On an inequality by N. Trudinger and J. Moser and related elliptic equations

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Abstract

It has been shown by Trudinger and Moser that for normalized functions u of the Sobolev space $\mathbb{W}^{1,N}(\Omega)$, where Ω is a domain in $\mathbb{R}^N$, the integral $\int_\Omega \exp(u^{\alpha_N N/(N-1)})dx$ remains uniformly bounded. Carleson and Chang proved that there exists a corresponding extremal function in the case that Ω is the unit ball in $\mathbb{R}^N$. In this paper we give a new proof, a generalization, and a new interpretation of this result. In particular, we give an explicit sequence which is maximizing for the above integral among all normalized "concentrating sequences ".

1 Introduction

1.1 Critical growth in $W^{1,q}$: the case $q = N$ versus $1 < q < N$

We recall the following facts: let $W_0^{1,q}(\Omega)$ denote the Sobolev space over a bounded domain $\Omega \subset \mathbb{R}^N$, with norm $||u||_q^q = \int_\Omega |\nabla u|^q \, dx$. Then, for $1 < q < N$ critical growth can be expressed by the following relations: Let

$$\sup_{||u||_q = 1} \int_\Omega |u|^p dx = s_{N,q}(p) \; ; \tag{1}$$

then

$$s_{N,q}(p) < \infty \quad \text{for} \quad 1 < p \le q^* = \frac{qN}{N-q}$$
$$s_{N,q}(p) = +\infty \quad \text{for} \quad p > q^*$$

The value of the *best Sobolev constant* $s_{N,q}(q^*)$ is explicit and independent of the domain Ω, and it is known that it is *never attained* in a smooth domain different from $\mathbb{R}^N$: $s_{N,q}(q^*, \Omega) = s_{N,q}(q^*, \mathbb{R}^N)$, and only $s_{N,q}(q^*, \mathbb{R}^N)$ is attained.

If $q = N$, then critical growth is given by the Trudinger-Moser inequality, which can be expressed as

$$\sup_{||u||_N=1} \int_\Omega (e^{\alpha u^{N/(N-1)}} - 1)dx = c_N(\alpha) \cdot |\Omega| \qquad (2)$$

(note the dependence on $|\Omega|$); then, denoting by ω_{N-1} the $(N-1)$-dimensional surface of the unit sphere in $\mathbb{R}^N$, one has

$$\begin{aligned} c_N(\alpha) < +\infty \quad &\text{for} \quad 0 < \alpha \le \alpha_N = N\omega_{N-1}^{1/(N-1)} \\ c_N(\alpha) = +\infty \quad &\text{for} \quad \alpha > \alpha_N \end{aligned}$$

In this note we consider general nonlinearities with subcritical and critical growth in the case $q = N$. We will suppose throughout this paper that

F1) $F \in \mathcal{C}^1(\mathbb{R})$
F2) F is increasing on $\mathbb{R}^+$, and $F(t) = F(|t|)$
F3) $0 \le F(t) \le e^{\alpha_N |t|^{N/(N-1)}} - 1$, for all $t \in \mathbb{R}$

Then we say, for dimension N

$$F \text{ has } \textit{subcritical growth} \text{ if } \quad \lim_{t\to\infty} \frac{F(t)}{e^{\alpha_N |t|^{N/(N-1)}}} = 0 \ ;$$

otherwise we say that F has *critical growth;* in this case we normalize to $\lim_{t\to\infty} F(t)e^{-\alpha_N|t|^{N/(N-1)}} = 1$, i.e. we say

$$F \text{ has } \textit{critical growth} \text{ if } \quad \lim_{t\to\infty} \frac{F(t)}{e^{\alpha_N |t|^{N/(N-1)}}} = 1 \ .$$

As is known from the cases $1 < q < N$, the notion of criticality is closely related to the behaviour of the functional on *concentrating sequences,* i.e. (in the case $q = N$) sequences $\{u_n\}$ converging weakly to 0 in $W_0^{1,N}(\Omega)$ and such that $|\nabla u_n|^N$ converges to a Dirac delta-function in measure. We make the

Definition 1 *A sequence $\{u_n\} \subset W_0^{1,N}(\Omega)$ is a normalized concentrating sequence, if*
a) $||u_n||_N = 1$
b) $u_n \rightharpoonup 0$ *, weakly in* $W_0^{1,N}(\Omega)$
c) $\exists\ x_0 \in \Omega$ *such that* $\forall\ \rho > 0 :$ $\int_{\Omega\setminus B_\rho(x_0)} |\nabla u_n|^N dx \to 0$

We first consider the behaviour of the functional $\int_\Omega F(u_n)dx$, with *subcritical* nonlinearities F, for normalized concentrating sequences $\{u_n\}$. We will prove:

Theorem 2 *If F has subcritical growth, then $\lim_{n\to\infty}\int_\Omega F(u_n)dx = 0$, for any normalized concentrating sequence $\{u_n\}$.*

By a concentration-compactness alternative of P.L. Lions it will be easy to conclude by Theorem 2 that

Theorem 3 *If F has subcritical growth, then*

$$\sup_{||u||_N=1}\int_\Omega F(u)dx$$

is attained.

Next, we consider critical growth; here we restrict the considerations to the case $\Omega = B_1(0)$, the unit ball in $\mathbb{R}^N$. Studying again the behaviour of the functional on normalized concentrating sequences, we find

Theorem 4 *Let $\Omega = B_1(0)$. If F has critical growth, then*

$$\lim_{n\to\infty}\int_\Omega F(u_n)dx \in [0, e^{1+\frac{1}{2}+\ldots+\frac{1}{N-1}}\,|\Omega|]$$

for any normalized concentrating sequence $\{u_n\}$. In particular, there exists an explicit normalized concentrating sequence $\{y_n\}$ with

$$\lim_{n\to\infty}\int_\Omega F(y_n)dx = e^{1+\frac{1}{2}+\ldots+\frac{1}{N-1}}\,|\Omega|\,.$$

We now turn to the question whether the supremum is attained in the case of critical growth. In an interesting and intricate paper Carleson and Chang [3] have shown that

$$\sup_{||u||_N=1}\int_{B_1(0)} e^{\alpha_N u^{N/(N-1)}}dx$$

is attained. For $N = 2$, this result was extended to general bounded domains Ω by Flucher [4], using symmetrization and conformal deformations.

Here we will prove more generally

Theorem 5 *Suppose that F satisfies F1-F3 and*

F4) $\quad F_\lambda(t) \geq e^{\alpha_N t^{N/(N-1)}} - 1 - \lambda\, t^{N/(N-1)}$.

Then, for $\lambda < \alpha_N$, one has

$$C_{N,\lambda} = \sup_{||u||_N=1} \int_{B_1(0)} F_\lambda(u)dx > e^{1+\frac{1}{2}+\ldots+\frac{1}{N-1}} |\Omega| \,,$$

and $C_{N,\lambda}$ is attained.

1.2 Application to the existence problem for related PDE

The relations (1) and (2) are important with regard to the solvability of the related differential equations with critical growth:

$$\begin{cases} -\Delta u = |u|^{2^*-2}u + g(u) & \text{in} \quad \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

respectively

$$\begin{cases} -\Delta u = h(u)e^{4\pi u^2} = e^{4\pi u^2 + \log(h(u))} & \text{in} \quad \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{4}$$

where g and h are functions with subcritical growth:

$$\frac{g(s)}{s^{2^*-1}} \to 0 \quad \text{as} \quad s \to \infty$$

respectively

$$\frac{\log(h(s))}{s^2} \to 0 \quad \text{as} \quad s \to \infty.$$

Solutions of (3) are given as critical points of the related functional

$$I(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - \frac{1}{2^*}|u|^{2^*} - G(u)\ dx$$

where $G(s) = \int_0^s g(r)dr$. Due to the fact that the supremum in (1) is not attained, one finds for the functional $I(u)$ *certain levels* at which the compactness condition of Palais-Smale fails. In the famous paper [2] of Brezis-Nirenberg it was shown that one can overcome this obstacle and prove existence if one shows, *using properties of the lower order term g*, that the critical levels of the functional avoid these non-compactness levels. To obtain such statements, one uses special sequences of functions obtained from the *maximizing sequence* for (1), which are explicit concentrating functions converging weakly to zero.

It has been shown in [1] by Adimurthi and in [5] that similar methods can be employed to prove existence results also for equation (2). Indeed, considering the associated functional to (4)

$$J(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - F(u)\ dx \tag{5}$$

where $F(s) = \int_0^s h(r)e^{4\pi r^2}dr$, one finds again levels of non-compactness; however, due to the fact that the best constant $c_2(4\pi)$ is attained, there is no natural concentrating sequence to be used to show that these levels are avoided. Thus, it is difficult to obtain *optimal existence results.* The sequence used in [1] and [5] is the so-called *Moser sequence*

$$z_n(t) = \frac{1}{\sqrt{2\pi}} \begin{cases} (\log n)^{1/2} & \text{if} \quad 0 \le |x| \le \frac{1}{n} \\ \frac{\log \frac{1}{|x|}}{(\log n)^{1/2}} & \text{if} \quad \frac{1}{n} \le |x| \le 1 \\ 0 & \text{if} \quad |x| > 1 \end{cases} \tag{6}$$

which was proposed by J. Moser in [8] to prove that the inequality (2) is sharp with respect to the constant 4π in the exponent. This sequence satisfies

$$\lim_{n\to\infty} \int_{B_1} (e^{4\pi u_n^2} - 1)dx = 2\pi\ . \tag{7}$$

while the *explicit concentrating sequence* $\{y_n\}$ with $||y_n|| = 1$ mentioned in Theorem 4 satisfies

$$\lim_{n\to\infty} \int_{B_1} (e^{4\pi u_n^2} - 1)dx = e\pi\ . \tag{8}$$

Since the proof of existence of solutions for equation (4) as done in [5] depends on the value of the limit (7) above, we can improve the result obtained using the limit (8).We will prove

Theorem 6 *Assume that* $h \in C(\mathbb{R})$ *and let* $f(s) = h(s)e^{4\pi s^2}$. *Assume that*

H1) $f(0) = 0$

H2) $\exists\ s_0 > 0$, $\exists\ M > 0$ *such that*

$0 < F(s) = \int_0^s f(r)dr \leq M|f(s)|\ , \quad \forall\ |s| \geq s_0$

H3) $0 < F(s) \leq \frac{1}{2}f(s)s\ ,\ \forall\ s \in \mathbb{R}\backslash\{0\}\ ,\ \forall\ x \in \Omega$

Then equation (4) has a solution provided that

$$\lim_{s\to\infty} h(s)s = \beta > \frac{1}{e\pi} \tag{9}$$

We recall that in [5] it was proved that (4) has a solution provided that $\lim_{s\to\infty} h(s)s = \beta > \frac{1}{2\pi}$. We refer to [6] for non-existence results concerning equation (4).

2 Proofs

2.1 Proof of Theorem 2:

Step 1. Let $\{u_n\}$ denote a normalized concentrating sequence as defined above. We may suppose that the concentration point is $0 \in \Omega$, and that $u \geq 0$ (since both $||u||_N$ and $F(u)$ do not change replacing u by $|u|$). We apply symmetrization (following J. Moser [8]), defining the radially symmetric function u^* as follows: let

$$m\{x\ |u^*(x) > \rho\} = m\{x \in \Omega\ |u(x) > \rho\} \quad \text{for every } \rho > 0.$$

Then u^* is a decreasing function in $|x|$, with $u^*(|x|) = 0$ for $|x| > R$, where $m(B_R(0)) = m(\Omega)$. By construction

$$\int_{B_R} F(u_n^*)dx = \int_\Omega F(u_n)dx\ ,$$

and it is known that

$$1 = \int_\Omega |\nabla u_n|^N\, dx \geq \int_{B_R} |\nabla u_n^*|^N\, dx \quad .$$

Setting $z_n {=} \frac{u_n^*}{||u_n^*||_N} \geq u_n^*$ we thus find, using the monotonicity of $F(t)$

$$\int_{B_R} F(u_n^*)dx \leq \int_{B_R} F(z_n)dx$$

Hence it suffices to show that $\int_{B_R} F(z_n)dx = \omega_{N-1} \int_0^R F(z_n(\rho))\rho^{N-1}d\rho \to 0$.

Step 2. To prove that $\int_0^R F(z_n)\rho^{N-1}d\rho \to 0$, we perform a change of variable which transforms the radial integral on $[0,R)$ into an integral on the half-line $[0,+\infty)$. Let

$$\rho = \mathrm{Re}^{-t/N} \qquad \text{and} \qquad w_n(t) = N^{(N-1)/N}\omega_{N-1}^{1/N} z_n(\rho) = \alpha_N^{(N-1)/N} z_n(\rho)\ .$$

Then $w_n(t)$ is an increasing function on $[0,\infty)$. One checks easily that

$$\int_0^\infty \left|w_n'(t)\right|^N dt = w_{N-1}\int_0^R \left|\frac{d}{d\rho}z_n(\rho)\right|^N \rho^{N-1}d\rho = \int_{B_R} |\nabla z_n(x)|^N\, dx$$

and

$$\int_0^\infty F(\frac{1}{\alpha_N^{(N-1)/N}} w_n(t))\ e^{-t}dt = \frac{N}{R^N}\int_0^R F(z_n(\rho))\rho^{N-1}d\rho \tag{10}$$

$$= \frac{1}{m(B_R)}\int_{B_R} F(z_n(x))dx$$

Clearly, since the sequence u_n is concentrating in x_0, the sequence z_n is concentrating in 0 and the sequence w_n in $+\infty, i.e.$ for any fixed $A>0$ we have $\int_0^A |w_n'|^N\, dt \to 0$.

We now distinguish the cases:

a) there exists $a_n \in (0,+\infty)$ with $w_n^{N/(N-1)}(a_n) = a_n - 2\log(a_n)$. Since for a given interval $[0,A]$ we have, for $\eta>0$ arbitrarily small, that

$$w_n^{N(N-1)}(t) \le t\ (\int_0^A \left|w_n'\right|^N dt)^{1/(N-1)} \le \eta t,\ \text{for } n \text{ large}\ ,$$

it follows that $a_n \to \infty$. Thus we have by assumption F3

$$\int_0^A F(\frac{1}{\alpha_N^{(N-1)/N}} w_n\,(t))e^{-t}dt \le \int_0^A (e^{w_n^{N/(N-1)}(t)} - 1)e^{-t}dt \le \int_0^A (e^{\eta t}-1)e^{-t}dt$$

$$= \frac{1}{1-\eta}(1-e^{(\eta-1)A}) - 1 + e^{-A}$$

Since A is arbitrarily large and $\eta>0$ arbitrarily small, we conclude that $\int_0^A F(\frac{1}{\alpha_N} w_n)e^{-t}dt \to 0$ as $n\to\infty$.

Next, consider

$$\int_A^{a_n} F(\frac{1}{\alpha_N^{(N-1)/N}} w_n(t))e^{-t}dt \le \int_A^{a_n} (e^{t-2\log t}-1)e^{-t}dt$$

$$= \frac{1}{A} - \frac{1}{a_n} + e^{-a_n} - e^{A} \to 0\ ,\ \text{as } n\to\infty$$

Finally, on $[a_n, +\infty)$ we have $w_n^{N(N-1)}(t) \geq a_n - 2\log(a_n)$, and hence, by the assumption of subcriticality, we have for any $\epsilon > 0$ fixed that $F(s) \leq \epsilon(e^{\alpha_N s^{N/(N-1)}} - 1)$, for all $s \geq a_n$ with n sufficiently large; thus, for n sufficiently large

$$\int_{a_n}^{\infty} F\left(\frac{1}{\alpha_N^{(N-1)/N}} w_n(t)\right) e^{-t} dt \leq \epsilon \int_{a_n}^{\infty} e^{w_n^{N(N-1)}(t) - t} dt \leq \epsilon\ c$$

Hence, the theorem is proved in case a).

b) for all $t \in \mathbb{R}$ holds: $w_n^{N/(N-1)}(t) < t - 2\log^+ t$. Then, arguing as in a) we have

$$\begin{aligned} &\int_0^{\infty} F(\frac{1}{\alpha_N^{(N-1)/N}} w_n(t)) e^{-t} dt \\ &\leq \int_0^{A} F(\frac{1}{\alpha_N^{(N-1)/N}} w_n(t)) e^{-t} dt + \int_A^{\infty} (e^{t - 2\log t} - 1) e^{-t} dt \end{aligned}$$

which can be made arbitrarily small.

2.2 Concentration-compactness

For the subsequent proofs we rely on the following concentration-compactness result of P.L. Lions [7]:

Proposition 7 *(P.L. Lions). Let Ω be a bounded domain in $\mathbb{R}^N$, and let $\{u_n\}$ be a sequence in $W_0^{1,N}(\Omega)$ such that $||u_n||_N \leq 1$ for all n. We may suppose that $u_n \rightharpoonup u$ weakly in $W_0^{1,N}(\Omega)$, $|\nabla u_n|^N \to \mu$ weakly in measure. Then either*
(i) $\mu = \delta_{x_0}$, the Dirac measure of mass 1 concentrated at some $x_0 \in \overline{\Omega}$, and $u \equiv 0$, or
(ii) there exists $\beta > \alpha_N$ such that the family $u_n = e^{u_n^{N/(N-1)}}$ is uniformly bounded in $L^\beta(\Omega)$ and thus
$\int_\Omega e^{\alpha_N |u_n|^{N/(N-1)}} \to \int e^{\alpha_N |u|^{N/(N-1)}}$ as $n \to \infty$. In particular, this is the case if u is different from 0.

2.3 Proof of Theorem 3:

Since $\sup_{||u||_N=1} \int_\Omega F(u)\,dx > 0$, we conclude by Theorem 2 that there cannot exist a normalized concentrating sequence which is maximizing. By the concentration-compactness alternative of P.L. Lions we infer that the supremum is attained.

2.4 Proof of Theorem 4:

We proceed by the following steps:
1) if $\{u_n\}$ is any normalized concentrating sequence in $W_0^{1,N}(B_1)$, then

$$\lim_{n\to\infty} \int_\Omega F(u_n)dx \le e^{1+\frac{1}{2}+\ldots+\frac{1}{N-1}}\,|\Omega|\ ;$$

2) give an *explicit* normalized concentrating sequence $\{y_n\}$ with

$$\lim_{n\to\infty} \int_\Omega F(y_n)dx = e^{1+\frac{1}{2}+\ldots+\frac{1}{N-1}}\,|\Omega|\ ;$$

1. Upper bound: In this proof we follow Carleson-Chang [3]. First note that by F3) and the transformation in section 2.1 it suffices to show that for any normalized concentrating sequence $\{u_n\} \in C^1[0,\infty)$, i.e $\int_0^\infty |u_n'|^N = 1$, $\int_0^A |u_n'|^N \to 0$, with $u_n(0) = 0$, $u_n'(t) \ge 0$ holds

$$\overline{\lim}_{n\to\infty} \int_0^\infty (e^{u_n^{N/(N-1)}-t} - 1)dt \le e^{1+\frac{1}{2}+\ldots+\frac{1}{N-1}}\ , \tag{11}$$

More precisely, we show:
If $\overline{\lim}_{n\to\infty} \int_0^\infty e^{u_n^{N/(N-1)}-t} > 2$, then $\{u_n\}$ has the following properties (cf. [3], p.117)
(a) if $a_n \in [1,\infty)$ denotes the first point with $u_n^{N/(N-1)}(a_n) = a_n - 2\log a_n$, then $a_n \to \infty$
(b) $\overline{\lim}_{n\to\infty} \int_{a_n}^\infty e^{u_n^{N/(N-1)}-t}dt \le e^{1+\frac{1}{2}+\ldots+\frac{1}{N-1}}$
(c) $\lim_{n\to\infty} \int_0^{a_n} e^{u_n^{N/(N-1)}-t}dt = 1$

Proof:
Estimate (11) follows clearly from (b) and (c).

Property a): following [3], we note that the point a_n exists, for n large enough; if not, $u_n^{N/(N-1)}(t) < t - 2\log^+(t)$ for all t and thus

$\lim_{n\to\infty}\int_0^\infty e^{u_n^{N/(N-1)}-t} \le 2$, contradicting the assumption.
Next, for each $A > 0$, we can choose numbers n_0 and $\eta > 0$ such that for all $t \in [0, A]$ and all $n \ge n_0$

$$u_n^{N(N-1)}(t) \le t\ (\int_0^A \left|u_n'\right|^N dt)^{1/(N-1)} \le \eta t < t - 2\log^+ t\ . \qquad (12)$$

This implies that $a_n \ge A$ for all $n \ge n_0$, and since A is arbitrary, we have $a_n \to \infty$ as $n \to \infty$.

Property c): Note that (12) implies also that $u_n \to 0$ uniformly on compact sets. Using that a_n is the first point where $u_n^{N/(N-1)}(t) = t - 2\log^+(t)$ we have the following estimate from above

$$\int_0^{a_n} e^{u_n^{N/(N-1)}(t)-t}dt \le e^\varepsilon \int_0^A e^{-t}dt + \int_A^{a_n} e^{-2\log t}dt$$
$$= e^\varepsilon(1-e^{-A}) + \left(\frac{1}{A} - \frac{1}{a_n}\right) \to 1 \quad \text{as } \varepsilon \to 0,\ \ A \to \infty$$

On the other hand, we can estimate from below

$$\int_0^{a_n} e^{u_n^{N/(N-1)}(t)-t}dt \ge \int_0^{a_n} e^{-t}dt = 1 - e^{-a_n} \to 1 \ \text{ as } \ a_n \to \infty$$

Property b): We recall that in [3], Lemma 2, the following result was proved (note the misprint there: the factor $e^{(c^n/n)((n-1)/n)^n\beta_n}$ should read $e^{(c^n/n)((n-1)/n)^{n-1}\beta_n}$):

Proposition 8 *(Carleson-Chang). For $a > 0$ and $\delta > 0$ given, suppose that $\int_a^\infty |w'|^N \le \delta$; then*

$$\int_a^\infty e^{w^{N/(N-1)}(t)-t}dt$$
$$\le \frac{1}{1-\delta^{\frac{1}{N-1}}} \exp\left(w^{\frac{N}{N-1}}(a)\left[1 + \frac{1}{N-1}\frac{\delta}{(1-\delta^{\frac{1}{N-1}})^{(N-1)}}\right] - a\right) \cdot e^{1+\frac{1}{2}+\ldots+\frac{1}{N-1}}$$

Proof. Apply this to $w(t) = u_n(t), a = a_n$, and $\delta = \delta_n = \int_{a_n}^\infty |u_n'|^N dt$. Thus we have

$$\int_{a_n}^\infty e^{u_n^{N/(N-1)}(t)-t}dt \le \frac{1}{1-\delta_n^{1/(N-1)}} e^{K_n} \cdot e^{1+\frac{1}{2}+\ldots+\frac{1}{N-1}} \qquad (13)$$

where $K_n = u_n^{N/(N-1)}(a_n)[1 + \frac{1}{N-1}\frac{\delta}{(1-\delta_n^{1/(N-1)})^{N-1}}] - a_n$. By

$$u_n^{N/(N-1)}(a_n) \le a_n \left(\int_0^{a_n} \left| u_n' \right|^N dt \right)^{1/(N-1)}$$

we have, using $\int_0^\infty |u_n'|^{N/(N-1)} = 1$, that

$$u_n^{N/(N-1)}(a_n) \le a_n(1-\delta_n)^{1/(N-1)}.$$

Thus

$$\begin{aligned}\delta_n &\le 1 - \frac{u_n^N(a_n)}{a_n^{N-1}} = 1 - (1 - \frac{2\log^+(a_n)}{a_n})^{N-1} \\ &\le (N-1)\frac{2\log^+(a_n)}{a_n} \to 0 \text{ as } n \to \infty\end{aligned}$$

and

$$\begin{aligned}K_n &= u_n^{N/(N-1)}(a_n)[1 + \frac{1}{N-1}\frac{\delta_n}{(1-\delta_n^{1/(N-1)})^{N-1}}] - a_n \\ &= (a_n - 2\log^+(a_n))(1 + \frac{1}{N-1}\delta_n + O(\delta_n^{N/(N-1)})) - a_n \\ &= -2\log^+(a_n) + a_n\frac{1}{N-1}\delta_n + O\left(\left(\frac{\log^N(a_n)}{a_n}\right)^{1/(N-1)}\right) \\ &\le O\left(\left(\frac{\log^N(a_n)}{a_n}\right)^{1/(N-1)}\right)\end{aligned}$$

Since $a_n \to \infty$ we see that $\overline{\lim}_{n\to\infty} K_n \le 0$. Thus, we get by (13)

$$\overline{\lim_{n\to\infty}} \int_{a_n}^\infty e^{u_n^{N/(N-1)}(t)-t} dt \le e^{1+\frac{1}{2}+\ldots+\frac{1}{N-1}}$$

■

2. The maximizing concentrating sequence: We first consider the function $F(t) = e^{\alpha_N t^{N/(N-1)}}$ with $\Omega = B_1(0)$. We want to produce an explicit normalized concentrating sequence of functions $\{y_n\} \in C[0,\infty)$, piecewise differentiable, with $y_n(0) = 0$ and $y_n'(t) \ge 0$ and such that

$$\lim_{n\to\infty} \int_0^\infty e^{y_n^{N/(N-1)}(t)-t} dt \to 1 + e^{1+\frac{1}{2}+\ldots+\frac{1}{N-1}}$$

For $n \in \mathbb{N}$ set $\delta_n = \frac{2\log n}{n}$, and let

$$y_n(t) = \begin{cases} \frac{t}{n^{1/N}} \, (1-\delta_n)^{\frac{N-1}{N}} & , \quad 0 \le t \le n \\ \frac{N-1}{(n(1-\delta_n))^{1/N}} \log \frac{A_n+1}{A_n+e^{-(t-n)/(N-1)}} + (n(1-\delta_n))^{\frac{N-1}{N}} & , \; n \le t \end{cases} \tag{14}$$

First note that $y_n(t)$ is continuous and piecewise differentiable; furthermore we have

$$\int_0^n |y_n'(t)|^N dt = (1-\delta_n)^{N-1}$$

We now choose A_n in (14) such that $\int_0^\infty |y_n'|^N \, dt = 1$, i.e.

$$\int_n^\infty |y_n'(t)|^N dt = 1 - (1-\delta_n)^{N-1} = (N-1)\delta_n + \sigma_N(\delta_n^2) \quad , \tag{15}$$

where

$$\sigma_N(s) = \begin{cases} 0 & , \quad N = 2 \\ O(s) & , \quad \mathrm{N} \ge 3 \end{cases}$$

We show that such a choice for A_n is possible, with

$$A_n = \frac{1}{n^2} \frac{1}{e^{1+\frac{1}{2}+\ldots+\frac{1}{N-1}}} + \begin{cases} O(1/n^4) & , \quad N = 2 \\ O(\log^2(n)/n^3) & , \quad N \ge 3 \end{cases} \tag{16}$$

Indeed, using the change of variables $s = t - n$, we have

$$\begin{aligned} &\int_n^\infty |y_n'(t)|^N dt \\ &= \int_n^\infty \frac{(N-1)^N}{n(1-\delta_n)} \left| \frac{d}{dt} [\log(A_n+1) - \log(A_n + e^{-(t-n)/(N-1)})] \right|^N dt \\ &= \int_n^\infty \frac{(N-1)^N}{n(1-\delta_n)} \left| \frac{\frac{1}{N-1} \; e^{-(t-n)/(N-1)}}{A_n + e^{-(t-n)/(N-1)}} \right|^N dt \end{aligned}$$

Next, setting $r = e^{s/(N-1)}$ yields

$$\begin{aligned} &\frac{(N-1)^N}{n(1-\delta_n)} \int_0^\infty \left| \frac{\frac{1}{N-1} e^{-s/(N-1)}}{A_n + e^{-s/(N-1)}} \right|^N ds \\ &= \frac{(N-1)^N}{n(1-\delta_n)} \int_1^\infty \left| \frac{\frac{1}{N-1}}{A_n r + 1} \right|^N \frac{N-1}{r} dr \\ &= \frac{N-1}{n(1-\delta_n)} \int_1^\infty \frac{1}{(A_n r+1)^N \; r} dr \quad . \end{aligned}$$

Finally, set $\rho = \frac{1}{r}$ to obtain

$$\begin{aligned}
&\frac{N-1}{n(1-\delta_n)} \int_1^\infty \frac{1}{(A_n r+1)^N\ r} dr \\
&= \frac{N-1}{n(1-\delta_n)} \int_0^1 \frac{\rho^{N-1}}{(A_n+\rho)^N} d\rho \\
&= \frac{N-1}{n(1-\delta_n)} \left(\log \frac{A_n+1}{A_n} - \sum_{k=1}^{N-1} \frac{1}{(N-k)(A_n+1)^{N-k}} \right)
\end{aligned}$$

By (15) this gives the condition

$$\begin{aligned}
&\frac{N-1}{n(1-\delta_n)} \left(\log \frac{A_n+1}{A_n} - \sum_{k=1}^{N-1} \frac{1}{(N-k)(A_n+1)^{N-k}} \right) \\
&= (N-1)\delta_n + \sigma_N(\delta_n^2) = (N-1)\frac{2\log n}{n} + \sigma_N(\frac{\log^2 n}{n^2}) \ ,
\end{aligned}$$

that is

$$\log \frac{A_n+1}{A_n} - \sum_{k=1}^{N-1} \frac{1}{(N-k)(A_n+1)^{N-k}} = \left(2\log n + \sigma_N(\frac{\log^2 n}{n}) \right) (1-\delta_n)$$

and hence

$$\frac{A_n+1}{A_n}\ \exp\left(- \sum_{k=1}^{N-1} \frac{1}{(N-k)(A_n+1)^{N-k}} \right) = n^2(1+\sigma_N(\frac{\log^2 n}{n})) \ , \quad (17)$$

and finally that

$$\frac{A_n+1}{A_n n^2} = e^{1+\frac{1}{2}+\ldots+\frac{1}{N-1}} + \left\{ \begin{array}{ll} O(1/n^2) \quad , & N=2 \\ O(\log^2(n)/n) \ \ , & N \geq 3 \end{array} \right. \quad , \quad \text{for } n \text{ large} \quad (18)$$

This yields (16).

3. The limit: We now calculate the limit of $\int_0^\infty e^{y_n^{N/(N-1)}-t} dt$.

Proposition 9 *Let $\{y_n\}$ denote the sequence (14). Then*

$$\int_0^\infty e^{y_n^{N/(N-1)}(t)-t} \to 1 + e^{1+\frac{1}{2}+\ldots+\frac{1}{N-1}} \quad , \quad \textit{as} \ \ n \to \infty$$

Proof. a) By the upper bound proved above we have (since y_n is a normalized concentrating sequence) that

$$\overline{\lim_{n\to\infty}} \int_0^\infty e^{y_n^{N/(N-1)}(t)-t} \le 1 + e^{1+\frac{1}{2}+\ldots+\frac{1}{N-1}}$$

We now prove the other inequality:
b) We first prove that there exists some $c > 0$ such that

$$\int_0^n e^{y_n^{N/(N-1)}(t)-t} \ge 1 + \frac{\Gamma(1+\frac{N}{N-1})}{n^{1/(N-1)}} - c\frac{\log n}{n^{N/(N-1)}}, \quad as \quad n \to \infty \ , \tag{19}$$

where Γ denotes the standard gamma-function. Indeed

$$\begin{aligned}
\int_0^n e^{y_n^{N/(N-1)}(t)-t} dt &= \int_0^n e^{(1-\delta_n)\ (\mathrm{t}^N/n)^{1/(N-1)}-t} dt \\
&\ge \int_0^n \left(1 + \frac{1-\delta_n}{n^{1/(N-1)}}\ t^{N/(N-1)}\right)\ e^{-t} dt \\
&\ge 1 - e^{-n} + \frac{1-\delta_n}{n^{1/(N-1)}} \left(\int_0^\infty t^{N/(N-1)}\ e^{-t} dt - \int_n^\infty e^{-t/2}\right) \\
&\ge 1 + \frac{\Gamma(1+\frac{N}{N-1})}{n^{1/(N-1)}} - c\frac{\log n}{n^{N/(N-1)}} \quad , \quad \text{as} \quad n \to \infty \ ,
\end{aligned}$$

by the definition of the gamma-function.

c) Next we prove

$$\int_n^\infty e^{y_n^{N/(N-1)}(t)-t} dt \ge e^{1+\frac{1}{2}+\ldots+\frac{1}{N-1}} + \begin{cases} O(\frac{1}{n^2}) & , \quad N=2 \\ O(\frac{\log^2 n}{n}) & , \quad N \ge 3 \end{cases} \tag{20}$$

We perform the change of variables $s = t - n$, and set

$$z_n(s) = \frac{N-1}{n^{1/N}(1-\delta_n)^{1/N}} \log \frac{A_n + 1}{A_n + e^{-s/(N-1)}} \quad ,$$

and

$$d_n = n(1-\delta_n) \ .$$

Then $\int_n^\infty e^{y_n^{N/(N-1)}-t} dt$ becomes

$$\int_0^\infty \exp\left([d_n^{\frac{N-1}{N}} + z_n(s)]^{\frac{N}{N-1}} - \ s - n \right) ds \tag{21}$$

$$\geq \int_0^\infty \exp\left(d_n + \frac{N}{N-1} z_n(s) \ d_n^{\frac{1}{N-1}} - \ s - n \right) ds$$

$$= \int_0^\infty \exp\left(\frac{N}{N-1} z_n(s) \ n^{\frac{1}{N}} (1-\delta_n)^{\frac{1}{N}} - n\delta_n - \ s \right) ds$$

$$= \int_0^\infty \exp\left(N \ \log \frac{A_n + 1}{A_n + e^{-s/(N-1)}} - n\delta_n - \ s \right) ds$$

$$= \frac{1}{n^2} \int_0^\infty \left(\frac{1 + A_n}{A_n + e^{-s/(N-1)}}\right)^N \ e^{-s} ds$$

$$= \frac{(1+A_n)^N}{n^2} \int_0^\infty \frac{1}{(1 + A_n e^{s/(N-1)})^N \ e^{-sN/(N-1)}} \ e^{-s} ds$$

$$= \frac{(1+A_n)^N}{n^2} \int_0^\infty \frac{e^{s/(N-1)}}{(1 + A_n e^{s/(N-1)})^N} ds$$

$$= \frac{(1+A_n)^N}{n^2} (N-1) \int_1^\infty \frac{1}{(1+A_n r)^N} dr$$

$$= \frac{(1+A_n)^N}{n^2} \ \frac{1}{A_n (1+A_n)^{N-1}}$$

$$= \frac{1 + A_n}{n^2 A_n} = e^{1+\frac{1}{2}+\ldots+\frac{1}{N-1}} + \begin{cases} O(\frac{1}{n^2}) & , \quad N = 2 \\ O(\frac{\log^2 n}{n}) & , \quad N \geq 3 \end{cases}$$

by relation (18). Hence the claim. ■

4. General nonlinearities F **:** Suppose now that $F(t)$ is a general nonlinearity with critical growth, satisfying hypotheses F1-F3. Then we may write

$$F(t) = e^{\alpha_N t^{N/(N-1)}} - 1 + G(t)$$

with

$$\frac{G(t)}{e^{\alpha_N t^{N/(N-1)}}} \to 0 \quad , \quad \text{as} \quad t \to \infty$$

Then, by Theorems 2 and 4 we have for any normalized concentrating sequence $\{u_n\}$

$$\lim_{n\to\infty}\int_\Omega F(u_n)dx = \lim_{n\to\infty}\int_\Omega (e^{\alpha_N u_n^{N/(N-1)}}-1)dx + \lim_{n\to\infty}\int_\Omega G(u_n)dx$$
$$\le e^{1+\frac{1}{2}+....+\frac{1}{N-1}}\,|\Omega|$$

while for the sequence $\{y_n\}$ given in (14) holds

$$\lim_{n\to\infty}\int_\Omega F\left(y_n\right) = \lim_{n\to\infty}\int_\Omega (e^{\alpha_N y_n^{N/(N-1)}}-1+G(y_n))dx = e^{1+\frac{1}{2}+....+\frac{1}{N-1}}\,|\Omega|$$

2.5 Proof of Theorem 5:

We show that under condition F4

$$C_{N,\lambda} = \sup_{\int_0^\infty |u'|^N=1}\int_0^\infty F_\lambda(\frac{1}{\alpha_N^{(N-1)/N}}u)e^{-t}dt > e^{1+\frac{1}{2}+...+\frac{1}{N-1}}$$

Indeed, by the estimates (19) and (21) we have for n sufficiently large

$$\int_0^\infty e^{y_n^{N/(N-1)}(t)-t}$$
$$= \int_0^n e^{y_n^{N/(N-1)}(t)-t} + \int_n^\infty e^{y_n^{N/(N-1)}(t)-t} \ge$$
$$\ge 1 + \frac{\Gamma(1+\frac{N}{N-1})}{n^{1/(N-1)}} - c\frac{\log n}{n^{N/(N-1)}} + e^{1+\frac{1}{2}+...+\frac{1}{N-1}} + \begin{cases} O(\frac{1}{n^2}) & , \quad N=2 \\ O(\frac{\log^2 n}{n}) & , \quad N\ge 3 \end{cases} .$$

Furthermore, we can estimate the term

$$\lambda\int_0^\infty |y_n|^{\frac{N}{N-1}}e^{-t}dt \le \frac{\lambda}{n^{1/(N-1)}}\int_0^n |t|^{N/(N-1)}e^{-t}dt + c\int_n^\infty ne^{-t}dt$$
$$\le \lambda\,\frac{\Gamma(1+\frac{N}{N-1})}{n^{1/(N-1)}} + c\frac{\log n}{n^{N/(N-1)}} .$$

Hence we obtain for $\lambda < 1$ and a suitably large n

$$C_{N,\lambda} = \sup_{\int|u'|^N=1}\int_0^\infty F_\lambda(\frac{1}{\alpha_N^{(N-1)/N}}u)e^{-t}dt \ge \int_0^\infty F_\lambda(\frac{1}{\alpha_N^{(N-1)/N}}y_n)e^{-t}dt$$
$$\ge e^{1+\frac{1}{2}+...+\frac{1}{N-1}} + (1-\lambda)\frac{\Gamma(1+\frac{N}{N-1})}{n^{1/(N-1)}} - c\frac{\log n}{n^{N/(N-1)}} + \begin{cases} O(\frac{1}{n^2}) & , \quad N=2 \\ O(\frac{\log^2 n}{n}) & , \quad N\ge 3 \end{cases}$$
$$> e^{1+\frac{1}{2}+...+\frac{1}{N-1}}.$$

Then, since by Theorem 4 there cannot exist a normalized concentrating sequence which is *maximizing* for $C_{N,\lambda}$, we conclude by the concentration-compactness theorem of P.L. Lions that $C_{N,\lambda}$ is attained.

Open problem: Show that $\sup_{||u||=1} \int_{B_1} F(u)dx$ is *not attained* for $F(t)$ of the form

$$F(t) = e^{\alpha_N t^{N/(N-1)}} - g(t)$$

with g subcritical, and

$$g(t) \geq t^{N/(N-1)} \quad .$$

2.6 Proof of Theorem 6:

We restrict attention to the radial case, i.e $\Omega = B_1(0) \subset \mathbb{R}^2$. Consider the functional $I(u) = \int_{B_1} [\frac{1}{2}|\nabla u|^2 - F(u)]\, dx$, where $F(s)$ is as in Theorem 6. Then we know by [5] that this functional satisfies the Palais-Smale condition $(PS)_c$ for $c < \frac{1}{2}$. By the remarks in section 2.1, we may assume that u_n is radially symmetric, and we can rewrite the functional in radial coordinates:

$$\int_0^1 [\frac{1}{2}|u_r|^2 - F(u)]\ 2\pi r dr$$

Cancelling the factor 2π we see that $\int_0^1 [\frac{1}{2}|u_r|^2 - \int_0^1 F(u)]\ rdr$ satisfies $(PS)_c$ for $c < \frac{1}{4\pi}$. Next, we perform a change of variables to transform the interval $(0,1)$ to the interval $(0,+\infty)$: Let

$$r = e^{-t/2}\ ,\ dr = -\frac{1}{2}e^{-t/2}dt\ ,\ u_t = u_r\frac{dr}{dt} = -\frac{1}{2}u_r e^{-t/2}$$

and hence we obtain

$$\int_0^{+\infty} [\frac{1}{2}|2u_t e^{t/2}|^2 - F(u)]\ \frac{1}{2}e^{-t}dt$$

Multiplying by 2 we see that the functional

$$\int_0^{\infty} [2|u_t|^2 - F(u)e^{-t}]dt$$

satsifies $(PS)_c$ for $c < \frac{1}{2\pi}$. Finally, substitute $y = 2\sqrt{\pi}u$ and multiply by π to obtain

$$J(y) = \int_0^{\infty} [\frac{1}{2}|y_t|^2 - \pi F(\frac{1}{2\sqrt{\pi}}y)e^{-t}]dt \qquad (22)$$

which satisfies again $(PS)_c$ for $c < \frac{1}{2}$.

2.6.1 Estimates for (P-S)

We now show that the functional $J(u)$ given by (22) (which satisfies $(PS)_c$ for $c < \frac{1}{2}$) has a critical level c with $c < \frac{1}{2}$ provided h satisfies condition (9).

Theorem 10 *Suppose that $f(s) = h(s)e^{4\pi s^2}$ satisfies H1-H3, and assume that $\lim_{s\to\infty} h(s)s > \frac{1}{e\pi}$. Then $J(u)$ has a critical level below $\frac{1}{2}$.*

Proof: As in [5], the critical level is given by the mountain pass theorem. To prove that the mountain pass level is below $\frac{1}{2}$ it suffices to show that there is a $w \in H_0^1, ||w|| = 1$, such that $\max_{t\geq 0} J(tw) < \frac{1}{2}$. In [5] we used the Moser sequence to show this. Here we use the sequence $\{y_n\}$ given in (14). So we assume, by way of contradiction, that for all $n \in \mathbb{N}$

$$\max_{s\geq 0} J(sy_n) = \int_0^\infty [\frac{1}{2}s_n^2|y_n'|^2 - \pi F(\frac{1}{2\sqrt{\pi}}s_n y_n)e^{-t}]dt \geq \frac{1}{2} \ .$$

This implies $s_n^2 \geq 1$. Furthermore, since $\frac{d}{ds}J(sy_n)|_{s=s_n} = 0$ we have for n sufficiently large, using condition (9)

$$\begin{aligned} s_n^2 &= \pi \int_0^\infty f(\frac{s_n}{2\sqrt{\pi}}y_n)\frac{s_n}{2\sqrt{\pi}}y_n e^{-t} \\ &= \pi \int_0^\infty h(\frac{s_n}{2\sqrt{\pi}}y_n)\frac{s_n}{2\sqrt{\pi}}y_n \cdot e^{s_n^2 y_n^2 - t} \\ &\geq (\beta - \epsilon)\pi \int_n^\infty e^{s_n^2(n-2\log n)-t} \ . \end{aligned}$$

We show that $s_n^2 \to 1$; assume that this is not so, i.e. suppose that there exists a subsequence of s_n with $s_n^2 \geq 1 + \delta$, for some $\delta > 0$. Then we have

$$s_n^2 \geq \ (\beta - \epsilon)\pi \int_n^\infty e^{(1+\delta)(n-2\log n)-t} = \ (\beta - \epsilon)\pi e^{\delta n - (1+\delta)2\log n}$$

This would imply that $s_n^2 \to +\infty$, which then yields a contradiction. Hence we must have

$$s_n^2 \ \to \ 1$$

We now estimate more precisely; fix $A > 0$ and set $[0, b_n) = \{t \in [0,\infty) : s_n y_n(t) < A\}$. Since $y_n(t) = \frac{t}{\sqrt{n}}\sqrt{1-\delta_n} \to 0$, for every fixed $t \geq 0$, we conclude that $b_n \to \infty$. Then we have

$$\begin{aligned} s_n^2 \geq (\beta - \epsilon)\pi \int_0^\infty e^{s_n^2 y_n^2 - t} + \pi \int_0^{b_n} f(\frac{s_n}{2\sqrt{\pi}}y_n)\frac{s_n}{2\sqrt{\pi}}y_n \qquad (23) \\ - (\beta - \epsilon)\pi \int_0^{b_n} e^{s_n^2 y_n^2 - t} \end{aligned}$$

The last integral in (23) goes to 1: indeed, we have

$$\int_0^{b_n} e^{-t} \le \int_0^{b_n} e^{s_n^2 u_n^2 - t} = \int_0^{b_\epsilon} e^{s_n^2 u_n^2 - t} + \int_{b_\epsilon}^{b_n} e^{s_n^2 u_n^2 - t} \ ,$$

where we choose for given $\epsilon > 0$ the number $b_\epsilon > 0$ such that

$$\int_{b_\epsilon}^{b_n} e^{s_n^2 y_n^2 - t} \le e^{s_n^2 A^2} \int_{b_\epsilon}^{b_n} e^{-t} \le \epsilon/2 \ , \ \forall n$$

Next, using that $y_n(t) \le \tau_n \to 0$ uniformly on $[0, b_\epsilon]$, choose N_ϵ sufficently large such that

$$\int_0^{b_\epsilon} e^{s_n^2 y_n^2 - t} \le e^{s_n^2 \tau_n^2} \int_0^{b_\epsilon} e^{-t} \le 1 + \epsilon/2 \ , \ \text{for } n \ge N_\epsilon$$

The second integral in (23) is positive, and in fact goes to zero (as can be seen using a similar argument). Hence we have in the limit, using theorem 4 for $N = 2$

$$\begin{aligned} 1 = \lim_{n\to\infty} s_n^2 &= (\beta - \epsilon)\pi \left[\lim_{n\to\infty} \int_0^\infty e^{s_n^2 y_n^2 - t} dt - 1 \right] \\ &\ge (\beta - \epsilon)\pi \left[\lim_{n\to\infty} \int_0^\infty e^{y_n^2 - t} dt - 1 \right] \\ &= (\beta - \epsilon)\ \pi e \end{aligned}$$

Thus, for $\beta > \frac{1}{e\pi}$ given, we obtain a contradiction, choosing $\epsilon > 0$ sufficiently small.

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