Positive solutions of a fourth-order semilinear problem involving critical growth*

Claudianor O. Alves

Universidade Federal de Campina Grande - Departamento de Matemática e Estatística

58109.970 Campina Grande - PB, Brazil, coalves@dme.ufpb.br

João Marcos do Ó

Universidade Federal da Paraíba - Departamento de Matemática

58059.900 João Pessoa - PB, Brazil, jmbo@mat.ufpb.br

Abstract

In this work we study the existence of positive solutions of the critical problem

$$(P) \qquad \Delta^2 u + a(x)u = |u|^{2^{**}-2}u, \ \mathbb{R}^N \text{ and } u \in D^{2,2}(\mathbb{R}^N),$$

where $2^{**} = 2N/(N-4)$, $N \geq 5$, $a \in L^{N/4}(\mathbb{R}^N)$ is a nonnegative continuous function and Δ^2 is the biharmonic operator. We also prove a global compactness result for the associated energy functional of problem (P), similar to that due to Struwe in [22]. The basic tool employed here is the concentration compactness due to P. L. Lions and a linking theorem on the cone of nonnegative functions.

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1 Introduction

The main purpose of this paper is to investigate the existence of positive solutions of the fourth-order critical problem

$$\Delta^2 u + a(x)u = |u|^{2^{**}-2}u, \ \mathbb{R}^N \text{ and } u \in D^{2,2}(\mathbb{R}^N), \tag{1.1}$$

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where $2^{**} = 2N/(N-4)$, $N \geq 5$, $a : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative continuous function with $a \in L^{N/4}(\mathbb{R}^N)$, Δ^2 is the biharmonic operator and $D^{2,2}(\mathbb{R}^N)$ is the closure of $C_o^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\Delta u|^2 \, dx \right)^{\frac{1}{2}}.$$

Problems involving critical growth in second-order semilinear and quasilinear problems have been object of intensive research in the last years, starting with the work of Brezis-Nirenberg [8]. See, for example, [4, 24] for semilinear elliptic equations and [5, 14] for quasilinear equations. For results involving biharmonic equations with critical growth, we refer to [3, 6, 10, 11, 13, 15, 19] and references therein.

Here we extend to problem (1.1) the results of the paper of Benci-Cerami [4]. Also, we prove a global compactness result similar to that due to Struwe in [21]. We use the variational method and our arguments make use of the Lions concentration-compactness principle the limit case (see [18]) and a linking theorem on the cone of positive functions. This global compactness result is crucial to investigate the behavior of the Palais-Smale sequence of the associated energy functional of problem (1.1). We would also like to mention that this kind of problem as well a global compactness for the p-Laplacian operator has been studied in [2].

We study the existence of solutions for problem (1.1), regarded as critical points of the associated energy functional $I : D^{2,2}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\Delta u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} a(x) u^2 dx - \frac{1}{2^{**}} \int_{\mathbb{R}^N} |u|^{2^{**}} \, dx.$$

The main theorem of this paper is stated as follows.

Theorem 1.1 *Let $a : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative continuous function such that*

$$a(x_o) > 0, \quad \textit{for some } x_o \in \mathbb{R}^N, \tag{1.2}$$

there are numbers $1 < p_1 < N/4 < p_2$ and, for $N \leq 7$, $p_2 < N/(8-N)$, such that

$$a \in L^s(\mathbb{R}^N), \ \textit{for all } \ s \in [p_1, p_2] \tag{1.3}$$

and

$$|a|_{N/4} < S(2^{4/N} - 1), \tag{1.4}$$

where S corresponds to the best constant for the embedding of $D^{2,2}(\mathbb{R}^N)$ in $L^{2^{**}}(\mathbb{R}^N)$. Then there exists a critical point $u_o \in D^{2,2}(\mathbb{R}^N)$ of the functional I with

$$S^{N/4}/N < I(u_o) < 2S^{N/4}/N.$$

Remark 1.2 *Assumptions like* (1.2) – (1.4) *are quite natural and have already appeared in the papers [2, 4] for the p–Laplacian and Laplacian operator, respectively. It should be remarked that assumption* (1.4) *seems to be just technical and it leaves room for improvement.*

This paper is organized as follows: In section 2 we list some elementary properties and prove the fundamental results for Palais-Smale Sequences. In section 3 we prove the main result of this paper, via a link theorem on the cone of nonnegative functions.

Notation. In this work we make use of the following notation.
We denote in a Banach space X the strong convergence by " $\to$ " and the weak convergence by " $\rightharpoonup$ ".
As usual, we say that a C^1-functional $\Phi : X \to I\!R$ satisfies the Palais-Smale condition at level c (the $(PS)_c$ condition for short) if every Palais-Smale sequence of Φ at level c, that is, $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$ in a dual space X^*, is relatively compact.
$B_R(p)$ denotes as usual the open ball of $I\!R^N$, centered at p and of radius R.
$L^r(I\!R^N)$, $1 \le r < \infty$ denote Lebesgue spaces and by $|u|_r = (\int_{I\!R^N} |u|^r\,dx)^{1/r}$ their norm.
We denote by S the best constant of the immersion, $D^{2,2}(I\!R^N) \hookrightarrow L^{2^{**}}(I\!R^N)$, that is,

$$S = \inf\left\{\int_{I\!R^N} |\Delta u|^2\,dx : u \in D^{2,2}(I\!R^N) \text{ with } \int_{I\!R^N} |u|^{2^{**}}\,dx = 1\right\}.$$

This infimum S is achieved by the functions $u_{\delta,y}$ given by

$$u_{\delta,y}(x) = \frac{C_N \delta^{(N-4)/4}}{\left[\delta + |x-y|^2\right]^{(N-4)/2}}, \quad C_N = [(N-4)(N-2)N(N+2)]^{(N-4)/8}, \tag{1.5}$$

for any $\delta > 0$ and $y \in I\!R^N$ (see [13, 18, 19, 25]).

2 Preliminary Results

This section supplies a basic tool needed to study the behavior of the Palais-Smale sequences of I, the associated energy functional of problem (1.1). For that matter we shall need the concentration-compactness principle due to P.L. Lions [18]. In what follows we enunciate a version adequate for our purposes (see also [10, 23, 22]).

Lemma 2.1 *Let* $\{u_n\} \subset D^{2,2}(I\!R^N)$ *with* $u_n \rightharpoonup u$ *weakly in* $D^{2,2}(I\!R^N)$. *Suppose* $\mu_n = |\Delta u_n|^2\,dx \rightharpoonup \mu$, $\nu_n = |u|^{2^{**}}\,dx \rightharpoonup \nu$ *weakly in the sense of measures where* μ *and* ν *are bounded non-negative measures on* $I\!R^N$. *Then, we have:*

1. *There exists some at most countable set* Λ, *a family* $\{y_i\}_{i\in\Lambda}$ *of distinct points in* $I\!R^N$ *and, a family* $\{\nu_i\}_{i\in\Lambda}$ *of positive numbers such that*

$$\nu = |u|^{2^{**}}\, dx + \sum_{i\in\Lambda} \nu_i \delta_{y_i},$$

where δ_x *is the Dirac-mass of mass* 1 *concentrated at* $x \in I\!R^N$

2. *In addition we have*

$$\mu \geq |\Delta u|^2\, dx + \sum_{i\in\Lambda} \mu_i \delta_{y_i},$$

for some family $\{\mu_i\}_{i\in\Lambda}$ *of positive numbers satisfying*

$$S\nu_i^{(N-4)/N} \leq \mu_i, \text{ for all } i \in \Lambda.$$

In particular, $\sum_{i\in\Lambda} \nu_i^{(N-4)/N} < \infty$.

Using Lemma 2.1, we investigate the behavior of the Palais-Smale sequences of the energy functional $I_\infty : D^{2,2}(I\!R^N) \to I\!R$ given by

$$I_\infty(u) = \frac{1}{2}\int_{I\!R^N} |\Delta u|^2\, dx - \frac{1}{2^{**}}\int_{I\!R^N} |u|^{2^{**}}\, dx$$

associated to the limiting problem

$$\Delta^2 u = |u|^{2^{**}-2}\, u, \ I\!R^N \text{ and } u \in D^{2,2}(I\!R^N). \tag{2.6}$$

The next result is crucial to do a careful study of the behavior of Palais-Smale sequence associated to the functional I. A version of this result for bounded domain and Laplacian operator was proved by Struwe in [22].

Lemma 2.2 *Let* $\{u_n\}$ *a Palais-Smale sequence for* I_∞ *such that* $u_n \rightharpoonup 0$ *and* $u_n \not\to 0$ *in* $D^{2,2}(I\!R^N)$. *Then, there exists a sequence* $\{R_n\} \subset I\!R$, $\{x_n\} \subset I\!R^N$, v_o *a nontrivial solution of the limiting problem* (P_∞) *and a Palais-Smale sequence* $\{w_n\}$ *for* I_∞ *such that for some subsequence of* $\{u_n\}$ *we have*

$$w_n(x) = u_n(x) - R_n^{(N-4)/2} v_o(R_n(x - x_n)) + o_n(1).$$

Proof. Let $\{u_n\}$ a Palais-Smale sequence for I_∞, that is,

$$I_\infty(u_n) \to c \text{ and } I'_\infty(u_n) \to 0 \ \text{ as } n \to \infty. \tag{2.7}$$

We can assume that

$$c \geq \frac{2}{N} S^{N/4} \tag{2.8}$$

because if $c < \frac{2}{N} S^{N/4}$, a similar arguments to that used by Brezis-Nirenberg in [8] shows that $u_n \to 0$ strongly in $D^{2,2}(\mathbb{R}^N)$. From (2.7), taking subsequence if necessary, we have

$$\frac{2}{N} \int_{\mathbb{R}^N} |\Delta u_n|^2 \, dx \to c \ \text{ as } n \to \infty, \tag{2.9}$$

then by (2.8) and (2.9),

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\Delta u_n|^2 \, dx \geq S^{N/4}.$$

Choose $\{x_n\} \subset \mathbb{R}^N$ and $\{R_n\} \subset \mathbb{R}$ such that

$$\sup_{y \in \mathbb{R}^N} \int_{B_{R_n^{-1}}(y)} |\Delta u_n|^2 \, dx = \int_{B_{R_n^{-1}}(x_n)} |\Delta u_n|^2 \, dx = \frac{1}{2L} S^{N/4},$$

where L is a natural number such that $B_2(0)$ is covered by L balls of radius 1, and scale

$$u_n \mapsto v_n(x) = R_n^{(4-N)/2} u_n(x/R_n + x_n).$$

Thus $I_\infty(u_n) = I_\infty(v_n)$ and

$$\sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |\Delta v_n|^2 \, dx = \int_{B_1(0)} |\Delta v_n|^2 \, dx = \frac{1}{2L} S^{N/4}.$$

Now, for each $\Phi \in D^{2,2}(\mathbb{R}^N)$ the following sequence

$$\widetilde{\Phi}_n(x) = R_n^{(N-4)/2} \Phi(R_n(x - x_n))$$

satisfies

$$\int_{\mathbb{R}^N} \Delta u_n \Delta \widetilde{\Phi}_n dx = \int_{\mathbb{R}^N} \Delta v_n \Delta \Phi dx$$

and

$$\int_{\mathbb{R}^N} |u_n|^{2^{**}-2} u_n \, \widetilde{\Phi}_n \, dx = \int_{\mathbb{R}^N} |v_n|^{2^{**}-2} v_n \, \Phi \, dx.$$

Hence,

$$I_\infty(v_n) \to c \text{ and } I'_\infty(v_n) \to 0, \ \text{ as } n \to \infty.$$

Thus for each bounded sequence $\{\phi_n\} \subset D^{2,2}(\mathbb{R}^N)$, we have

$$I'_\infty(v_n)(\phi_n) \to 0 \text{ as } n \to \infty. \tag{2.10}$$

Considering $v_o \in D^{2,2}(I\!R^N)$, the weak limit of $\{v_n\} \subset D^{2,2}(I\!R^N)$, we have v_o is a solution of $(P)_\infty$ and according to Lemma 2.1, we may assume $\nu_n = \mid v_n \mid^{2^{**}} dx \rightharpoonup \nu$ and $\mu_n = \mid \Delta v_n \mid^2 dx \rightharpoonup \mu$ weakly in the sense of measures, with

$$\begin{aligned} \nu &= \mid v_o \mid^{2^{**}} + \sum_{i \in \Lambda} \nu_i \delta_{y_i}, \\ \mu &\geq \mid \Delta v_o \mid^2 + \sum_{i \in \Lambda} \mu_i \delta_{y_i}, \end{aligned}$$

where Λ is at most countable set and

$$S\nu_i^{(N-4)/N} \leq \mu_i, \text{ for all } i \in \Lambda. \tag{2.11}$$

Claim 2.3 *Λ is empty or finite.*

Proof of claim 2.3 Next we take as a test function $v_n\psi$, where $\psi \in C_0^\infty(I\!R^N, [0,1])$, such that $\psi \equiv 1$ on $B(y_k, \epsilon)$, $\psi \equiv 0$ on $I\!R^N - B(y_k, 2\epsilon)$, $\mid \nabla\psi \mid \leq 2/\epsilon$ and $\mid \Delta\psi \mid \leq 2/\epsilon^2$. Hence

$$\int_{I\!R^N} \Delta v_n \Delta(v_n\psi) dx = \int_{I\!R^N} \mid v_n \mid^{2^{**}} \psi dx + o_n(1). \tag{2.12}$$

We observe that

$$\begin{aligned} &\int_{I\!R^N} \Delta v_n \Delta(v_n\psi) dx = \\ &\int_{I\!R^N} \mid \Delta v_n \mid^2 \psi dx + 2\int_{I\!R^N} \Delta v_n \nabla v_n \nabla\psi dx + \int_{I\!R^N} v_n \Delta v_n \Delta\psi dx. \end{aligned}$$

Since

$$\begin{aligned} \mid \textstyle\int_{I\!R^N} \Delta v_n \nabla v_n \nabla\psi dx \mid &\leq \left(\textstyle\int_{I\!R^N} \mid \Delta v_n \mid^2 dx\right)^{1/2} \left(\textstyle\int_{I\!R^N} \mid \nabla v_n \mid^2 \mid \nabla\psi \mid^2 dx\right)^{1/2} \\ &\leq C \left(\textstyle\int_{B(y_k, 2\epsilon)} \mid \nabla v_n \mid^2 \mid \nabla\psi \mid^2 dx\right)^{1/2} \end{aligned}$$

and

$$\begin{aligned} \lim_{n \to +\infty} \int_{B(y_k, 2\epsilon)} \mid \nabla v_n \mid^2 \mid \nabla\psi \mid^2 dx &= \textstyle\int_{B(y_k, 2\epsilon)} \mid \nabla v_o \mid^2 \mid \nabla\psi \mid^2 dx \\ &\leq C \left(\textstyle\int_{B(y_k, 2\epsilon)} \mid \nabla v_o \mid^{2N/(N-2)} dx\right)^{(N-2)/N}, \end{aligned}$$

where $C > 0$ and $C_1 > 0$ are constants independent of n, we see that

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} \int_{I\!R^N} \Delta v_n \nabla v_n \nabla\psi dx = 0.$$

Similarly, since $\Delta\psi_\epsilon \sim \epsilon^{-2}$ and $v_n \to v_o$ in $L^2_{loc}(I\!R^N)$, we have

$$\begin{aligned}\lim_{n\to\infty} \mid \int_{I\!R^N} u_n \Delta v_n \Delta\psi dx \mid &\leq C \lim_{n\to\infty} \left(\int_{B(y_i,2\epsilon)} {v_n}^2 \mid \Delta\psi \mid^2 dx\right)^{1/2} \\ &\leq C \left(\int_{B(y_i,2\epsilon)} {v_o}^2 \mid \Delta\psi \mid^2 dx\right)^{1/2} \\ &\leq C \left(\int_{B(y_i,2\epsilon)} \mid v_o \mid^p\right)^{1/p} \left(\int_{B(y_i,2\epsilon)} \mid \Delta\psi \mid^{N/2}\right)^{2/N} \\ &\leq C_1 \left(\int_{B(y_i,2\epsilon)} \mid v_o \mid^p\right)^{1/p},\end{aligned}$$

for some positive constants C and C_1 independent of ϵ. Consequently,

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} \int_{I\!R^N} v_n \Delta v_n \Delta\psi dx = 0.$$

Letting $n \to \infty$ and then $\epsilon \to 0$ in (2.12) we obtain $\mu_k = \nu_k$, which together with (2.11) imply that $\nu_k \geq S^{N/4}$. From this it is easy to see that Λ is empty or finite.

Hereafter we denote $B_r = B_r(0)$ and $\Gamma = \{y_i \;:\; |y_i| > 1\}$. Next our objective is to show that v_o is nontrivial. If we assume by contradiction that v_o is trivial, for all $\phi \in C_o^\infty(I\!R^N \setminus \Lambda)$,

$$\int_{I\!R^N} |v_n|^{2^{**}} \phi dx \to 0 \tag{2.13}$$

and by (2.10) and (2.13),

$$\int_{I\!R^N} |\Delta v_n|^2 \phi dx \to 0. \tag{2.14}$$

Let ρ be a fixed real number such that

$$0 < \rho < \min\{\text{dist}(\Gamma, \overline{B}_1(0)), 1\},$$

and

$$\Phi_n(x) = \Phi(x) v_n(x),$$

where $\Phi \in C_o^\infty(B_{1+\rho}, [0,1])$ is a cut-off function such that $\Phi \equiv 1$ in $B_{1+\rho/3}$ and $\Phi \equiv 0$ in $I\!R^N \backslash B_{1+2\rho/3}$. We remark that

$$\int_{B_{1+\rho}\backslash B_{1+\rho/3}} |\Delta\Phi_n|^2 dx \to 0, \tag{2.15}$$

because by (2.14),

$$\int_{B_{1+\rho}\backslash B_{1+\rho/3}} |\Delta v_n|^2 dx \to 0. \tag{2.16}$$

Using the fact that $\{\Phi_n\}$ is a bounded sequence, we have

$$I'_\infty(v_n)(\Phi_n) \to 0 \text{ as } n \to \infty$$

hence

$$\int_{B_{1+\rho}} \Delta v_n \Delta \Phi_n dx - \int_{B_{1+\rho}} |v_n|^{2^{**}-2} v_n \Phi_n dx = o_n(1)$$

and we find

$$\int_{B_{1+\rho/3}} |\Delta \Phi_n|^2 + \int_{B_{1+\rho}\backslash B_{1+\rho/3}} \Delta v_n \Delta \Phi_n - \int_{B_{1+\rho/3}} |\Phi_n|^{2^{**}} - \int_{B_{1+\rho}\backslash B_{1+\rho/3}} |v_n|^{2^{**}} \Phi = o_n(1).$$

From (2.13), (2.15) and this last fact we conclude that

$$\int_{B_{1+\rho/3}} |\Delta \Phi_n|^2 dx - \int_{B_{1+\rho/3}} |\Phi_n|^{2^{**}} dx = o_n(1)$$

or equivalently

$$\int_{I\!R^N} |\Delta \Phi_n|^2 dx - \int_{I\!R^N} |\Phi_n|^{2^{**}} dx = o_n(1). \tag{2.17}$$

Now, using (2.17) and the definition of S, we have

$$\|\Phi_n\|^2 \left[1 - S^{-\frac{2^{**}}{2}} \|\Phi_n\|^{2^{**}-2}\right] \le o_n(1). \tag{2.18}$$

Using the estimate

$$\|\Phi_n\|^2 \le \int_{B_{1+\rho/3}} |\Delta v_n|^2 dx + o_n(1) \le \int_{B_2} |\Delta v_n|^2 dx + o_n(1)$$

we obtain

$$\|\Phi_n\|^2 \le L \int_{B_1} |\Delta v_n|^2 dx + o_n(1) = \frac{1}{2} S^{N/4} + o_n(1). \tag{2.19}$$

Combining (2.18) and (2.19) it follows

$$\limsup_{n\to\infty} \|\Phi_n\|^2 \left[1 - \left(\frac{1}{2}\right)^{4/(N-4)}\right] \le 0$$

and we can conclude that

$$\Phi_n \to 0 \text{ in } D^{2,2}(I\!R^N).$$

Using again the definition of sequence $\{\Phi_n\}$ we have

$$\lim_{n\to\infty} \int_{B_1} |\Delta v_n|^2 dx = 0,$$

contradicting the equality

$$\int_{B_1} |\Delta v_n|^2 dx = \frac{1}{2L} S^{N/4}, \quad \forall n \in I\!N,$$

thus v_o is nontrivial.

To conclude the lemma, let $\Psi \in C_o^\infty(I\!R^N,[0,1])$ such that $\Psi(x)=1$ in $B_1(0)$, $\Psi(x)=0$ in $I\!R^N\backslash B_2$, and let

$$w_n(x) = u_n(x) - R_n^{(N-4)/2} v_o(R_n(x-x_n))\Psi(\bar{R}_n(x-x_n))$$

where the sequence $\{\bar{R}_n\} \subset I\!R$ is chosen satisfy $R_n\left(\bar{R}_n\right)^{-1} \to \infty$. Using the same arguments explored by Struwe in [22] we finish the proof of Lemma 2.2. ∎

Remark 2.4 *If in Lemma 2.2 the sequence $\{u_n\}$ is nonnegative, then the function v_o is nonnegative. Moreover, if $\{u_n\}$ is a Palais-Smale sequence at level $c = S^{\frac{N}{4}}/N$, we have that v_o assume the best constant S, thus (see [18, 25]) there exist $\delta > 0$ and $y \in I\!R^N$ such that $v_o(x) = u_{\delta,y}(x)$, for all $x \in I\!R^N$. Therefore, for all $x \in I\!R^N$*

$$u_n(x) = w_n(x) + S^{(N-4)/8}\Phi_{\delta_n,y_n}(x) + o_n(1)$$

where $\Phi_{\delta_n,y_n} = S^{(4-N)/8}u_{\delta_n,y_n}$ for some $y_n \in I\!R^N$ and $\delta_n > 0$.

The next result is a technical lemma and its proof we can be found in Brezis-Lieb [7] (see also Alves [2]).

Lemma 2.5 *$A : I\!R^K \to I\!R^K$; $A(y) = |y|^{p-2}\, y$ and $\eta_n : I\!R^N \to I\!R^K$ such that $\eta_n(x) \to 0$ almost everywhere, $\eta_n \in (L^p(I\!R^N))^K$ $(p \geq 2)$ and $|\eta_n|_{(L^p(I\!R^N))^K} \leq C$. Then we have*

$$\int_{I\!R^N} |A(\eta_n + w) - A(\eta_n) - A(w)|^{p/(p-1)}\, dx = o_n(1),$$

for each $w \in (L^p(I\!R^N))^K$ fixed.

Next we study the behavior of the Palais-Smale sequence of the functional I associated to (1.1).

Theorem 2.6 (A Global Compactness Result) *Let $\{u_n\} \subset D^{2,2}(I\!R^N)$ be a Palais-Smale sequence of the functional I. Then, or $\{u_n\}$ possesses a strongly convergent subsequence, or else there exists a finite sequence $\{z_o^1, ..., z_o^k\}$ of nontrivial solutions for the problem* (2.6) *such that*

$$\|u_n\|^2 \to \|u_o\|^2 + \sum_{j=1}^{k} \left\|z_o^j\right\|^2$$

and

$$I(u_n) \to I(u_o) + \sum_{j=1}^{k} I_\infty(z_o^j)$$

where u_o is the weak limit of sequence $\{u_n\}$ in $D^{2,2}(I\!R^N)$.

Proof. First note that $u_n(x) \to u_o(x)$ almost everywhere in $I\!R^N$ and hence u_o is a solution of (1.1). Suppose that u_n does not converge to u_o in $D^{2,2}(I\!R^N)$, and let $\{z_n^1\} \subset D^{2,2}(I\!R^N)$ given by $z_n^1 = u_n - u_o$. Then

$$z_n^1 \rightharpoonup 0 \text{ but } z_n \not\to 0 \text{ in } D^{2,2}(I\!R^N)$$

and

$$I_\infty(z_n^1) = I(u_n) - I(u_o) + o_n(1) \tag{2.20}$$

because we have

$$\left\|z_n^1\right\|^2 = \|u_n - u_o\|^2 = \|u_n\|^2 - \|u_o\|^2 + o_n(1)$$

and by Brezis and Lieb [7]

$$\left|z_n^1\right|_{2^{**}}^{2^{**}} = |u_n - u_o|_{2^{**}}^{2^{**}} = |u_n|_{2^{**}}^{2^{**}} - |u_o|_{2^{**}}^{2^{**}} + o_n(1)$$

and

$$\int_{I\!R^N} a(x){u_n}^2 dx = \int_{I\!R^N} a(x){u_o}^2 dx + o_n(1),$$

because u_n converges weakly to u_o in $L^{N/4}$, since u_n^2 is bounded in $L^{N/(N-4)}$ and $u_n(x) \to u_o(x)$ almost everywhere in $I\!R^N$.

Moreover, by Lemma 2.5 we have

$$i_{1,n} = \int_{I\!R^N} \left| |u_n|^{2^{**}-2} u_n - |(u_n - u_o)|^{2^{**}-2} (u_n - u_o) - |u_o|^{2^{**}-2} u_o \right|^{2^{**}/(2^{**}-1)} dx \to 0$$

and

$$i_{2,n} = \int_{I\!R^N} a(x) \left|u_n - u_o\right|^2 dx \to 0.$$

Thus we obtain

$$I'_\infty(z_n^1) = I'(u_n) - I'(u_o) + o_n(1). \tag{2.21}$$

Since $I'(u_o)=0$, using (2.20) and (2.21), we conclude that $\{z_n^1\}$ is a Palais-Smale sequence of the functional I_∞. From Lemma 2.2, we have $\{R_{n,1}\}\subset I\!R$, $\{x_{n,1}\}\subset I\!R^N$, z_o^1 a nontrivial solution of (2.6) and a Palais-Smale sequence $\{z_n^2\}$ for I_∞ given by

$$z_n^2(x)=z_n^1(x)-R_{n,1}^{(N-4)/2}z_o^1(R_{n,1}(x-x_{n,1}))+o_n(1).$$

If we define

$$v_n^1(x)=R_{n,1}^{(4-N)/2}z_n^1(x/R_{n,1}+x_n)$$

and

$$\widetilde{z}_n^2(x)=v_n^1(x)-z_o^1(x)+o_n(1),$$

we have

$$\begin{aligned} v_n^1 &\rightharpoonup z_o^1\\ I_\infty(v_n^1) &= I_\infty(z_n^1)+o_n(1)\\ I'_\infty(v_n^1) &= o_n(1). \end{aligned}$$

Thus

$$\left\|\widetilde{z}_n^2\right\|^2=\left\|v_n^1\right\|^2-\left\|z_o^1\right\|^2+o_n(1),$$

that is,

$$\left\|\widetilde{z}_n^2\right\|^2=\left\|z_n^1\right\|^2-\left\|z_o^1\right\|^2+o_n(1),$$

which implies

$$\left\|\widetilde{z}_n^2\right\|^2=\|u_n\|^2-\|u_o\|^2-\left\|z_o^1\right\|^2+o_n(1),$$

consequently

$$I_\infty(\widetilde{z}_n^2)=I_\infty(v_n^1)-I_\infty(z_o^1)+o_n(1)=I(u_n)-I(u_o)-I_\infty(z_o^1)+o_n(1)$$

and

$$I'_\infty(\widetilde{z}_n^2)=I'_\infty(v_n^1)-I'_\infty(z_o^1)+o_n(1)=o_n(1).$$

If $\widetilde{z}_n^2\to 0$ in $D^{2,2}(I\!R^N)$, the proof is complete. Otherwise, we can iterating the above produce with the help of Lemma 2.2 to get a finite sequence $\{\widetilde{z}_o^1,...,\widetilde{z}_o^k\}$ of nontrivial solutions for the problem (2.6) satisfying

$$\left\|\widetilde{z}_n^j\right\|^2\geq S^{N/4},\quad j=1,...,k \tag{2.22}$$

and

$$\left\|\widetilde{z}_n^k\right\|^2=\|u_n\|^2-\|u_o\|^2-\sum_{j=1}^{k-1}\left\|z_o^j\right\|^2+o_n(1). \tag{2.23}$$

Then

$$I_\infty(\widetilde{z}_n^k) = I(u_n) - I(u_o) - \sum_{j=1}^{k-1} I_\infty(z_o^j) + o_n(1). \tag{2.24}$$

We notice that this iteration must terminate at some finite index k, because from (2.22) and (2.23) we have

$$0 \le \left\|\widetilde{z}_n^k\right\|^2 \le \|u_n\|^2 - \|u_o\|^2 - \sum_{j=1}^{k-1} S^{N/4} = \|u_n\|^2 - \|u_o\|^2 - (k-1)S^{N/4} + o_n(1). \tag{2.25}$$

which implies, for k sufficiently large, that

$$\limsup_{n\to\infty} \left\|\widetilde{z}_n^k\right\|^2 \le 0,$$

and hence $\widetilde{z}_n^k \to 0$ in $D^{2,2}(I\!\!R^N)$. ∎

Corollary 2.7 *Let $\{u_n\}$ be a Palais-Smale sequence for I at level $c \in (0, 2S^{N/4}/N)$. Then $\{u_n\}$ has a subsequence strongly convergent in $D^{2,2}(I\!\!R^N)$.*

Next we have a regularity result which it will be used to prove a compactness criterion.

Lemma 2.8 *Let $u \in D^{2,2}(I\!\!R^N)$ be a distributional solution of*

$$\Delta^2 u = V(x)u \quad in\ \mathcal{D}'(I\!\!R^N), \tag{2.26}$$

where $V(x) \in L^{N/4}(I\!\!R^N)$. Then $u \in L^p(I\!\!R^N)$ for all $p \ge 2N/(N-4)$, $(N \ge 5)$.

Proof. The proof follows adapting arguments as those of Lemma B1 of [23] and applying the Calderon-Zigmund inequality (see Theorem 9.9 in [16]). ∎

Proposition 2.9 *Let $\{u_n\}$ it be a nonnegative Palais-Smale sequence for I at level $c \in (2S^{N/4}/N, 4S^{N/4}/N)$. Then $\{u_n\}$ has a subsequence strongly convergent in $D^{2,2}(I\!\!R^N)$.*

Proof. If not, by Theorem 2.6,

$$I(u_n) \to I(u_o) + \sum_{j=1}^{k} I_\infty(z_o^j)$$

where u_o is the weak limit of sequence $\{u_n\}$ in $D^{2,2}(I\!R^N)$. Thus u_o is nontrivial, because if $u_o = 0$ we get

$$I(u_n) \to \sum_{j=1}^{k} I_\infty(z_o^j)$$

hence $k = 1$, since $I_\infty(z_o^j) \geq 2S^{N/4}/N$, for $j = 1 \dots k$. On the other hand, as a consequence of Lemma 2.8 and Remark 2.4, we get that z_0^1 is a classical solution of (2.6) with $I_\infty(z_o^1) = 2S^{N/4}/N$. Thus

$$I(u_n) \to 2S^{N/4}/N,$$

which is a contradiction with $I(u_n) \to \ c \in (2S^{N/4}/N, 4S^{N/4}/N)$.

■

Let $f : D^{2,2}(I\!R^N) \to I\!R$;

$$f(u) = \int_{I\!R^N} (|\Delta u|^2 + a(x)u^2)dx$$

and

$$\mathcal{M} = \{u \in D^{2,2}(I\!R^N); \int_{I\!R^N} |u|^{2^{**}} dx = 1\}.$$

We notice that $\{u_n\} \subset \mathcal{M}$ satisfies

$$f(u_n) \to c \text{ and } f' |_{\mathcal{M}} (u_n) \to 0$$

if and only if $v_n \doteq c^{(N-4)/8} u_n$ satisfies

$$I(v_n) \to 2c^{N/4}/N \text{ and } I'(v_n) \to 0.$$

Corollary 2.10 *If there exists a nonnegative sequence* $\{u_n\} \subset \mathcal{M}$ *such that*

$$f(u_n) \to c \textit{ and } f' |_{\mathcal{M}} (u_n) \to 0,$$

for $c \in (S, 2^{4/N}S)$ *, then the functional* f *has a critical point* $u \in D^{2,2}(I\!R^N)$ *at level* c*.*

Remark 2.11 *The Corollary 2.10 implies that (1.1) has at least a positive solution. Moreover, it shows that* $(PS)_c$ *condition holds in the cone of the positive functions to functional* $f|_{\mathcal{M}}$ *for all* $c \in (S, 2^{4/N}S)$*.*

3 Proof of the Main Theorem

In this section we will show the existence of positive solution for problem (1.1). To this end we do arguments which are similar in spirit to those addressed in [4].

Here we consider the family of functions (see 1.5)

$$\Phi_{\delta,y} = S^{(4-N)/8} u_{\delta,y} \in D^{2,2}(I\!R^N),$$

which satisfies: $\|\Phi_{\delta,y}(x)\|^2 = S$, $|\Phi_{\delta,y}(x)|_{2^{**}} = 1$ and $\Phi_{\delta,y} \in L^q(I\!R^N)$, for all $q \in (N/(N-4), 2^{**}]$.

Lemma 3.1 *For each $y \in I\!R^N$ and $q \in (N/(N-4), 2^{**})$, we have*

$$|\Delta\Phi_{\delta,y}|_\infty \to +\infty \text{ as } \delta \to 0, \tag{3.27}$$

$$|\Delta\Phi_{\delta,y}|_\infty \to 0 \text{ as } \delta \to +\infty, \tag{3.28}$$

$$|\Phi_{\delta,y}|_q \to 0 \text{ as } \delta \to 0, \tag{3.29}$$

$$|\Phi_{\delta,y}|_q \to +\infty \text{ as } \delta \to +\infty. \tag{3.30}$$

Proof. Using the definition of the function $\Phi_{\delta,y}$,

$$|\Delta\Phi_{\delta,y}|(x) = \frac{C_N}{S^{(N-4)/8}} \frac{(N-4)\delta^{(N-4)/4}\left[2|x-y|^2 + N\delta\right]}{\left[\delta + |x-y|^2\right]^{N/2}}$$

then

$$|\Delta\Phi_{\delta,y}|_\infty = \frac{C_N}{S^{(N-4)/8}} N(N-4)\delta^{-N/4}$$

and consequently (3.27) and (3.28) hold. Since

$$|\Phi_{\delta,y}|_q^q = \left(\frac{C_N}{S^{(N-4)/8}}\right)^q \delta^{(N-4)(2^{**}-q)/4} \int_{I\!R^N} \left[1 + |x|^2\right]^{(4-N)q/2} dx,$$

and $2^{**} > q$, we have that (3.29) and (3.30) hold. ■

Lemma 3.2 *The infimum* $\inf\{f(u) : u \in \mathcal{M}\}$ *is never achieved.*

Proof. It is easy to see that $\inf\{f(u) : u \in \mathcal{M}\} \geq S$. In fact the equality holds, because by Hölder's inequality, with $2q \in (N/(N-4), 2^{**})$ and $1/q + 1/t = 1$,

$$f(\Phi_{\delta,0}) \leq S + |a|_t |\Phi_{\delta,0}|_{2q}^2,$$

and by Lemma 3.1 - (3.29), $|\Phi_{\delta,0}|_{2q} \to 0$ as $\delta \to 0$.

Therefore, if we assume that there is $v \in \mathcal{M}$ such that $f(v) = S$, we obtain a contradiction, because

$$S \leq \|v\|^2 < f(v) = S.$$

■

Lemma 3.3 *For each* $\epsilon > 0$,

$$\int_{I\!R^N \setminus B_\epsilon} |\Delta \Phi_{\delta,o}|^2 \, dx \to 0 \text{ as } \delta \to 0$$

Proof. Using the definition of $\Phi_{\delta,o}$, we obtain

$$\int_{I\!R^N \setminus B_\epsilon} |\Delta \Phi_{\delta,o}|^2 \, dx \leq \frac{C_N^2 (N-4)^2 \delta^{(N-4)/2}}{S^{(N-4)/4}} \int_\epsilon^\infty \frac{(r^2 + N\delta)^2}{r^{N+1}} dr \to 0 \text{ as } \delta \to 0.$$

■

Lemma 3.4 *For each* $\epsilon > 0$, *there exists* $\underline{\delta} = \underline{\delta}(\epsilon) > 0$ *and* $\overline{\delta} = \overline{\delta}(\epsilon) > 0$ *such that, for all* $\delta \in (0, \underline{\delta}] \cup [\overline{\delta}, +\infty)$ *we have*

$$\sup_{y \in I\!R^N} f(\Phi_{\delta,y}) < S + \epsilon.$$

Proof. For each $y \in I\!R^N$ fixed, we distinguish two cases:

(i) $s \in (N/4, p_2)$. By Hölder's Inequality,

$$\int_{I\!R^N} a(x) |\Phi_{\delta,y}|^2 \, dx \leq |a|_s \, |\Phi_{\delta,o}|_{2t}^2, \quad \text{for } y \in I\!R^N,$$

where $t = s/(s-1)$. Then, by Lemma 3.1, there exists $\underline{\delta} > 0$, such that for all $\delta \in (0, \underline{\delta}]$

$$\sup_{y \in I\!R^N} \int_{I\!R^N} a(x) |\Phi_{\delta,y}|^2 \, dx < \epsilon.$$

Thus, for all $\delta \in (0, \underline{\delta}]$,

$$\sup_{y \in I\!R^N} f(\Phi_{\delta,y}) \leq S + \sup_{y \in I\!R^N} \int_{I\!R^N} a(x) |\Phi_{\delta,y}|^2 \, dx \leq S + \epsilon.$$

(ii) $s \in (p_1, N/4)$. In this case we have $2t = 2s/(s-1) > 2^{**}$ and

$$\begin{aligned} \textstyle\int_{I\!R^N} a(x) |\Phi_{\delta,y}|^2 \, dx &\leq |a|_s \, |\Phi_{\delta,o}|_\infty^{(2t-2^{**})/t} \, |\Phi_{\delta,o}|_{2^{**}}^{2^{**}/t} \\ &\leq \left(\tfrac{C_N}{S^{(N-4)/8}}\right)^{(2t-2^{**})/t} |a|_s \, \delta^{(4-N)(2t-2^*)/4t}. \end{aligned}$$

Thus there exists $\overline{\delta} > 0$, such that for all $\delta \in [\overline{\delta}, +\infty)$,

$$\sup_{y \in I\!\!R^N} \int_{I\!\!R^N} a(x) \left|\Phi_{\delta,y}\right|^2 dx < \epsilon$$

and hence for all $\delta \in [\overline{\delta}, +\infty)$,

$$\sup_{y \in I\!\!R^N} f(\Phi_{\delta,y}) \le S + \sup_{y \in I\!\!R^N} \int_{I\!\!R^N} a(x) \left|\Phi_{\delta,y}\right|^2 dx \le S + \epsilon.$$

■

Lemma 3.5 *Assume that* $|a|_{N/4} < S(2^{4/N} - 1)$. *Then*

$$\sup_{y \in I\!\!R^N \ \delta \in (0,\infty)} f(\Phi_{\delta,y}) < 2^{4/N} S.$$

Proof. Using the definition of $\Phi_{\delta,y}$ and Hölder's inequality,

$$f(\Phi_{\delta,y}) = S + \int_{I\!\!R^N} a(x) \left|\Phi_{\delta,y}\right|^2 dx \le S + |a|_{N/4}\,,$$

thus

$$\sup_{y \in I\!\!R^N \ \delta \in (0,\infty)} f(\Phi_{\delta,y}) \le S + |a|_{N/4}\,.$$

Finally, using that $|a|_{N/4} < S(2^{4/N} - 1)$, we obtain

$$\sup_{y \in I\!\!R^N \ \delta \in (0,\infty)} f(\Phi_{\delta,y}) < 2^{4/N} S.$$

■

Now we consider the function $\alpha : D^{2,2}(I\!\!R^N) \to I\!\!R^{N+1}$ given by

$$\alpha(u) = \frac{1}{S} \int_{I\!\!R^n} \left(\frac{x}{|x|}, \sigma(x)\right) |\Delta u|^2 \, dx = (\beta(u), \gamma(u)),$$

where

$$\begin{aligned} \sigma(x) &= \begin{cases} 0 & \text{if } |x| < 1 \\ 1 & \text{if } |x| \ge 1; \end{cases} \\ \beta(u) &= \frac{1}{S} \int_{I\!\!R^N} \frac{x}{|x|} \, |\Delta u|^2 \, dx; \\ \gamma(u) &= \frac{1}{S} \int_{I\!\!R^N} \sigma(x) \, |\Delta u|^2 \, dx. \end{aligned}$$

Lemma 3.6 *For $|y| \geq 1/2$, we have*

$$\beta(\Phi_{\delta,y}) = y/|y| + o_\delta(1) \quad as\ \delta \to 0.$$

Proof. Fixed $\epsilon > 0$, by Lemma 3.3 there exists $\widehat{\delta} > 0$ such that for all $\delta \in (0, \widehat{\delta})$,

$$\frac{1}{S}\int_{I\!R^N \setminus B_\epsilon(y)} |\Delta \Phi_{\delta,y}|^2\, dx < \epsilon,$$

then

$$\left|\beta(\Phi_{\delta,y}) - \frac{1}{S}\int_{B_\epsilon} \frac{x}{|x|}\, |\Delta \Phi_{\delta,y}|^2\, dx\right| < \epsilon. \tag{3.31}$$

If $\epsilon > 0$ is small and $|y| \geq 1/2$, for all $x \in B_\epsilon(y)$ we get

$$\left|\frac{x}{|x|} - \frac{y}{|y|}\right| < 2\epsilon.$$

Thus we have

$$\left|\frac{y}{|y|} - \frac{1}{S}\int_{B_\epsilon(y)} \frac{x}{|x|}\, |\Delta \Phi_{\delta,y}|^2\, dx\right| \leq$$

$$\left|\frac{1}{S}\int_{B_\epsilon(y)} \left(\frac{x}{|x|} - \frac{y}{|y|}\right) |\Delta \Phi_{\delta,y}|^2\, dx\right| + \left|\frac{1}{S}\int_{I\!R^N \setminus B_\epsilon(y)} \frac{y}{|y|}\, |\Delta \Phi_{\delta,y}|^2\, dx\right|$$

and hence

$$\left|\frac{y}{|y|} - \frac{1}{S}\int_{B_\epsilon(y)} \frac{x}{|x|}\, |\Delta \Phi_{\delta,y}|^2\, dx\right| < 2\epsilon + \epsilon = 3\epsilon.$$

This fact together with (3.31) imply that

$$\left|\beta(\Phi_{\delta,y}) - \frac{y}{|y|}\right| < 4\epsilon,$$

for all $|y| \geq 1/2$ and $\delta \in (0, \widehat{\delta})$. ■

Now we consider the following set

$$\Upsilon = \{u \in \mathcal{M} \cap \Sigma : \alpha(u) = (0, 1/2)\},$$

where

$$\Sigma = \{u \in D^{2,2}(I\!R^N); u \geq 0\}.$$

Notice that Υ is a nonempty set, because $\Phi_{\delta,o}$ belongs to the cone Σ and for all $\delta > 0$ we have

$$\begin{aligned} \int_{B_\epsilon} \frac{x}{|x|}\, |\Delta \Phi_{\delta,o}|^2\, dx &= 0 \\ \frac{1}{S}\int_{I\!R^N} \sigma(x)\, |\Delta \Phi_{\delta,o}|^2\, dx &= \gamma(\Phi_{\delta,o}), \end{aligned}$$

and

$$\gamma(\Phi_{\delta,o}) \to 0 \text{ as } \delta \to 0 \text{ and } \gamma(\Phi_{\delta,o}) \to 1 \text{ as } \delta \to \infty,$$

thus there exists $\delta_1 > 0$ such that

$$(\beta(\Phi_{\delta_1,o}), \gamma(\Phi_{\delta_1,o})) = (0, 1/2).$$

Lemma 3.7 *We have*

$$c_o = \inf_{u \in \Upsilon} f(u) > S.$$

Proof. It is obvious that $c_o \geq S$. To prove this lemma we suppose by contradiction that $c_o = S$. Thus there exists a sequence of nonnegative functions $\{u_n\} \subset D^{2,2}(I\!R^N)$ such that

$$|u_n|_{2^{**}} = 1 \text{ and } \alpha(u_n) = (0, 1/2)$$

and

$$f(u_n) \to S.$$

Using Remark 1 we obtain a sequence of points $y_n \subset I\!R^N$, a sequence of positive numbers δ_n and a sequence of functions $\omega_n \subset D^{2,2}(I\!R^N)$ converging strongly to 0 in $D^{2,2}(I\!R^N)$ such that

$$u_n(x) = \omega_n(x) + \Phi_{\delta_n,y_n}(x), \;\; \forall x \in I\!R^N$$

Since $\omega_n \to 0$ in $D^{2,2}(I\!R^N)$, from the definition of α,

$$\alpha(\omega_n + \Phi_{\delta_n,y_n}) = \alpha(\Phi_{\delta_n,y_n}) + o_n(1).$$

Therefore

$$\beta(\Phi_{\delta_n,y_n}) \to 0, \;\; n \to +\infty \tag{3.32}$$

$$\gamma(\Phi_{\delta_n,y_n}) \to 1/2, \;\; n \to +\infty \tag{3.33}$$

Going if necessary to a subsequence, one of the following cases occurs for (Φ_{δ_n,y_n}) as $n \to +\infty$,

$$\delta_n \to +\infty, \tag{3.34}$$

$$\delta_n \to \delta_0 > 0, \tag{3.35}$$

$$\delta_n \to 0 \text{ and } y_n \to y_0 \text{ with } \mid y_0 \mid < 1/2, \tag{3.36}$$

$$\delta_n \to 0 \text{ and } \mid y_n \mid \geq 1/2. \tag{3.37}$$

Now we shall prove that none of these possibilities can be true. Assume that (3.34) holds, then by Lemma 3.1,

$$\gamma(\Phi_{\delta_n,y_n}) = \frac{1}{S}\int_{I\!R^N\setminus B_1(0)} |\Delta\Phi_{\delta_n,y_n}|^2\,dx = 1 - \frac{1}{S}\int_{B_1(0)} |\Delta\Phi_{\delta_n,y_n}|^2\,dx = 1 - o_n(1)$$

which implies a contradiction with (3.33). If (3.35) holds, then $|y_n| \to +\infty$, because otherwise $\{\Phi_{\delta_n,y_n}\}$ would converge strongly in $D^{2,2}(I\!R^N)$, thus the same would be true for $\{u_n\}$, and therefore $f(u) = S$ for some $u \in \mathcal{M}$, which it is a contradiction with Lemma 3.2. Thus,

$$\begin{aligned}
\gamma(\Phi_{\delta_n,y_n}) &= \gamma(\Phi_{\delta_0,y_n}) + o_n(1)\\
&= \tfrac{1}{S}\textstyle\int_{I\!R^N} \sigma(x)\,|\Delta\Phi_{\delta_0,y_n}|^2\,dx - o_n(1)\\
&= \tfrac{1}{S}\textstyle\int_{I\!R^N} \sigma(x-y_n)\,|\Delta\Phi_{\delta_0,0}|^2\,dx\\
&= 1 + \tfrac{1}{S}\textstyle\int_{B_1(y_n)} \sigma(x)\,|\Delta\Phi_{\delta_0,y_n}|^2\,dx\\
&= 1 - o_n(1)
\end{aligned}$$

which implies a contradiction with (3.33). If (3.36) holds, thus

$$\gamma(\Phi_{\delta_n,y_n}) = \frac{1}{S}\int_{I\!R^N\setminus B_1(0)} |\Delta\Phi_{\delta_n,y_n}|^2\,dx = \frac{1}{S}\int_{I\!R^N\setminus B_1(y_n)} |\Delta\Phi_{\delta_n,0}|^2\,dx = o_n(1)$$

which implies a contradiction with (3.33). If (3.37) holds, by Lemma 3.6,

$$\beta(\Phi_{\delta,y_n}) = y_n/\mid y_n\mid + o_\delta(1) \ \text{ as } \delta \to 0,$$

which it is a contradiction with (3.32). ■

The next three lemmas follow using the Change Variable Theorem and lemmas that we showed until this moment.

Lemma 3.8 *There exists $\delta_1 \in (0,1/2)$ such that*

$$f(\Phi_{\delta_1,y}) < (S+c_o)/2, \quad \textit{for } y \in I\!R^N, \tag{3.38}$$

$$\gamma(\Phi_{\delta_1,y}) < 1/2, \quad \textit{for } |y| < 1/2, \tag{3.39}$$

$$\left|\beta(\Phi_{\delta_1,y}) - \frac{y}{|y|}\right| < 1/4, \quad \textit{for } |y| \geq 1/2. \tag{3.40}$$

Proof. First we remark that (3.38) follows from Lemmas 3.3 and 3.6, and (3.40) holds, because Lemma 3.6. To show (3.39) we record the following equality

$$\gamma(\Phi_{\delta,y}) = \frac{1}{S}\int_{|x|\geq 1} |\Delta\Phi_{\delta,y}|^2\,dx = \frac{1}{S}\int_{|x-y|\geq 1} |\Delta\Phi_{\delta,o}|^2\,dx,$$

thus

$$\gamma(\Phi_{\delta,y}) = 1 - \frac{1}{S}\int_{B_1(y)} |\Delta\Phi_{\delta,o}|^2\, dx \to 0 \text{ as } \delta \to 0.$$

This yields (3.39). ∎

Lemma 3.9 *There exists $\delta_2 > 1/2$ such that for all $y \in I\!R^N$,*

$$f(\Phi_{\delta_2,y}) < (S + c_o)/2, \tag{3.41}$$

$$\gamma(\Phi_{\delta_2,y}) > 1/2. \tag{3.42}$$

Proof. From Lemmas 3.4 and 3.7 we show (3.41). And, by similar argument as used in the proof of Lemma 3.8, we show (3.42). ∎

Lemma 3.10 *There exists $R > 0$ such that for $y \in I\!R^N \setminus B(0,R)$ and $\delta \in [\delta_1, \delta_2]$,*

$$f(\Phi_{\delta,y}) < (S + c_o)/2, \tag{3.43}$$

$$(\beta(\Phi_{\delta,y})\,|y)_{I\!R^N} > 0. \tag{3.44}$$

Proof. To show this lemma we use the definition of $\Phi_{\delta,y}$ and the same arguments explored in [4]. ∎

Now we consider the map $Q : I\!R^N \times (0,\infty) \to D^{2,2}(I\!R^N)$ given by

$$Q(y,\delta) = \Phi_{\delta,y}$$

and the following sets

$$\begin{aligned} V &= B_R(0) \times (\delta_1, \delta_2),\\ \Theta &= Q(\overline{V}),\\ H &= \{h \in C(\Sigma \cap \mathcal{M}, \Sigma \cap \mathcal{M}) : h(u) = u \text{ if } f(u) < (S + c_o)/2\},\\ \Gamma &= \{A \subset \Sigma \cap \mathcal{M} : A = h(\Theta), \text{ for some } h \in H\}. \end{aligned}$$

Lemma 3.11 *Let $F : \overline{V} \to I\!R^{N+1}$ given by*

$$F(y,\delta) = (\alpha \circ Q)(y,\delta) = \frac{1}{S}\int_{I\!R^N} (\frac{x}{|x|}, \sigma(x))\, |\Delta\Phi_{\delta,y}|^2\, dx.$$

Then

$$deg(F, V, (0, 1/2)) = 1.$$

Proof. Consider the homotopy

$$Z(t,.) = tF + (1-t)Id_V$$

which satisfies $(0,1/2) \notin Z(t,\partial V)$, that is, for every $(y,\delta) \in \partial V$, we have

$$(t\beta(\Phi_{\delta,y}) + (1-t)y, t\gamma(\Phi_{\delta,y}) + (1-t)\delta) \neq (0,1/2).$$

To prove this fact we distinguish the cases.

(i) If $| y |< 1/2$, from (3.39) we have $\gamma(\Phi_{\delta_1,y}) < 1/2$, thus for all $t \in [0,1]$,

$$t\gamma(\Phi_{\delta_1,y}) + (1-t)\delta < 1/2.$$

(ii) If $1/2 \leq| y \leq R$, from (3.40),

$$\left|\beta(\Phi_{\delta_1,y}) - \frac{y}{| y |}\right| < \frac{1}{4},$$

hence, for all $t \in [0,1]$,

$$\begin{aligned} | t\beta(\Phi_{\delta_1,y}) + (1-t)y | &\geq | t\frac{y}{| y |} + (1-t)y | - | t\beta(\Phi_{\delta_1,y}) - t\frac{y}{| y |} | \\ &\geq t + (1-t) | y | - t/4 \\ &\geq 1/2. \end{aligned}$$

(iii) If $| y |\leq R$, from (3.42), $\gamma(\Phi_{\delta_1,y}) > 1/2$, thus for all $t \in [0,1]$,

$$t\gamma(\Phi_{\delta_1,y}) + (1-t)\delta > 1/2.$$

(iv) If $| y |= R$, from 3.44, for all $\delta \in [\delta_1, \delta_2]$ and $t \in [0,1]$,

$$(t\beta(\Phi_{\delta,y}) + (1-t)y \,| y)_{I\!R^N} > (1-t) | y |^2 = (1-t)R^2.$$

Therefore, by the homotopy invariance of the topological degree,

$$deg(F, V, (0,1/2)) = deg(Id_V, V, (0,1/2)) = 1.$$

■

Lemma 3.12 *$A \cap \Upsilon \neq \emptyset$, for all $A \in \Gamma$.*

Proof. Consider the map

$$F_h = \alpha \circ h \circ Q : I\!\!R^N \times (0,\infty) \to I\!\!R^N \times (0,\infty).$$

We claim that $F_h(y,\delta) = F(y,\delta)$, for all $(y,\delta) \in \partial V$. In fact, if $(y,\delta) \in \partial V$, by (3.41), (3.38) and (3.43), we have

$$f(\Phi_{y,\delta}) < (c_0+\delta)/2,$$

thus

$$h(\Phi_{y,\delta}) = \Phi_{y,\delta}.$$

So, for all $(y,\delta) \in \partial V$,

$$F_h(y,\delta) = \alpha(h(\Phi_{y,\delta})) = \alpha(\Phi_{y,\delta}) = F(y,\delta).$$

Therefore, by the properties of the degree theory and Lemma 3.8 we have

$$deg(F_h, V, (0,1/2))) = deg(F, V, (0,1/2))) = 1,$$

which implies that for all $h \in H$, there is $(y,\delta) \in V$ such that

$$\alpha(h(\Phi_{\delta,y})) = (0,1/2).$$

■

Proof of Theorem 1.1 Consider the minimax level

$$c = \inf_{A\in\Gamma} \sup_{u\in A} f(u),$$

and

$$K_c = \{u \in \Sigma \cap \mathcal{M} : f(u) = c \text{ and } f'\,|_{\mathcal{M}}\,(u) = 0\}.$$

It is easy to see that the proof of Theorem 1.1 is a consequence of the following.

Claim 3.13 *$S < c < 2^{4/N}S$ and K_c is nonempty.*

Proof of claim 3.13 Using the definition of minimax level c and Lemma 3.5, we have

$$c \le \sup_{u\in\Theta} f(u) \le \sup_{y\in I\!\!R^N \;\; \delta\in(0,\infty)} f(\Phi_{\delta,y}) < 2^{4/N}S. \tag{3.45}$$

On the other hand, by Lemma 3.11, $A \cap \Upsilon \neq \emptyset$ for all $A \in \Gamma$, thus

$$c \ge \inf_{u\in\Upsilon} f(u) = c_o > S. \tag{3.46}$$

From (3.45) and (3.46)

$$S < c < 2^{\frac{4}{N}} S.$$

Suppose now $K_c = \emptyset$. For each $s \in I\!R$, denote

$$f^s = \{u \in \Sigma \cap \mathcal{M} : f(u) \leq s\}.$$

By Corollary 2.10, the Palais-Smale condition holds in

$$\{u \in \Sigma \cap \mathcal{M} : S < f(u) < 2^{2/N} S\}.$$

Thus, using the deformation lemma there is $\epsilon_o > 0$ and $\eta \in C([0,1] \times \Sigma \cap \mathcal{M}, \Sigma \cap \mathcal{M})$ such that

$$\eta(t,u) = u \text{ for } t = 0 \text{ or } (t,u) \in (0,1) \times f^{c-\epsilon_o} \cup (\Sigma \cap \mathcal{M} \setminus f^{c+\epsilon_o})$$

and

$$\eta(1, f^{c+\epsilon_o/2}) \subset \eta(1, f^{c-\epsilon_o/2}).$$

Let $A_0 \in \Gamma$ such that

$$c < \sup_{u \in A_0} f(u) < c + \epsilon_o/2,$$

thus $\eta(1, A_0) \in \Gamma$ and $\sup_{u \in \eta(1,A_0)} f(u) < c - \epsilon_0/2$ which is a contradiction and we concluded the proof of Theorem 1.1.

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