PERGAMON

Nonlinear Analysis 46 (2001) 495–510

Nonlinear Analysis

www.elsevier.com/locate/na

Local mountain-pass for a class of elliptic problems in $\mathbb{R}^N$ involving critical growth

C.O. Alves[a], João Marcos do Ó[b,*,1], M.A.S. Souto[a,1]

[a] *Departamento de Matemática e Estatística - Univ. Fed. Paraíba, 58109.970 Campina Grande, Pb, Brazil*

[b] *Departamento de Matemática, Univ. Fed. Paraíba, 58059.900 João Pessoa, Pb, Brazil*

Received 8 April 1999; accepted 19 February 2000

Keywords: Elliptic problems; Critical growth; Variational methods; Mountain-pass theorem; Schrödinger equation; Standing wave solutions

1. Introduction

In this paper we shall be concerned with the existence and the concentration behavior of positive bound-state solutions (solutions with bounded energy) for the problem

$$-\varepsilon^2 \Delta u + V(z)u = f(u) + u^{2^*-1}, \quad \text{in } \mathbb{R}^N,$$

$$u \in C^2(\mathbb{R}^N) \cap H^1(R^N),\ u(z) > 0 \quad \text{for all } z \in \mathbb{R}^N, \tag{P_ε}$$

where $\varepsilon > 0$; $2^* = 2N/(N-2)$, $N \geq 3$, is the critical Sobolev exponent; $V : \mathbb{R}^N \to \mathbb{R}$ is a locally Hölder continuous function satisfying

$$V(z) \geq \alpha > 0 \quad \text{for all } z \in \mathbb{R}^N, \tag{V_*}$$

and

$$\inf_{\Omega} V < \inf_{\partial\Omega} V \tag{V_{**}}$$

* Corresponding author.

E-mail address: jmbo@mat.ufpb.br (J.M. do Ó).

[1] Partially supported by CNPq/Brazil.

PII: S0362-546X(00)00125-5

for some bounded domain $\Omega \subset \mathbb{R}^N$; and the nonlinearity $f : \mathbb{R}_+ \to \mathbb{R}$ is locally Lipschitz and such that

(f_1) $f(s) = o_1(s)$ near the origin;
(f_2) there are q_1, $q_2 \in (1, 2^* - 1)$, $\lambda > 0$ such that

$$f(s) \geq \lambda s^{q_1} \quad \text{for all } s > 0 \text{ and } \lim_{s\to\infty} \frac{f(s)}{s^{q_2}} = 0$$

(when $N = 3$, we need $q_1 > 2$, otherwise we require a sufficiently large λ);
(f_3) for some $\theta \in (2, q_2 + 1)$ we have

$$0 < \theta F(s) \leq f(s)s \quad \text{for all } s > 0,$$

where $F(s) = \int_0^s f(t)\,dt$;
(f_4) the function $f(s)/s$ is increasing for $s > 0$.

Since we are interested in positive solutions, we define $f(s) = 0$ for $s < 0$.
Let us state our main result:

Theorem 1. *Suppose that the potential V satisfies (V_*)–(V_{**}) and f satisfies (f_1)–(f_4). Then there is an $\varepsilon_0 > 0$ such that problem (P_ε) possesses a positive bound state solution u_ε, for all $0 < \varepsilon < \varepsilon_0$. Moreover, u_ε possesses at most one local (hence global) maximum z_ε in $\mathbb{R}^N$, which is inside Ω, such that*

$$\lim_{\varepsilon\to 0^+} V(z_\varepsilon) = V_0 = \inf_\Omega V.$$

Besides, there are C and ζ, positive constants such that

$$u_\varepsilon(x) \leq C \exp\left(-\zeta \left|\frac{x - z_\varepsilon}{\varepsilon}\right|\right) \quad \text{for all } x \in \mathbb{R}^N.$$

We would like to remark that this kind of equation in (P_ε) arises from the problem of obtaining standing waves solutions of the nonlinear Schrödinger equation

$$\mathrm{i}\varepsilon\frac{\partial\psi}{\partial t} = -\varepsilon^2\Delta\psi + (V(z) + E)\psi - |\psi|^{-1}h(|\psi|)\psi \quad \text{in } \mathbb{R}^N \qquad (S_\varepsilon)$$

where $h(s) = f(s) + s^{2^*-1}$. A standing wave solution to problem (S_ε) is one in the form $\psi(x,t) = \exp(-\mathrm{i}\varepsilon^{-1}Et)u(x)$. In this case u is a solution of (P_ε).

Some recent works have treated this problem in the subcritical case and we cite a couple of them.

Floer and Weinstein [7] have studied the problem (S_ε) in the case $N = 1$, $h(s) = s^3$ and bounded potentials with nondegenerate critical point. They show that for small ε, this problem has a solution which concentrates around each nondegenerate point (see also [18] for a related work).

Roughly speaking, in this work, concentration behavior around the origin of a function means that it has the form $\psi(\varepsilon x)$ where ψ is a C^2 function with exponential decay.

Under the potential condition

$$\inf_{z\in\mathbb{R}^N} V < \liminf_{|z|\to\infty} V(z), \tag{1}$$

Rabinowitz [14] has proved the existence of positive ground-state solutions (solutions with minimal energy) for (P_ε), in the case where $h(s)$ behaves like $s^p, 1 < p < 2^* - 1$, for small ε. The remaining concentration behavior result in this subcritical case was obtained by Wang [17].

Alves and Souto in [2] have established existence and concentration behavior of ground-state solutions when the nonlinearity has the critical form $h(s) = \lambda s^q + s^{2^*-1}$, $1 < q < 2^* - 1$, under condition (1).

In the very interesting article [5], del Pino and Felmer have obtained the complete treatment (existence and concentration behavior of solutions) with the potential under conditions (V_*) and (V_{**}). They have obtained bound-state solutions but not ground-state solutions, and this is reasonable, because some problems under condition (V_{**}) do not admit any ground-state solution (see, for example, Theorem 4 of [2]). From this reason we cannot look for minimax critical points of the energy functional $J_\varepsilon : E \to \mathbb{R}$,

$$J_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} (\varepsilon^2|\nabla u|^2 + V(z)u^2)\,\mathrm{d}z - \int_{\mathbb{R}^N} F(u_+)\,\mathrm{d}z - \frac{1}{2^*}\int_{\mathbb{R}^N} u_+^{2^*}\,\mathrm{d}z$$

(where $u^+ = \max\{u, 0\}$), defined on the Hilbert space

$$E = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(z)u^2\,\mathrm{d}z < \infty \right\},$$

associated to (P_ε).

A way to solve this problem is to modify the nonlinearity into one more convenient in order to apply the mountain-pass theorem. Namely, we will consider the following Carathéodory function:

$$g(z,s) = \begin{cases} \chi_\Omega(f(s) + s^{2^*-1}) + \chi_D \tilde{f}(s) & \text{if } s \geq 0, \\ 0 & \text{if } s < 0, \end{cases}$$

where

$$\tilde{f}(s) = \begin{cases} f(s) + s^{2^*-1} & \text{if } s \leq a, \\ \dfrac{\alpha s}{k} & \text{if } s > a, \end{cases}$$

$k > \theta(\theta-2)^{-1} > 1$, $a > 0$ is such that $f(a) + a^{2^*-1} = k^{-1}a\alpha$, $D = \mathbb{R}^N \setminus \bar{\Omega}$ and χ_A denotes the characteristic function of subset A of $\mathbb{R}^N$. Similar modified nonlinearity has been used by del Pino and Felmer in [5] to study the subcritical case.

We have organized the paper as follows. In the second section, using the mountain-pass theorem, we shall prove that the functional associated to the modified problem possesses a critical point. In the following section we shall prove Theorem 1 by a couple of lemmas which show that for small ε this critical point of the modified functional has a concentration behavior and it is also critical point of functional J_ε.

2. The modified functional

In this section we will consider the energy functional $J : E \to \mathbb{R}^N$ given by

$$J(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} G(z,u)\,\mathrm{d}z,$$

where

$$\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(z)u^2)\,\mathrm{d}z.$$

Here $G(z,s) = \chi_\Omega(F(s) + \frac{1}{2^*}s^{2^*}) + \chi_D\tilde{F}(s)$ and $\tilde{F}(s) = \int_0^s \tilde{f}(t)\,\mathrm{d}t$.

Notice that, using (f_1)–(f_4), it is easy to check that

(g_1) $g(z,s) = f(s) + s^{2^*-1} = o(s)$, near the origin, uniformly in $z \in \mathbb{R}^N$;

(g_2) $g(z,s) \le f(s) + s^{2^*-1}$ for all $s > 0$, $z \in \mathbb{R}^N$;

(g_3) $0 < \theta G(z,s) \le g(z,s)s$, for all $z \in \Omega$, $s > 0$ or $z \in D$ and $s \le a$;

and

$$0 \le 2G(z,s) \le g(z,s)s \le \frac{1}{k}V(z)s^2 \quad \text{for all } z \in D,\ s > 0,$$

(g_4) the function $s^{-1}g(z,s)$ is increasing in $s > 0$ for each z fixed.

Lemma 2. *J has the mountain-pass geometry.*

Proof. Let η be a nonzero function in $C_0^\infty(\Omega)$, such that $\eta \ge 0$. Then for all $t > 0$

$$J(t\eta) \le \frac{t^2}{2}\|\eta\|^2 - \frac{t^{2^*}}{2^*}\int_{\mathbb{R}^N} \eta^{2^*}\,\mathrm{d}z.$$

Of course, $J(t\eta) \to -\infty$ as $t \to \infty$. As the usual way, from (g_2) $J(u) = \frac{1}{2}\|u\|^2 + o(\|u\|)$, near origin. This completes the proof. □

Now in view of Lemma 2, we can apply a version of the mountain-pass theorem without $(P.S.)$ condition (see [11]) to obtain a sequence $\{u_n\}$ such that

$$J(u_n) = c + o_n(1) \quad \text{and} \quad |J'(u_n)| = o_n(1), \tag{2}$$

where c is the minimax level of functional J given by

$$0 < c = \inf_{g \in \Gamma} \max_{0 \le t \le 1} J(g(t)), \tag{3}$$

where $\Gamma = \{g \in C^0([0,1];E) : \text{such that } g(0) = 0 \text{ and } J(g(1)) \le 0\}$. Throughout this work $o_n(1)$ denotes a sequence converging to zero as $n \to \infty$.

Remark 1. As in [5,6], we shall use the equivalent characterization of c more adequate to our purpose, given by

$$c = \inf_{v \in E \setminus \{0\}} \max_{t \ge 0} J(tv).$$

Furthermore, it is easy to check that for each non-negative $v \in E - \{0\}$ there is a unique $t_0 = t_0(v)$ such that

$$J(t_0 v) = \max_{t \geq 0} J(tv).$$

We denote by S the best Sobolev constant for the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$.

Lemma 3. *There is a* $v \in E \setminus \{0\}$ *such that*

$$\max_{t>0} J(tv) < \frac{1}{N} S^{N/2},$$

therefore we have this estimate for the minimax level (3) *of* J:

$$c < \frac{1}{N} S^{N/2}.$$

Proof. For each $h > 0$, consider the function

$$\psi_h(z) = \frac{[N(N-2)h]^{(N-2)/4}}{(h + |z|^2)^{(N-2)/2}}.$$

We recall that ψ_h satisfies the problem

$$-\Delta u = u^{2^*-1}, \quad \mathbb{R}^N,$$

$$u(z) > 0, \quad \int_{\mathbb{R}^N} |\nabla u|^2 \, \mathrm{d}z < \infty$$

and

$$\int_{\mathbb{R}^N} |\nabla \psi_h|^2 \, \mathrm{d}z = \int_{\mathbb{R}^N} \psi_h^{2^*} \, \mathrm{d}z = S^{N/2}$$

(see [16]). Now, consider $v_h(z) = (\varphi\psi_h(z)/\|\varphi\psi_h\|_{L^{2^*}})$ where $\varphi \in C_0^\infty(\mathbb{R}^N)$, $0 \leq \varphi(z) \leq 1$ and

$$\varphi(z) = \begin{cases} 1, & z \in B_1, \\ 0, & z \notin B_2, \end{cases}$$

where $B_1 \subset B_2 \subset\subset \Omega$ are concentric balls of radius ρ and 2ρ, respectively. From condition (f_2) we have

$$J(tv_h) \leq \frac{t^2}{2} \int_{B_2} (|\nabla v_h|^2 + \|V\|_{L^\infty(\Omega)} v_h^2) \, \mathrm{d}z - \frac{\lambda t^{q_1+1}}{q_1+1} \int_{B_2} v_h^{q_1+1} \, \mathrm{d}z - \frac{t^{2^*}}{2^*}. \tag{4}$$

Using the same arguments explored in [12], there exists $h > 0$ such that

$$\max_{t \geq 0} \left\{ \int_{B_2} \left[\frac{t^2}{2} (|\nabla v_h|^2 + \|V\|_{L^\infty(\Omega)} v_h^2) - \frac{\lambda t^{q_1+1}}{q_1+1} v_h^{q_1+1} \right] \mathrm{d}z - \frac{t^{2^*}}{2^*} \right\} < \frac{1}{N} S^{N/2}. \tag{5}$$

Therefore, from (4) and (5) we have that

$$\max_{t \geq 0} J(tv_h) < \frac{1}{N} S^{N/2}$$

and the proof of the lemma is complete. $\square$

Lemma 4. *Every sequence* $\{u_n\}$ *satisfying* (2) *is bounded in* E.

Proof. To check this, observe that using (g_3) we have

$$\begin{aligned}
J(u_n) - \frac{1}{\theta}J'(u_n) &= \left(\frac{1}{2}-\frac{1}{\theta}\right)\|u_n\|^2 + \frac{1}{\theta}\int_{\mathbb{R}^N}[u_n g(z,u_n) - \theta G(z,u_n)]\,\mathrm{d}z \\
&\geq \left(\frac{1}{2}-\frac{1}{\theta}\right)\|u_n\|^2 + \frac{1}{\theta}\int_{D}[u_n g(z,u_n) - \theta G(z,u_n)]\,\mathrm{d}z \\
&\geq \left(\frac{1}{2}-\frac{1}{\theta}\right)\|u_n\|^2 + \left(\frac{2-\theta}{\theta}\right)\int_{D} G(z,u_n)\,\mathrm{d}z \\
&\geq \left(\frac{1}{2}-\frac{1}{\theta}\right)\|u_n\|^2 + \left(\frac{2-\theta}{2k\theta}\right)\int_{D} V(z)u_n^2\,\mathrm{d}z \\
&= \left(\frac{\theta-2}{2\theta}\right)\int_{\mathbb{R}^N}\left[|\nabla u_n|^2 + \left(1-\frac{1}{k}\right)V(z)u_n^2\right]\mathrm{d}z. \qquad (6)
\end{aligned}$$

Applying (2) in inequality (6) we have the upper bound of $\|u_n\|$. □

Lemma 5. *There is a sequence* $\{z_n\}\subset\mathbb{R}^N$ *and* $R>0$, $\beta>0$ *such that*

$$\int_{B_R(z_n)} u_n^2\,\mathrm{d}z \geq \beta.$$

Proof. Suppose by contradiction that the lemma does not hold. Then by Lion's result (see [10] or [4]) it follows that

$$\int_{\mathbb{R}^N}|u_n|^q\,\mathrm{d}z = o_n(1), \quad \text{as } n\to\infty \quad \text{for all } 2<q<2^*$$

and then

$$\int_{\mathbb{R}^N} F(u_n)\,\mathrm{d}z = \int_{\mathbb{R}^N} u_n f(u_n)\,\mathrm{d}z = o_n(1).$$

This implies that

$$\int_{\mathbb{R}^N} G(z,u_n)\,\mathrm{d}z \leq \frac{1}{2^*}\int_{\Omega\cup\{u_n\leq a\}}(u_n^+)^{2^*}\,\mathrm{d}z + \frac{\alpha}{2k}\int_{D\cap\{u_n>a\}}u_n^2\,\mathrm{d}z + o_n(1) \qquad (7)$$

and

$$\int_{\mathbb{R}^N} u_n g(z,u_n)\,\mathrm{d}z = \int_{\Omega\cup\{u_n\leq a\}}(u_n^+)^{2^*}\,\mathrm{d}z + \frac{\alpha}{k}\int_{D\cap\{u_n>a\}}u_n^2\,\mathrm{d}z + o_n(1). \qquad (8)$$

From equality (8) and $J'(u_n)\cdot u_n = o_n(1)$, we conclude that

$$\|u_n\|^2 - \frac{\alpha}{k}\int_{D\cap\{u_n>a\}}u_n^2\,\mathrm{d}z + o_n(1) = \int_{\Omega\cup\{u_n\leq a\}}(u_n^+)^{2^*}\,\mathrm{d}z. \qquad (9)$$

Let $\ell \geq 0$ be such that

$$\|u_n\|^2 - \frac{\alpha}{k}\int_{D\cap\{u_n>a\}} u_n^2\,\mathrm{d}z \to \ell.$$

It is easy to check that $\ell > 0$, otherwise we have $u_n \to 0$ which contradicts $c > 0$. From (9)

$$\int_{\Omega\cup\{u_n\leq a\}} (u_n^+)^{2^*}\,\mathrm{d}z \to \ell.$$

From inequality (7) and $J(u_n) = c + o_n(1)$ we have

$$\ell \leq Nc \tag{10}$$

and hence $\ell > 0$. Now, using the definition of the constant S, we have

$$\|u_n\|^2 - \frac{\alpha}{k}\int_{D\cap\{u_n>a\}} u_n^2\,\mathrm{d}z \geq S\left(\int_{\Omega\cup\{u_n\leq a\}} u_n^{2^*}\,\mathrm{d}z\right)^{2/2^*}.$$

Taking the limit in the above inequality, as $n \to \infty$, we achieve that

$$\ell \geq S\ell^{2/2^*},$$

which, together with (10), implies

$$c \geq \frac{1}{N}S^{N/2},$$

which contradicts Lemma 3. □

Lemma 6. *The sequence $\{z_n\}$ is bounded in $\mathbb{R}^N$.*

Proof. For each $\rho > 0$ consider a differentiable function ψ_ρ such that

$$\psi_\rho(z) = \begin{cases} 0 & \text{if } |z| \leq \rho, \\ 1 & \text{if } |z| \geq 2\rho \end{cases}$$

and $|\nabla\psi_\rho(z)| \leq C\rho^{-1}$, for all $z \in \mathbb{R}^N$. Use $J'(u_n)(\psi_\rho u_n) = o_n(1)$ to obtain

$$\begin{aligned}\alpha\left(1 - \frac{1}{k}\right)\int_{\mathbb{R}^N} u_n^2\psi_\rho &\leq \int_{\mathbb{R}^N}\left[|\nabla u_n|^2 + \left(V(z) - \frac{\alpha}{k}\right)u_n^2\right]\psi_\rho\,\mathrm{d}z \\ &= -\int_{\mathbb{R}^N} u_n\nabla\psi_\rho\nabla u_n\,\mathrm{d}z \\ &\quad + \int_{\mathbb{R}^N}\left[g(z,u_n)u_n - \frac{\alpha}{k}u_n^2\right]\psi_\rho\,\mathrm{d}z + o_n(1).\end{aligned}$$

If ρ is large enough, we have $\Omega \subset B_\rho(0)$. Furthermore, from (g_3) we get

$$\alpha\left(1 - \frac{1}{k}\right)\int_{\mathbb{R}^N} u_n^2\psi_\rho\,\mathrm{d}z \leq \frac{C}{\rho}\|u_n\|_{H^1}^2 + o_n(1). \tag{11}$$

The inequality (11) completes the proof. □

Using standard arguments, up to subsequence, we may assume that there is $u \in H$ such that

$$u_n \rightharpoonup u \text{ in } \quad H, \quad u_n \to u \text{ in } L^q_{\mathrm{loc}}(\mathbb{R}^N) \quad \text{and} \quad u_n(x) \to u(x) \quad \text{a.e. in } \mathbb{R}^N.$$

From Lemmas 5 and 6, u is nontrivial and using the weak convergence of u_n we can see that u is a critical point of J. Finally, from Remark 1, (g_3) and Fatou's lemma

$$\begin{aligned} c \le J(u) &= J(u) - \frac{1}{2}J'(u)u \\ &= \frac{1}{2}\int_{\mathbb{R}^N} [ug(z,u) - 2G(z,u)]\,\mathrm{d}z \\ &\le \liminf_n \left\{\frac{1}{2}\int_{\mathbb{R}^N} [u_n g(z,u_n) - 2G(z,u_n)]\,\mathrm{d}z\right\} \\ &= \liminf_n \left\{J(u_n) - \frac{1}{2}J'(u_n)u_n\right\} = c \end{aligned}$$

and then u is a solution with minimal energy $J(u) = c$.

We have proved up this moment the following result:

Proposition 7. *For all $\varepsilon > 0$, there is a positive critical point $u_\varepsilon \in E$ associated to the functional*

$$J_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} (\varepsilon^2|\nabla u|^2 + V(z)u^2\,\mathrm{d}z) - \int_{\mathbb{R}^N} G(z,u)\,\mathrm{d}z$$

at the level

$$c_\varepsilon = \inf_{v\in E\setminus\{0\}} \max_{t\ge 0} J_\varepsilon(tv).$$

3. Proof of Theorem 1

In order to proof Theorem 1, let us fix some notations. First we suppose, without loss of generality that $\partial\Omega$ is smooth and $0 \in \Omega$. Furthermore,

$$V(0) = V_0 = \inf_{\Omega} V.$$

We will denote by $I_0 : H^1(\mathbb{R}^N) \to \mathbb{R}$ the functional given by

$$I_0(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V_0 u^2)\,\mathrm{d}x - \int_{\mathbb{R}^N} \left[F(u) + \frac{1}{2^*}(u^+)^{2^*}\right]\,\mathrm{d}x,$$

associated to the autonomous problem

$$-\Delta u + V_0 u = f(u) + |u|^{2^*-2}u, \quad \text{in } \mathbb{R}^N. \tag{12}$$

It is known that under assumptions (f_1)–(f_4), (12) possesses a ground-state solution ω at the level

$$c_0 = I_0(\omega) = \inf_{v\in H^1\setminus\{0\}} \max_{t\ge 0} I_0(tv) \tag{13}$$

(see [1] for instance). Furthermore,

$$c_0 < \frac{1}{N} S^{N/2}.$$

Remark 2. The dependence of the mountain-pass level c_0 on the potential V_0 is continuous and increasing (see [14]).

From a result due to Gidas et al. [8], any solution of problem (12) must be spherically symmetric about some point in $\mathbb{R}^N$ and $\partial u/\partial r < 0$ for all $r > 0$, where r is the radial coordinate.

Let I_ε denote the energy functional

$$I_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(\varepsilon x)u^2)\,\mathrm{d}x - \int_{\mathbb{R}^N} G(\varepsilon x, u)\,\mathrm{d}x,$$

defined in

$$E_\varepsilon = \left\{ u \in H^1(\mathbb{R}^N)\colon \int_{\mathbb{R}^N} V(\varepsilon x)u^2\,\mathrm{d}x < \infty \right\},$$

associated to the problem

$$-\Delta u + V(\varepsilon x)u = g(\varepsilon x, u) \quad \text{in } \mathbb{R}^N. \qquad (MP_\varepsilon)$$

From Proposition 7, the family of nonnegative functions

$$v_\varepsilon(x) = u_\varepsilon(z) = u_\varepsilon(\varepsilon x), \quad z = \varepsilon x$$

is such that each v_ε is a critical point of I_ε at the level

$$b_\varepsilon = I_\varepsilon(v_\varepsilon) = \inf_{v \in E_\varepsilon \setminus \{0\}} \max_{t \geq 0} I_\varepsilon(tv).$$

It is easy to check that $b_\varepsilon = \varepsilon^{-N} c_\varepsilon$. Furthermore, from Lemma 3, for each $\varepsilon > 0$ we have

$$b_\varepsilon < \frac{1}{N} S^{N/2}. \qquad (14)$$

Moreover,

Lemma 8. $\limsup_{\varepsilon \to 0} b_\varepsilon \leq c_0$, *the mountain-pass minimax level of* I_0.

Proof. Fix ω defined in (13) and consider $\omega_\varepsilon(x) = \varphi(\varepsilon x)\omega(x)$, where φ is the function defined in Lemma 3. Here we assume that $B_1 = B_\rho(0)$, $B_2 = B_{2\rho}(0) \subset \Omega$. It is easy to see that $\omega_\varepsilon \to \omega$ in $H^1(\mathbb{R}^N)$, $I_0(\omega_\varepsilon) \to I_0(\omega)$, as $\varepsilon \to 0^+$, and the support of ω_ε is in $\Omega_\varepsilon = \{x \in \mathbb{R}^N; \varepsilon x \in \Omega\}$. By definition of b_ε we have

$$b_\varepsilon \leq \max_{t>0} I_\varepsilon(t\omega_\varepsilon) = I_\varepsilon(t_\varepsilon \omega_\varepsilon)$$

$$= \frac{t_\varepsilon^2}{2} \int_{\mathbb{R}^N} [|\nabla \omega_\varepsilon|^2 + V(\varepsilon x)\omega_\varepsilon^2]\,\mathrm{d}x - \int_{\mathbb{R}^N} \left[F(t_\varepsilon \omega_\varepsilon) + \frac{t_\varepsilon^{2^*}}{2^*} \omega_\varepsilon^{2^*} \right] \mathrm{d}x, \qquad (15)$$

for some $t_\varepsilon > 0$. It is easy to verify that $t_\varepsilon \to 1$ as $\varepsilon \to 0^+$. On the other hand,

$$I_\varepsilon(t_\varepsilon\omega_\varepsilon) = I_0(t_\varepsilon\omega_\varepsilon) + \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^N}(V(\varepsilon x) - V_0)\omega_\varepsilon^2\,\mathrm{d}x. \tag{16}$$

Since $V(\varepsilon x)$ is bounded on the support of ω_ε, by the Lebesgue Dominated Convergence Theorem and (15) and (16), we conclude the proof. □

Notice that $I_\varepsilon(v_\varepsilon) \le c_0 + o_\varepsilon(1)$, where $o_\varepsilon(1)$ goes to zero as $\varepsilon \to 0$. From (V_*) and (g_2), we have

$$I_\varepsilon(u) \ge \bar{I}(u) \doteq \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + \alpha u^2)\,\mathrm{d}x - \int_{\mathbb{R}^N}\left[F(u) + \frac{1}{2^*}(u^+)^{2^*}\right]\mathrm{d}x.$$

Hence, b_ε is bounded from below by $\bar{c} > 0$, the minimax level of functional $\bar{I}$.

Now, using similar arguments as of Lemma 5, we have the following result.

Lemma 9. *There are $\varepsilon_0 > 0$, a family $\{y_\varepsilon\}_{\{0<\varepsilon\le\varepsilon_0\}} \subset \mathbb{R}^N$ and positive constants R, β such that*

$$\int_{B_R(y_\varepsilon)} v_\varepsilon^2\,\mathrm{d}x \ge \beta, \quad \textit{for all } 0 < \varepsilon \le \varepsilon_0.$$

Lemma 10. *$\varepsilon y_\varepsilon$ is bounded in $\mathbb{R}^N$. Moreover, $dist(\varepsilon y_\varepsilon, \Omega) \le \varepsilon R$*

Proof. For $\delta > 0$, we define $K_\delta = \{x \in \mathbb{R}^N\colon dist(x,\Omega) \le \delta\}$. We set $\phi_\varepsilon(x) = \phi(\varepsilon x)$, where $\phi \in C^\infty(\mathbb{R}^N,[0,1])$ is such that

$$\phi(x) = \begin{cases} 1, & x \notin K_\delta, \\ 0, & x \in \Omega \end{cases}$$

and $|\nabla\phi| \le C\delta^{-1}$. Taking $v_\varepsilon\phi_\varepsilon$ as test function, using property (g_3) and the fact that the support of ϕ_ε does not intercept Ω_ε, we obtain

$$\begin{aligned}\alpha\left(1 - \frac{1}{k}\right)\int_{\mathbb{R}^N} v_\varepsilon^2\phi_\varepsilon\,\mathrm{d}x &\le \int_{\mathbb{R}^N}\left[|\nabla v_\varepsilon|^2 + \left(V(\varepsilon x) - \frac{\alpha}{k}\right)v_\varepsilon^2\right]\phi_\varepsilon\,\mathrm{d}x \\ &= -\int_{\mathbb{R}^N} v_\varepsilon\nabla\phi_\varepsilon\nabla v_\varepsilon\,\mathrm{d}x + \int_{\mathbb{R}^N}\left[v_\varepsilon\tilde{f}(v_\varepsilon) - \frac{\alpha}{k}v_\varepsilon^2\right]\phi_\varepsilon\,\mathrm{d}x \\ &\le -\int_{\mathbb{R}^N} v_\varepsilon\nabla\phi_\varepsilon\nabla v_\varepsilon\,\mathrm{d}x.\end{aligned}$$

So we have

$$\alpha\left(1 - \frac{1}{k}\right)\int_{\mathbb{R}^N} v_\varepsilon^2\phi_\varepsilon\,\mathrm{d}x \le C\delta^{-1}\varepsilon\|v_\varepsilon\|_{H^1}^2. \tag{17}$$

If for some sequence $\varepsilon_n \searrow 0$ we have

$$B_R(y_{\varepsilon_n}) \cap \{x \in \mathbb{R}^N; \varepsilon_n x \in K_\delta\} = \emptyset \tag{18}$$

then

$$\alpha\left(1-\frac{1}{k}\right)\int_{B_R(y_{\varepsilon_n})} v_{\varepsilon_n}^2\,\mathrm{d}x \le C\delta^{-1}\varepsilon_n\|v_{\varepsilon_n}\|_{H^1}^2,$$

which contradicts Lemma 9. Thus (18) does not hold, that is, for all ε there is an x such that $\varepsilon x\in K_\delta$ and $|x-y_\varepsilon|\le R$. It is easy to verify that this implies $dist(\varepsilon y_\varepsilon,\Omega)\le\varepsilon R+\delta$. From this fact we conclude the proof. □

From Lemma 10 we can suppose that the family $\{y_\varepsilon\}$, defined in Lemma 9, can be taken such that $\varepsilon y_\varepsilon\in\Omega$ for all $\varepsilon\in(0,\varepsilon_0]$. Indeed, if not, we replace y_ε by $\varepsilon^{-1}x_\varepsilon$, where x_ε comes from Lemma 10, so that $|\varepsilon y_\varepsilon-x_\varepsilon|\le\varepsilon R$. Observing that $|y_\varepsilon-(x_\varepsilon/\varepsilon)|\le R$, we can replace R by $2R$ in Lemma 9.

Let us consider the following subset of $\mathbb{R}^N$:

$$E_\varepsilon=\{x\in\mathbb{R}^N\colon v_\varepsilon(x)\ge a \text{ and } \varepsilon x\notin\Omega\}.$$

Observe that, in fact, we want to show that E_ε is empty if ε is small. Let F_ε be the following translation of E_ε:

$$F_\varepsilon=\{x\in\mathbb{R}^N\colon v_\varepsilon(x+y_\varepsilon)\ge a \text{ and } \varepsilon x+\varepsilon y_\varepsilon\notin\Omega\}.$$

It is easy to see that $|E_\varepsilon|=|F_\varepsilon|$ ($|A|$ indicates the Lebesgue measure of the subset A).

Lemma 11. *The following limits hold*:
(i) $\lim_{\varepsilon\to0}V(\varepsilon y_\varepsilon)=V_0$;
(ii) $\lim_{\varepsilon\to0}b_\varepsilon=c_0$;
(iii) $\lim_{\varepsilon\to0}|E_\varepsilon|=0$.

Proof. If $\varepsilon_n\searrow0$ and $y_n=y_{\varepsilon_n}$ are such that $\varepsilon_n y_n\to x_0$, we must prove that $V(x_0)=V_0$. We already know that $x_0\in\bar\Omega$, that is, $V(x_0)\ge V_0$. Let us set $v_n(x)=v_{\varepsilon_n}(x)$, $\omega_n(x)=v_{\varepsilon_n}(x+y_n)$, $E_n=E_{\varepsilon_n}$ and $F_n=F_{\varepsilon_n}$. From Lemma 9 we have

$$\int_{B_R(0)}\omega_n^2\,\mathrm{d}x\ge\beta>0,\quad\text{for all } n,$$

$$-\Delta\omega_n+V(\varepsilon_n x+\varepsilon_n y_n)\omega_n=g(\varepsilon_n x+\varepsilon_n y_n,\omega_n)\quad\text{in }\mathbb{R}^N$$

and $\|\omega_n\|_{H^1}=\|v_n\|_{H^1}$ is bounded. Let $\omega\in H^1(\mathbb{R}^N)$ such that $\omega_n\rightharpoonup\omega$ in $H^1(\mathbb{R}^N)$. We have $\omega\ge0$, $\omega\ne0$ and

$$-\Delta\omega+V(x_0)\omega=\chi(x)[f(\omega)+\omega^{2^*-1}]+(1-\chi(x))\tilde f(\omega)=\tilde g(x,\omega)\quad\text{in }\mathbb{R}^N,\tag{19}$$

where $\chi(x)=\lim_n\chi_\Omega(\varepsilon_n x+\varepsilon_n y_n)$ almost everywhere in $\mathbb{R}^N$. It is easy to verify that if $x_0\in\Omega$ we have $\chi(x)=1$ for all $x\in\mathbb{R}^N$ and therefore ω is a solution of the problem

$$-\Delta\omega+V(x_0)\omega=f(\omega)+\omega^{2^*-1}\quad\text{in }\mathbb{R}^N.\tag{20}$$

On the other hand, if $x_0\in\partial\Omega$, without loss of generality, we may suppose that the outer normal vector ν in x_0 is $(1,0,\dots,0)$. Let $P=\{x\in\mathbb{R}^N\colon x_1<0\}$. Observe that on P, $\chi\equiv1$. In fact, for each $x\in P$,

$$\varepsilon_n x+\varepsilon_n y_n\in\Omega,\quad\text{for all large } n.$$

Then in both cases, $\tilde{g}(x,s)=f(s)+s^{2^*-1}$ for all $x \in P$. This implies that the functional energy $\tilde{I}$ associated to problem (19) has the same minimax level $\tilde{c}$ of the functional $\tilde{I}_{x_0}$ associated to problem (20). In effect, for all $u \in H^1(\mathbb{R}^N)$ we have $\tilde{I}_{x_0}(u) \le \tilde{I}(u)$ and then $C_{x_0} \le \tilde{c}$ (where C_{x_0} is the minimax level of $\tilde{I}_{x_0}$). On the other hand $\tilde{I}_{x_0}(u)=\tilde{I}(u)$, for all u with support inside P.

From the autonomous case (see Remark 2) theory, $c_0 \le \tilde{I}(\omega)$. It is easy to check, from Fatou's Lemma and Lemma 8, that

$$
\begin{aligned}
c_0 \le \tilde{I}(\omega) &= \frac{1}{2}\int_{\mathbb{R}^N} [\omega\tilde{g}(x,\omega) - 2\tilde{G}(x,\omega)]\,\mathrm{d}x \\
&\le \liminf_{n\to\infty}\left\{\frac{1}{2}\int_{\mathbb{R}^N\setminus F_n} [\omega_n g(\varepsilon_n x + \varepsilon_n y_n, \omega_n) - 2G(\varepsilon_n x + \varepsilon_n y_n, \omega_n)]\,\mathrm{d}x\right\} \\
&\le \liminf_{n\to\infty}\left\{\frac{1}{2}\int_{\mathbb{R}^N\setminus E_n} [v_n g(\varepsilon_n x, v_n) - 2G(\varepsilon_n x, v_n)]\,\mathrm{d}x\right\} \\
&= \liminf_{n\to\infty}\left\{I_{\varepsilon_n}(v_{\varepsilon_n}) - \frac{1}{2}I'_{\varepsilon_n}(v_{\varepsilon_n})v_{\varepsilon_n}\right\} \le c_0,
\end{aligned} \tag{21}
$$

where $\tilde{G}$ denotes the primitive of $\tilde{g}$. Thus, (ii) follows from (21).

Suppose that limit (i) does not hold, that is $V(x_0) > V_0$. It comes from Remark 2,

$$c_0 < \tilde{c} \le \tilde{I}(\omega) = c_0$$

which is a contradiction, then $V(x_0) = V_0$. To show the part (iii) we have from (21) that

$$\lim_{n\to\infty} \frac{1}{2}\int_{\mathbb{R}^N\setminus E_n} [v_n g(\varepsilon_n x, v_n) - 2G(\varepsilon_n x, v_n)]\,\mathrm{d}x = c_0. \tag{22}$$

The same approach used in the whole $\mathbb{R}^N$ instead of $\mathbb{R}^N\setminus E_n$ in inequality (21) also shows that

$$\lim_{n\to\infty} \frac{1}{2}\int_{\mathbb{R}^N} [v_n g(\varepsilon_n x, v_n) - 2G(\varepsilon_n x, v_n)]\,\mathrm{d}x = c_0. \tag{23}$$

Since (22) and (23) hold, we have

$$\lim_{n\to\infty} \frac{1}{2}\int_{E_n} [v_n g(\varepsilon_n x, v_n) - 2G(\varepsilon_n x, v_n)]\,\mathrm{d}x = 0.$$

But, from definition of $\tilde{F}$ we have

$$\int_{E_n} [v_n g(\varepsilon_n x, v_n) - 2G(\varepsilon_n x, v_n)]\,\mathrm{d}x = \left[\frac{\alpha}{k}a^2 - 2F(a) - 2\frac{a^{2^*}}{2^*}\right]|E_n| \ge 0$$

and this implies the limit (iii). □

From the proof of Lemma 11, ω_n converges in the weak sense to a ω that is a solution of problem (12) satisfying $I_0(\omega) = c_0$, that is, ω is a ground state solution of

(12). From that proof we also have that $b_{\varepsilon_n} \to c_0$ and that $g(\varepsilon_n x + \varepsilon_n y_n, s)$ converges uniformly over compacts to $f(s) + s^{2^*-1}$. Moreover, from (g_3)

$$\int_{F_\varepsilon} \omega_\varepsilon g(\varepsilon x + \varepsilon y_\varepsilon, \omega_\varepsilon)\,\mathrm{d}x \le \frac{\alpha}{k}\int_{F_\varepsilon} \omega_\varepsilon^2\,\mathrm{d}x \le \frac{\alpha}{k}\|\omega_\varepsilon\|_{L^{2^*}}^2 |F_\varepsilon|^{(2^*-2)/2^*} = o_\varepsilon(1). \tag{24}$$

In the same way we have

$$\int_{F_\varepsilon} G(\varepsilon x + \varepsilon y_\varepsilon, \omega_\varepsilon)\,\mathrm{d}x = o_\varepsilon(1). \tag{25}$$

Also from (22) and the definition of g we have

$$\begin{aligned} 2c_0 + o_n(1) &= \int_{\mathbb{R}^N\setminus F_n} [\omega_n g(\varepsilon_n x + \varepsilon_n y_n, \omega_n) - 2G(\varepsilon_n x + \varepsilon_n y_n, \omega_n)]\,\mathrm{d}x \\ &= \int_{\mathbb{R}^N\setminus F_n} [\omega_n f(\omega_n) - 2F(\omega_n)]\,\mathrm{d}x + \left(1 - \frac{2}{2^*}\right)\int_{\mathbb{R}^N\setminus F_n} \omega_n^{2^*}\,\mathrm{d}x, \end{aligned}$$

which implies that

$$\int_{\mathbb{R}^N\setminus F_n} \omega_n^{2^*}\,\mathrm{d}x \to \int_{\mathbb{R}^N} \omega^{2^*}\,\mathrm{d}x, \tag{26}$$

and

$$\int_{\mathbb{R}^N\setminus F_n} [\omega_n f(\omega_n) - 2F(\omega_n)]\,\mathrm{d}x \to \int_{\mathbb{R}^N} [\omega f(\omega) - 2F(\omega)]\,\mathrm{d}x.$$

From limit (26) we have proved the following result.

Lemma 12. *$\omega_\varepsilon \chi_{\mathbb{R}^N\setminus F_\varepsilon}$ converges to ω in $L^{2^*}(\mathbb{R}^N)$.*

In order to prove the concentration of the solution, we state the following two fundamental propositions. The first one is an adequate version of a result due to Brezis and Kato [3] (see also [15] for the details).

Proposition 13. *Let $v \in H_0^1(\Lambda)$, $\Lambda \subset \mathbb{R}^N$ open, satisfying*

$$-\Delta v + (b(x) - q(x))v = \tilde{f}(x, v) \quad \text{in } \Lambda,$$

where $\tilde{f}\colon \Lambda \times \mathbb{R} \to \mathbb{R}_+$ is a Carathéodory function such that

$$0 \le \tilde{f}(x,s) \le C_{\tilde{f}}(s^r + s), \quad \text{for all } s > 0,\ x \in \Lambda,$$

$b \in C(\Lambda, [0, +\infty))$, $q \in L^{N/2}(\Lambda)$, $1 < r < (N+2)/(N-2)$. Then $v \in L^p(\Lambda)$ for all $2 \le p < \infty$, and there is a positive constant C_p depending on p, q and $C_{\tilde{f}}$ such that

$$\|v\|_{L^p(\Lambda)} \le C_p \|v\|_{H^1(\Lambda)}.$$

Moreover, the dependence on q of C_p can be given uniformly on Cauchy sequences q_k in $L^{N/2}$.

The next proposition is a very particular version of Theorem 8.17 in [9].

Proposition 14. *Suppose that $t > N$, $\tilde{g} \in L^{t/2}(\Lambda)$ and $u \in H^1(\Lambda)$ satisfies in the weak sense*

$$-\Delta u \le \tilde{g}(x) \quad \text{in } \Omega$$

where Λ is an open subset of $\mathbb{R}^N$. Then for any ball $B_{2R}(y) \subset \Lambda$,

$$\sup_{B_R(y)} u \le C(\|u^+\|_{L^2(B_{2R}(y))} + \|\tilde{g}\|_{L^{t/2}(B_{2R}(y))})$$

where C depends on N, t and R.

From Lemma 12, the sequence $\omega_n^{2^*-2}$ is a Cauchy sequence in $L^{N/2}$. Using Proposition 13, with $q(x) = \omega_n^{2^*-2}\chi_\Omega(\varepsilon_n x + \varepsilon_n y_n) \in L^{N/2}$; $b(x) = V(\varepsilon_n x + \varepsilon_n y_n)$ and $\tilde{f}(x,s) = g(\varepsilon_n x + \varepsilon_n y_n, s) - \omega_n^{2^*-1}\chi_\Omega(\varepsilon_n x + \varepsilon_n y_n)$, we have $\omega_n \in L^t$ for all $t \ge 2$ and

$$\|\omega_n\|_{L^t} \le C_t \|\omega_n\|_{H^1} \tag{27}$$

where C_t does not depend on n.

Still from Lemma 12,

$$\lim_{R\to\infty} \int_{|x|\ge R} \omega_n^2 \, dx + \int_{|x|\ge R} \omega_n^{2^*} \, dx = 0 \quad \text{uniformly on } n. \tag{28}$$

We will apply Proposition 14 to the following inequality:

$$-\Delta\omega_n \le \tilde{g}_n(x) = g(\varepsilon_n x + \varepsilon_n y_n, \omega_n) \quad \text{in } \mathbb{R}^N. \tag{29}$$

Combining (27) with the fact that $\|\omega_n\|_{H^1}$ bounded, we have a $t > N$ such that $\|\tilde{g}_n\|_{L^t} \le C$, for all n. Using Proposition 14 in inequality (29), for all $y \in \mathbb{R}^N$

$$\sup_{B_1(y)} \omega_n \le C(\|\omega_n\|_{L^2(B_2(y))} + \|\tilde{g}_n\|_{L^t(B_2(y))}), \tag{30}$$

which implies an uniform bound for $\|\omega_n\|_{L^\infty}$ and consequently, we have an uniform bound for $\|\omega_\varepsilon\|_{L^\infty}$ for $0 < \varepsilon < \varepsilon_0$. Moreover, combining the limit (28) with inequality (30) we reach

$$\lim_{|x|\to\infty} \omega_n(x) = 0 \quad \text{uniformly on } n \tag{31}$$

and we have for $\omega_\varepsilon(x) = v_\varepsilon(x + y_\varepsilon)$

$$\lim_{|x|\to\infty} \omega_\varepsilon(x) = 0 \quad \text{uniformly on } \varepsilon \in (0, \varepsilon_0]. \tag{32}$$

From limit (32) there is a $\rho > 0$ such that $\omega_\varepsilon(x) \le a$ for all $|x| \ge \rho$ for all n, that is

$$-\Delta\omega_\varepsilon + V(\varepsilon x + \varepsilon y_\varepsilon)\omega_\varepsilon = f(\omega_\varepsilon) + \omega_\varepsilon^{2^*-1} \quad \text{in } |x| \ge \rho.$$

On the other hand, if $|x| \le \rho$, $g(\varepsilon x + \varepsilon y_\varepsilon, s) = f(s) + s^{2^*-1}$ when $\Omega_\varepsilon \supset B_\rho(0)$, then

$$-\Delta\omega_\varepsilon + V(\varepsilon x + \varepsilon y_\varepsilon)\omega_\varepsilon = f(\omega_\varepsilon) + \omega_\varepsilon^{2^*-1} \quad \text{in } \mathbb{R}^N$$

for all $\varepsilon \in (0, \varepsilon_0]$.

Notice that, up this moment, we have obtained the existence of solutions u_ε for problem (P_ε). In order to prove the concentration behavior of these solutions, we shall prove:

Lemma 15. *If ε_0 is sufficiently small ω_ε possesses at most one local (hence global) maximum $x_\varepsilon \in \mathbb{R}^N$.*

Proof. If y_1 is a local maximum of ω_n we must have

$$\lambda\omega_n(y_1)^{q_1-1} + \omega_n(y_1)^{2^*-2} \geq \alpha. \tag{33}$$

From limit (32), it is sufficient to consider the problem in a fixed ball $B_R(0)$ of $\mathbb{R}^N$.

Since $\|\omega_n\|_{L^\infty}$ is uniformly bounded, elliptic regularity theory implies that ω_n converges in $C^2_{\text{loc}}(\mathbb{R}^N)$ to ω. Let $x_n \in B_R(0)$ a global maximum of ω_n. The translation of ω_n, $\bar{\omega}_n(x) = \omega_n(x + x_n)$, attains its global maximum at the origin. Proceeding with $\bar{\omega}_n$ as we have proceeded with ω_n, it is easy to see that $\bar{\omega}_n$ converges to ω in $C^2_{\text{loc}}(\mathbb{R}^N)$. Now, by Lemma 4.2 in [13], if n is sufficient large, $\bar{\omega}_n$ possesses no critical points other than the origin. □

At this moment we can consider the sequence $\{y_\varepsilon\}$ in Lemma 11, as the sequence of global maximum points of v_ε. For ε sufficiently small, u_ε attains its global maximum at an unique $z_\varepsilon \in \mathbb{R}^N$ which must satisfy

$$u_\varepsilon(z_\varepsilon) = v_\varepsilon(y_\varepsilon), \quad \varepsilon y_\varepsilon = z_\varepsilon. \tag{34}$$

Then from Lemma 11

$$\lim_{\varepsilon\to 0} V(z_\varepsilon) = V_0.$$

Finally ω_ε has an exponential decay:

Lemma 16. *There are $C > 0$ and $\zeta > 0$ such that*

$$\omega_\varepsilon(x) \leq C\mathrm{e}^{-\zeta|x|} \quad \textit{for all } x \in \mathbb{R}^N.$$

Proof. From limit (31) and (f_1) there is a $R_0 > 0$ such that

$$f(\omega_\varepsilon(x)) + \omega_\varepsilon(x)^{2^*-1} \leq \frac{V_0}{2}\omega_\varepsilon(x) \quad \text{for all } |x| \geq R_0. \tag{35}$$

Fix $\varphi(x) = M\mathrm{e}^{-\zeta|x|}$ with $\zeta^2 < V_0/2$ and $M\mathrm{e}^{-\zeta R_0} \geq \omega_\varepsilon(x)$ for all $|x| = R_0$. It is easy to verify that

$$\Delta\varphi \leq \zeta^2\varphi \quad \text{for all } x \neq 0. \tag{36}$$

Define $\varphi_\varepsilon = \varphi - \omega_\varepsilon$. Using (35) and (36) we have

$$-\Delta\varphi_\varepsilon + \frac{V_0}{2}\varphi_\varepsilon \geq 0 \quad \text{in } |x| \geq R_0,$$

$$\varphi_\varepsilon \geq 0 \quad \text{on } |x| = R_0,$$

$$\lim_{|x|\to\infty} \varphi_\varepsilon(x) = 0.$$

The classical maximum principle implies that $\varphi_\varepsilon \geq 0$ in $|x| \geq R_0$ and we conclude

$$\omega_\varepsilon(x) \leq M\mathrm{e}^{-\zeta|x|}, \quad \text{for all } |x| \geq R_0 \text{ and } \varepsilon \in (0, \varepsilon_0].$$

The proof is complete. □

Using (34) and Lemma 16, we have

$$u_\varepsilon(x) = v_\varepsilon(\varepsilon) = \omega_\varepsilon(\varepsilon x - y_\varepsilon) = \omega_\varepsilon(\varepsilon x - \varepsilon z_\varepsilon)$$
$$\leq C \exp\left(-\zeta \left|\frac{x - z_\varepsilon}{\varepsilon}\right|\right) \quad \text{for all } x \in \mathbb{R}^N.$$

We have completed the proof of Theorem 1.

Acknowledgements

This work was done while the second author was visiting the Courant Institute of Mathematical Sciences. He thanks Professor Louis Nirenberg and all the faculty and staff of CIMS-NYU for their kind hospitality.

This work was done during the third author's visit to the Department of Mathematics at Rutgers University. He thanks Professor Yanyan Li for his kind hospitality.

References

[1] C.O. Alves, D.C. de Morais Filho, M.A.S. Souto, Radially symmetric solutions for a class of critical exponent elliptic problems in $\mathbb{R}^N$, Electron. J. Differential Equations 1996 (1996) 1–12.
[2] C.O. Alves, M.A. Souto, On existence and concentration behavior of ground state solutions for a class of problems with critical growth, preprint.
[3] H. Brezis, T. Kato, Remarks on the Schrödinger operator with regular complex potentials, J. Math. Pures Appl. 58 (1979) 137–151.
[4] V. Coti-Zelati, P.H. Rabinowitz, Homoclinic type solutions for a semilinear elliptic PDE on $\mathbb{R}^N$, Comm. Pure Appl. Math. 45 (1992) 1217–1269.
[5] M. del Pino, P.L. Felmer, Local Mountain Pass for semilinear elliptic problems in unbounded domains, Calc. Var. 4 (1996) 121–137.
[6] W.Y. Ding, W-M. Ni, On the existence of positive entire solutions of a semilinear elliptic equation, Arch. Ration. Mech. Anal. 91 (1986) 283–308.
[7] A. Floer, A. Weinstein, Nonspreading wave packets for the cubic Schrödinger equations with bounded potential, J. Funct. Anal. 69 (1986) 397–408.
[8] B. Gidas, W-M. Ni, L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic problem in $\mathbb{R}^N$, Math. Analysis and Appl., Part A, Adv. in Math. Suppl. Studies 7A, L. Nachbin (Ed.), Academic Press, 1981, pp. 369–402.
[9] D. Gilbarg, N.S. Trudinger, Elliptic partial differential equation of second order, Springer, Berlin, 1998.
[10] P.-L. Lions, The concentration–compactness principle in the calculus of variations, the locally compact case, Part II, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984) 223–283.
[11] J. Mawhin, M. Willen, Critical point theory and Hamiltonian systems, Springer, New York, 1989.
[12] O.H. Miyagaki, On a class of semilinear elliptic problem in $\mathbb{R}^N$ with critical growth, Nonlinear Anal. 29 (1997) 773–781.
[13] W.M. Ni, I. Takagi, On the shape of least-energy solutions to semilinear Neumann problem, Comm. Pure Appl. Math. 44 (1991) 819–851.
[14] P.H. Rabinowitz, On a class of nonlinear Schrödinger equations, Z. Angew Math. Phys. 43 (1992) 272–291.
[15] M.A.S. Souto, On the shape and the location of the peaks of least-energy solutions to semilinear Dirichlet problems with critical growth, preprint.
[16] G. Talenti, Best constants in Sobolev Inequality, Ann. Mat. Pura Appl. 110 (1976) 353–372.
[17] X. Wang, On concentration of positive bound states of nonlinear Schrödinger equations, Comm. Math. Phys. 153 (1993) 229–244.
[18] Yan Yan Li, On a singularity perturbed elliptic equation, Adv. Differential Equations 2 (1997) 955–980.