\left\{\int_{\mathbb{R}^N}\bigl(|\nabla u|^2 + a(x)u^2\bigr)\,dx\,;\; u \in C_0^\infty\bigl(\mathbb{R}^N\bigr),\, \int_{\mathbb{R}^N} u^2\,dx = 1\right\}.$$

Let $\{R_j\}$ be an increasing sequence of numbers such that $R_j \to \infty$, as $j \to \infty$ and $R_j > R$ for each j. For a given bounded domain $\Omega \subset \mathbb{R}^N$ we denote by $H_0^1(\Omega)$ a Sobolev space on Ω equipped with the norm

$$\|u\|^2_{H_0^1(\Omega)} \;=\; \int_\Omega |\nabla u|^2\,dx\,.$$

By $H^1\bigl(\mathbb{R}^N\bigr)$ we denote a Sobolev space on $\mathbb{R}^N$ with norm

$$\|u\|^2_{H^1(\mathbb{R}^N)} \;=\; \int_{\mathbb{R}^N}\bigl(|\nabla u|^2 + u^2\bigr)\,dx\,.$$

Let λ_{R_j} be the first eigenvalue of the operator $-\Delta u + a(x)u$ in $B\bigl(0,R_j\bigr)$ with the Dirichlet boundary conditions, that is,

$$\lambda_{R_j} \;=\; \inf\left\{\int_{B(0,R_j)}\bigl(|\nabla u|^2 + a(x)u^2\bigr)\,dx\,;\; u \in H_0^1\bigl(B(0,R)\bigr),\, \int_{B(0,R_j)} u^2\,dx = 1\right\}.$$

It follows from these variational definitions of $\lambda_1(a)$ and λ_{R_j} that

(6.1) $$\lambda_1(a) \;\leq\; \lambda_{R_{j+1}} \;\leq\; \lambda_{R_j} \text{ for all } \; j\,.$$

In this section we establish the existence of a solution of problem $\big(1_{\lambda,\mu}\big)$. We shall use the method of sub– and supersolutions,

We recall that a function $u \in H^1_{\text{loc}}\big(\mathbb{R}^N\big)$ is a supersolution (in a weak sense) for problem $\big(1_{\lambda,\mu}\big)$ if for every $\phi \in C_0^\infty\big(\mathbb{R}^N\big)$, $\phi \ge 0$ on $\mathbb{R}^N$, we have

$$\int_{\mathbb{R}^N} \big(\nabla u \nabla \phi + a(x) u \phi\big)\, dx \;\ge\; \int_{\mathbb{R}^N} \big(\lambda u^q + \mu u^p\big)\phi\, dx\,.$$

The definition of a subsolution is obtained by reversing the above inequality.

Theorem 6.1. *Suppose that (a) holds and that $\lambda_1(a) > 0$. Then*

(i) *for each $\lambda_0 > 0$ there exists $\mu_0 > 0$ such that for $0 < \lambda \le \lambda_0$ and $0 < \mu \le \mu_0$ problem $\big(1_{\lambda,\mu}\big)$ has a solution;*

(ii) *for each $\mu_0 > 0$ there exists $\lambda_0 > 0$ such that for $0 < \lambda \le \lambda_0$ and $0 < \mu \le \mu_0$ problem $\big(1_{\lambda,\mu}\big)$ has a solution.*

Proof. We start by constructing a supersolution. For each $j \ge 1$ we consider the Dirichlet problem

$$(7_j) \qquad \begin{cases} -\Delta u_j + a(x) u_j \;=\; 1 \quad \text{in} \quad B\big(0, R_j\big)\,, \\ u_j(x) \;=\; 0 \quad \text{on} \quad \partial B\big(0, R_j\big)\,. \end{cases}$$

For each j problem $\big(7_j\big)$ has a solution u_j in $C^2\big(B\big(0,R_j\big)\big)$ which is Hölder continuous up to the boundary of $B\big(0,R_j\big)$. Let ϕ_j be the eigenfunction corresponding to λ_{R_j}. We assume that ϕ_j are normalized so that $0 \le \phi_j(x) \le 1$ on $\overline{B\big(0,R_j\big)}$. It follows from the Harnack inequality that there exists a constant $k > 0$ such that $\phi_j(x) \ge k$ for all $x \in \overline{B(0,R)}$ and for each j. Next we choose a constant $C > 0$ such that

$$(6.2) \qquad 1 - C\lambda_{R_j}\phi_j(x) - \frac{a(x)}{\delta} \;\le\; 0$$

for all $x \in B(0,R)$ and each j. We obviously have

$$(6.3) \qquad 1 - C\lambda_{R_j}\phi_j(x) - \frac{a(x)}{\delta} \;\le\; 1 - \frac{a(x)}{\delta} \;\le\; 0$$

for all $x \in B\big(0,R_j\big) - B(0,R)$. It follows from (6.2) and (6.3) that

$$-\Delta\left(u_j - C\phi_j - \frac{1}{\delta}\right) + a\left(u_j - C\phi_j - \frac{1}{\delta}\right) \;=\; 1 - C\lambda_{R_j}\phi_j - \frac{a}{\delta} \;\le\; 0$$

in $B\big(0,R_j\big)$. Since (5.1) holds and $\lambda_1(a) > 0$ we can apply the maximum principle to obtain

$$u_j(x) \;\le\; C\phi_j(x) + \frac{1}{\delta} \;\le\; C + \frac{1}{\delta}$$

for $x \in \overline{B\big(0,R_j\big)}$. We now extend each u_j by 0 outside $B\big(0,R_j\big)$ and this modified sequence is denoted again by $\big\{u_j\big\}$. Since the sequence $\{u_j\}$ is uniformly bounded we can apply the interior Schauder estimates to obtain a subsequence, denoted again by $\{u_j\}$, such that

$$u_j \longrightarrow u\,, \quad Du_j \longrightarrow Du \quad \text{and} \quad D^2u_j \longrightarrow D^2u$$

as $j \to \infty$ uniformly on each bounded subset of $\mathbb{R}^N$. By the Harnack inequality we can assume that $u > 0$ on $\mathbb{R}^N$. As in in the proof of Theorem 5.1 given $\mu_0 > 0$ we can find $\lambda_0 > 0$ and M such that for each $0 < \lambda \le \lambda_0$ and $0 < \mu \le \mu_0$ we have

$$M \;\ge\; \lambda M^q \, \|u\|_\infty^q + \mu M^p \, \|u\|_\infty^p \,.$$

Consequently, letting $U = Mu$, we see that

$$M \;=\; -\Delta(uM) + a(Mu) \;\ge\; \lambda M^q u^q + M^p u^p \quad \text{in} \quad \mathbb{R}^N\,,$$

which means that U is a supersolution for $(1_{\lambda,1})$. We now proceed to the construction of a subsolution. Here we follow the method from the paper [BD]. We fix $\lambda \in (0, \lambda_0]$ and let w be a solution of the problem

$$\begin{cases} -\Delta u + a(x)u \;=\; \lambda u^q \quad \text{in} \quad B(0,1)\,, \\ u(x) \;=\; 0 \quad \text{on} \quad \partial B(0,1)\,. \end{cases}$$

According to [BO] this problem has a solution $w \in C^2(B(0,1) \cap C^{1,\beta}\big(\overline{B(0,1)}\big)$ for some $\beta \in (0,1)$ and $\frac{du}{d\nu} < 0$ on $\partial B(0,1)$, where ν is an outward normal on $\partial B(0,1)$. We now define

$$v(x) \;=\; \begin{cases} w(x) & \text{for} \quad x \in B(0,1)\,, \\ 0 & \text{for} \quad x \in \mathbb{R}^N - B(0,1)\,. \end{cases}$$

We now check that v is a subsolution. Indeed, let $\phi \in C_0^\infty\big(\mathbb{R}^N\big)$, $\phi \ge 0$ on $\mathbb{R}^N$. Then

$$\begin{aligned} \int_{\mathbb{R}^N} \big(\nabla v \nabla \phi + a v \phi - \lambda v^q \phi\big)\, dx \;&=\; \int_{B(0,1)} \big(\nabla w \nabla \phi + a w \phi - \lambda w^q \phi\big)\, dx \\ &=\; \int_{B(0,1)} \big(-\Delta w + a w - \lambda w^q\big) \phi \, dx + \int_{\partial B(0,1)} \frac{\partial w}{\partial \nu}\, \phi \, dS_x \\ &=\; \int_{\partial B(0,1)} \frac{\partial w}{\partial \nu}\, \phi \, dS_x \\ &\le\; 0\,. \end{aligned}$$

We now choose $\gamma \in (0,1)$ so that $\gamma v \le U$ on $\mathbb{R}^N$. It is easy to check that γv is also a subsolution. As in [BD] we show that problem $\big(1_{\lambda,\mu}\big)$, for $\lambda \in (0, \lambda_0]$, has a solution $u(x)$ satisfying

$$\gamma v(x) \;\le\; u(x) \;\le\; U(x) \quad \text{on} \quad \mathbb{R}^N\,. \qquad \square$$

In the final part of this section we consider the case $\lambda_1(a) < 0$. We assume that the coefficient $a(x)$ is bounded and locally Hölder continuous on $\mathbb{R}^N$ and

(e) $a(x) \ge 0$ on $\mathbb{R}^N - B(0,R)$ for some $R > 0$.

If $\lambda_1(a) < 0$, then without loss of generality we may assume that $\lambda_{R_j} < 0$ for all j, where $\{R_j\}$ is a sequence from the first part of this section. Using the method of sub- and supersolutions we can establish the existence of a solution to the problem

$$(1_{\lambda,\mu}) \qquad \begin{cases} -\Delta u + a(x)u \ = \ \gamma u + \lambda u^q + \mu u^p \quad \text{in} \quad \mathbb{R}^N \,, \\ u(x) \ > \ 0 \quad \text{on} \quad \mathbb{R}^N \,, \end{cases}$$

where $\gamma < \lambda_1(a)$.

A supersolution can be obtained by considering the sequence of solutions $\{u_j\}$ of problems

$$\begin{cases} -\Delta u + a(x)u - \gamma u \ = \ 1 \quad \text{in} \quad B(0, R_j) \,, \\ u(x) \ = \ 0 \quad \text{on} \quad \partial B(0, R_j) \,. \end{cases}$$

To obtain a uniform bound for $\{u_j\}$ we choose constants $C > 0$ and $K > 0$ such that

$$1 - aK - \lambda_{R_j} C\phi_j + \gamma C\phi_j + \gamma K \ \le \ 1 - aK + \gamma K \ \le \ 1 + \gamma K \ \le \ 0$$

on $B(0, R_j) - B(0, R)$ and

$$1 - aK - \lambda_{R_j} C\phi_j + \gamma C\phi_j + \gamma K \ \le \ 1 - aK + (\gamma - \lambda_{R_j})C\phi_j \ \le \ 0$$

on $B(0, R)$. This is possible as we may assume that $\phi_j(x) \ge k$ on $B(0, R)$ for some constant $k > 0$ and for each j. It follows from the last two estimates that

$$-\Delta(u_j - C\phi_j - K) + a(u_j - C\phi_j - K) - \gamma(u_j - C\phi_j - K) \ \le \ 0$$

in $B(0, R_j)$. From this we deduce by the maximum principle that

$$u_j(x) \ \le \ C\phi_j(x) + K \quad \text{on} \quad B(0, R_j) \,.$$

As in the proof of Theorem 6.1 we show that a subsequence of $\{u_j\}$ converges to a function u and a multiplication of u by a suitable positive constants gives a supersolution for $(1_{\lambda,\mu,\gamma})$.

To construct a subsolution let, u be a solution to the Dirichlet problem

$$\begin{cases} -\Delta u + a(x)u - \gamma u \ = \ \lambda u^q \quad \text{in} \quad B(0, 1) \,, \\ u(x) \ = \ 0 \quad \text{on} \quad \partial B(0, 1) \,. \end{cases}$$

As in the proof of Theorem 6.1 we extend u by 0 outside $B(0, 1)$ and multiply the extension by a suitable constant $\alpha \in (0, 1)$. The resulting function is a subsolution on $\mathbb{R}^N$.

We are now in position to formulate the following existence result.

Theorem 6.2. *Suppose that* (*e*) *holds and that* $\lambda_1(a) < 0$ *and* $\gamma < \lambda_1(a)$. *Then the assertion of Theorem* 6.1 *holds for problem* $(1_{\lambda,\mu,\gamma})$.

7. Concave nonlinearity at resonance

This section is devoted to the problem

$$(1_{b^-})\qquad \begin{cases} -\Delta u + a(x)u \;=\; -b(x)u^q \quad \text{in} \quad \mathbb{R}^N\,, \\ u(x) \;>\; 0 \quad \text{on} \quad \mathbb{R}^N\,, \end{cases}$$

where $0 < q < 1$, $N \geq 3$ and $b(x)$ is continuous function on $\mathbb{R}^N$ satisfying $0 < b_0 \leq b(x) \leq B$ on $\mathbb{R}^N$ for some constants b_0 and B. Throughout this section it is assumed that the coefficient $a(x)$ is continuous and bounded on $\mathbb{R}^N$ and moreover $a^- \in L^{\frac{N}{2}}(\mathbb{R}^N)$.

One can also consider the problem

$$(1_{b^+})\qquad \begin{cases} -\Delta u + a(x)u \;=\; b(x)u^q \quad \text{in} \quad \mathbb{R}^N\,, \\ u(x) \;>\; 0 \quad \text{on} \quad \mathbb{R}^N\,. \end{cases}$$

It was observed in [BD] that for the solvability of problem (1_{b^-}) (respectively (1_{b^+})) the assumption $\lambda_1(a) < 0$ (respectively $\lambda_1(a) > 0$) is needed. The authors of the paper [BD] established the existence of solutions of problems (1_{b^-}) and (1_{b^+}) in the case where $b \equiv 1$ on $\mathbb{R}^N$. As in [BD] the existence of a solution of problem (1_{b^+}) can be obtained by the method of sub and supersolutions and will be given at the end of this section. First, in this section we construct a solution of problem (1_{b^-}) using a variational method based on a constrained minimization. To establish the relative compactness of a minimizing sequence the authors of [BD] used the concentration–compactness principle [LI]. In order to apply this method in our case some assumptions on a behaviour of $b(x)$ for large $|x|$ are needed. In this paper we avoid this by using the concentration–compactness principle at infinity [CH2] which can also be extended to the case considered in [BD].

To use a variational technique we introduce a Sobolev space E defined as the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|\phi\| \;=\; \|\phi\|_{q+1} + \|\nabla\phi\|_2\,.$$

Here $\|\cdot\|_p$ denotes the norm of the Lebesgue space $L^p(\mathbb{R}^N)$, that is, $\|u\|_p^p = \int_{\mathbb{R}^N} |u|^p\,dx$, $1 \leq p < \infty$. As in [BD] we observe that

$$\|u\|_2^2 \;\leq\; C\,\|\nabla u\|_2^{2^*} + \|u\|_{q+1}^{q+1} \tag{7.1}$$

for all $u \in E$, where $2^* = \frac{2N}{N-2}$. This inequality shows that E is continuously embedded in $H^1(\mathbb{R}^N)$.

Let

$$M \;=\; \left\{ u \in E\,;\; \int_{\mathbb{R}^N} \left(|\nabla u|^2 + au^2\right) dx = -1 \right\}$$

and set

$$m \;=\; \inf\left\{ \int_{\mathbb{R}^N} b(x)\,|u(x)|^{q+1}\,dx\,;\; u \in M \right\}.$$

If $\lambda_1(a) < 0$, then $M \neq \emptyset$.

Theorem 7.1. *Suppose that $\lambda_1(a) < 0$. Then problem $\left(1_{b^-}\right)$ admits a solution.*

Proof. A solution of problem $\left(1_{b^-}\right)$ will be obtained as a minimizer for m. Let $\{u_n\} \subset E$ be a minimizing sequence for m. We commence by showing that $\{\|\nabla u_n\|_2\}$ is bounded. In the contrary case we assume that $\|\nabla u_n\|_2 \to \infty$ and set $v_n = \frac{u_n}{\|\nabla u_n\|_2}$. Then $\int_{\mathbb{R}^N} b(x)\,|v_n|^{q+1}\,dx \to 0$ as $n \to \infty$ and we also may assume that $v_n \rightharpoonup 0$ in $H^1\left(\mathbb{R}^N\right)$. We also have

$$1 + \int_{\mathbb{R}^N} a^+\,|v_n|^2\,dx \;=\; \int_{\mathbb{R}^N} a^-\,|v_n|^2\,dx + o(1)\,.$$

Using the fact that $a^- \in L^{\frac{N}{2}}\left(\mathbb{R}^N\right)$, we see that $\int_{\mathbb{R}^N} a^-\,|v_n|^2\,dx \to 0$ and we get a contradiction. Since $\{\|\nabla u_n\|_2\}$ is bounded, the inequality (6.4) yields that $\{u_n\}$ is bounded in $H^1\left(\mathbb{R}^N\right)$. Therefore we may assume that $u_n \rightharpoonup u$ in $H^1\left(\mathbb{R}^N\right)$ and $u_n \to u$ in $L^p_{\mathrm{loc}}\left(\mathbb{R}^N\right)$ for $2 \leq p < 2^*$. As before we have $\int_{\mathbb{R}^N} a^- u_n^2\,dx \to \int_{\mathbb{R}^N} a^- u^2\,dx$. We now show that $u \not\equiv 0$ on $\mathbb{R}^N$. In the contrary case $u_n \rightharpoonup 0$ in $H^1\left(\mathbb{R}^N\right)$ and obviously $\int_{\mathbb{R}^N} a^- u_n^2\,dx \to 0$. Hence

$$\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx + \int_{\mathbb{R}^N} a^+ u_n^2\,dx \;=\; -1 + \int_{\mathbb{R}^N} a^- u_n^2\,dx \;\longrightarrow\; -1\,,$$

which is impossible. We claim that

$$m \;=\; \int_{\mathbb{R}^N} b(x)\,|u|^{q+1}\,dx\,. \tag{7.2}$$

To establish this claim we introduce, as in [CH1], a quantity α_∞ defined by

$$\alpha_\infty \;=\; \lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|\geq R} b(x)\,|u_n(x)|^{q+1}\,dx\,.$$

This quantity measures the loss of mass at infinity of a weakly convergent sequence in $H^1\left(\mathbb{R}^N\right)$. As in [CH2] by the Sobolev embedding theorem we have

$$m \;=\; \int_{\mathbb{R}^N} b(x)\,|u(x)|^{q+1}\,dx + \alpha_\infty\,.$$

It is sufficient to show that $\alpha_\infty = 0$. Arguing indirectly we must have

$$0 \;<\; \int_{\mathbb{R}^N} b(x)\,|u(x)|^{q+1}\,dx \;<\; m\,. \tag{7.3}$$

Since

$$\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx + \int_{\mathbb{R}^N} a^+ u_n^2\,dx \;=\; -1 + \int_{\mathbb{R}^N} a^- u_n^2\,dx\,,$$

by virtue of a lower semicontinuity of norm with respect to a weak convergence we see that

$$\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \int_{\mathbb{R}^N} a^+ u^2\,dx \;\leq\; -1 + \int_{\mathbb{R}^N} a^- u^2\,dx\,,$$

that is,

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} a u^2 \, dx \ \le \ -1 \, .$$

Due to (7.3), we cannot have

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} a u^2 \, dx \ = \ -1 \, .$$

Therefore

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} a u^2 \, dx \ < \ -1 \, .$$

Thus there exists $s \in (0,1)$ such that

$$\int_{\mathbb{R}^N} |\nabla (su)|^2 \, dx + \int_{\mathbb{R}^N} a (su)^2 \, dx \ = \ -1 \, .$$

Consequently, according to (7.3)

$$m \ \le \ \int_{\mathbb{R}^N} s^{q+1} b(x) \, |u|^{q+1} \, dx \ < \ \int_{\mathbb{R}^N} b(x) \, |u|^{q+1} \, dx \ < \ m$$

which is impossible. This means that $\alpha_\infty = 0$, that is, $m = \int_{\mathbb{R}^N} b(x) \, |u|^{q+1} \, dx$ and by the previous part of the proof $u \in M$. Since $|u|$ is also a minimizer, we may assume that u is nonnegative on $\mathbb{R}^N$ and the strict positivity of u is a consequence of the Harnack inequality. □

We close this paper with an existence result for the problem $\left(1_{b^+}\right)$ under the assumption that (a) holds and $\lambda_1(a) > 0$. In Section 4 we already pointed out that this problem has a solution which is bounded from below and above by positive constants.

Theorem 7.2. *Suppose that* (a) *holds and that* $\lambda_1(a) > 0$*. Then problem* $\left(1_{b^+}\right)$ *has a solution.*

Proof. We use the method of sub– and supersolutions and follow some ideas from the paper [BD]. Let $0 < \lambda < \lambda_1(a)$ and $v \in H^1\left(\mathbb{R}^N\right) \cap C^1\left(\mathbb{R}^N\right)$ be a positive function. By $u(x)$ we denote a positive solution of the equation

$$-\Delta u + a(x) u - \lambda u \ = \ v \quad \text{in} \quad \mathbb{R}^N$$

(see [BD]). We look for a supersolution of the form $w = \mu u + \nu$, where $\mu > 0$ and $\nu > 0$ are sufficiently large constants. The function w is a supersolution if and only if

$$\lambda \mu u + \mu v + a \nu \ \ge \ b (\mu u + \nu)^q \quad \text{on} \quad \mathbb{R}^N \, .$$

We choose $\bar{\nu} > 0$ such that

$$B \nu^q \ \ge \ B(\nu + t)^q - \lambda t$$

for each $t \ge 0$ and $\nu \ge \bar{\nu}$, where $B = \sup_{x \in \mathbb{R}^N} b(x)$. Taking $\bar{\nu}$ larger, if necessary, we may assume that $a(x)\nu \ge B\nu^q$ for each $\nu \ge \bar{\nu}$ and $x \in \mathbb{R}^N - B(0,R)$. It then follows that

$$(7.4) \qquad a(x)\nu \ \ge \ B\nu^q \ \ge \ B(\nu + u(x)\mu)^q - \lambda \mu u(x) \ \ge \ b(x)(\nu + \mu u(x))^q - \lambda \mu u(x)$$

for all $\nu \geq \bar{\nu}$ and $x \in \mathbb{R}^N - B(0,R)$. Since $u > 0$ on $\mathbb{R}^N$, there exists $\delta_1 > 0$ such that $u(x) \geq \delta_1$ for $x \in B(0,R)$. Therefore there exists $\bar{\mu} > 0$ such that

$$b(x)\big(\mu \|u\|_{\infty,B(0,R)} + \bar{\nu}\big)^q + \bar{\nu}\,\|a^-\|_{\infty,B(0,R)} \;\leq\; \mu\delta_1\lambda$$

for all $\mu \geq \bar{\mu}$. Hence for all $x \in B(0,R)$ and $\mu \geq \bar{\mu}$ we get

$$\begin{aligned} b(x)(\mu u + \bar{\nu})^q - \bar{\nu} a(x) \;&\leq\; b(x)\big(\mu \|u\|_{\infty,B(0,R)} + \bar{\nu}\big)^q + \bar{\nu}\,\|a^-\|_{\infty,B(0.R)} \\ &\leq\; \mu\delta_1\lambda \\ &\leq\; \lambda\mu u(x)\,, \end{aligned}$$

which means that

$$\lambda\mu u(x) + \bar{\nu} a(x) \;\geq\; b(x)(\mu u(x) + \bar{\nu})^q \tag{7.5}$$

for each $x \in B(0,R)$ and all $\mu \geq \bar{\mu}$. Thus by virtue of (7.4) and (7.5) $\bar{u} = \bar{\mu} u + \bar{\nu}$ is a supersolution. A construction of a subsolution w such that $w \leq \bar{u}$ on $\mathbb{R}^N$ is the same as in the proof of Theorem 6.1 and is omitted. □

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