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Quasilinear Elliptic Equations with Exponential Nonlinearities

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Abstract

Using a variational approach we study the existence and multiplicity of solutions for the following class of Dirichlet problems:

$$-div(a(|\nabla u|^N)|\nabla u|^{N-2}\nabla u) = f(x,u), \text{ in } \Omega, \quad u = 0, \text{ on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^N$ with $N \geq 2$ and the nonlinearity $f(x,u)$ has subcritical growth on Ω, i.e., for all $\alpha > 0$

$$\lim_{|u|\to\infty} f(x,u)\exp(-\alpha |u|^{\frac{N}{N-1}}) = 0.$$

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1 Introduction

The purpose of this paper is to study the existence and multiplicity of solutions for the following class of quasilinear elliptic problems:

$$\begin{cases} -div(a(|\nabla u|^N)|\nabla u|^{N-2}\nabla u) = f(x,u), \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega, \end{cases} \tag{1}$$

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where Ω is a bounded smooth domain in $\mathbb{R}^N$ with $N \geq 2$ and the nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and has subcritical growth on Ω, i.e., for all $\alpha > 0$

$$\lim_{|u| \to \infty} \frac{| f(x, u) |}{\exp(\alpha \mid u \mid^{\frac{N}{N-1}})} = 0, \quad uniformly\ on\ x \in \Omega.$$

Problems involving subcritical growth, in bounded domains in $\mathbb{R}^2$, have been studied recently by de Figueiredo, Miyagaki and Ruf [4]. In this paper we improve and generalize the existence results in [4] involving subcritical growth. For this purpose it is considered a more general class of nonlinearities and operators of elliptic type. Indeed, the main goal of this paper is to study this more general class of elliptic problems using the topological min-max- approach, as it is done in the earlier work of Ambrosetti-Rabinowitz [1] (see [7], for a complete reference). In order to prove the compactness condition of the functional associated to problem (1), we assume the following condition on the function f, namely:

(H_1) there are constants $\mu > N$ and $R > 0$ such that for $\forall x \in \Omega$, $| u | \geq R$,

$$0 < \mu F(x, u) \leq u f(x, u),$$

where F is the primitive of f.

Remarks i) Integrating condition (H_1) we obtain positive constants c, d such that

$$F(x, u) \geq c \mid u \mid^{\mu} - d. \tag{2}$$

Notice that assumption (H_1) with $N = 2$ is the usual Ambrosetti-Rabinowitz superlinearity condition (cf.[1]).

ii) In [4], in order to obtain a compactness condition, conditions different from (H_1) were assumed, which in our context do not seem natural. Here, instead, we make assumption (H_1) and explore more thoroughly the notion of subcriticality defined above. We remark that such a condition has been motivated by the following result due to Trudinger and Moser (cf. [6],[8]) :

$$\exp(\alpha \mid u \mid^{\frac{N}{N-1}}) \in L^1, \ \forall \ u \in W_0^{1,N}(\Omega), \ \forall \alpha > 0$$

and

$$\sup_{\|u\|_{W_0^{1,N}} \leq 1} \int \exp(\alpha \mid u \mid^{\frac{N}{N-1}}) \leq C(N) \in \mathbb{R}, \text{ if } \alpha \leq \alpha_N,$$

where $\alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$ and ω_{N-1} is the $(N-1)$-dimensional surface of the unit sphere. Such a result allows us to treat problem (1) variationally in the Sobolev space $W_0^{1,N}(\Omega)$.

We now state the following hypotheses on the function a, which we shall assume throughout the paper.

(a_1) $a : \mathbb{R}^+ \to \mathbb{R}$ is continuous

(a_2) There exist positive constants $p \in (1, N]$, b_1, b_2, c_1, c_2 such that

$$c_1 + b_1 u^{N-p} \le u^{N-p} a(u^N) \le c_2 + b_2 u^{N-p} \quad \forall u \in \mathbb{R}^+.$$

(a_3) The function $k : \mathbb{R} \to \mathbb{R}$, $k(u) = a(| u |^N) \, | u |^{N-2} u$ is strictly increasing and

$$k(u) \to 0 \quad as \quad u \to 0^+.$$

Remarks i) We remark that the class of elliptic operators considered here has been studied recently in works of Hirano [5] and Ubilla [9], where the function f is assumed to have polynomial growth.

ii) An important example of problem (1) to keep in mind throughout this paper, is given when $a(u) \equiv \alpha + \beta u^{\frac{p-N}{p}}$, with $N \ge p$, $\alpha > 0$ and $\beta \ge 0$, which corresponds to the problem

$$-\Delta_N u - \Delta_p u = f(x, u), \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $\Delta_p u \equiv div(| \nabla u |^{p-2} \nabla u)$ is the so-called $p-$Laplacian (see other examples in the last section).

Consider the following nonlinear eigenvalue problem for the $p-$Laplacian

$$- \Delta_p u = \lambda \, | u |^{p-2} u, \quad u \in W_0^{1,p}(\Omega).$$

It is well known (cf. [2]) that there exist a smallest $\lambda_1(p) > 0$ and an associated function $v_1 > 0$ in Ω that solve this problem. Moreover we have the following variational characterization

$$\lambda_1(p) = \inf \left\{ \int | \nabla u |^p \, dx : u \in W_0^{1,p}(\Omega), \ \int | u |^p \, dx = 1 \right\}.$$

We shall denote by λ_i the $i-$th eigenvalue of problem $(-\Delta, H_0^1(\Omega))$.

Now we state the main results which will be proved here.

Theorem 1 *Assume that f is continuous, it has subcritical growth and satisfies (H_1), and that a satisfies (a_1), (a_2) and (a_3), with $N b_2 < \mu b_1$. Furthermore, assume that*

$$(H_2) \qquad \limsup_{|u| \to 0} \frac{pF(x, u)}{| u |^p} < (c_1 + b_1 \delta_p(N)) \lambda_1(p), \quad uniformly \ on \ x \in \Omega.$$

where $\delta_p(N) = 1$ if $N = p$ and $\delta_p(N) = 0$ if $N \neq p$. Then, problem (1) has a nontrivial weak solution. Moreover, if $f(x,u)$ is an odd function in u, then problem (1) has an unbounded sequence of weak solutions.

For our next theorem we assume that $N = p = 2$. Thus, in this case condition (a_2) can be rewritten as follows: there exist constants b_1 $b_2 > 0$ such that

$$b_1 \leq a(u) \leq b_2, \quad \forall u \in \mathbb{R}^+. \tag{3}$$

Theorem 2 *Assume that f is continuous, it has subcritical growth and satisfies (H_1), and that a satisfies (a_1), (a_2) and (a_3), with $2b_2 < \mu b_1$. Furthermore, suppose that*

(H_3) $\exists \delta > 0,\ \exists \lambda_k \leq \gamma < \lambda_{k+1}$ *such that* $F(x,u) \leq \frac{b_1}{2}\gamma u^2$, *a.e.* $x \in \Omega$, $\forall \mid u \mid \leq \delta$.

(H_4) $\qquad F(x,u) \geq \frac{b_2}{2}\lambda_k u^2 \quad$ *a.e.* $x \in \Omega. \quad \forall u \in \mathbb{R}$.

Then, problem (1) has a nontrivial weak solution. Moreover, if instead of (H_4) we assume that $f(x,u)$ is an odd function in u, then problem (1) has an unbounded sequence of weak solutions.

2 The variational formulation

Note that if the function f is continuous and has subcritical growth, then there exist positive constants C and β such that

$$\mid f(x,u) \mid \leq C \exp(\beta \mid u \mid^{\frac{N}{N-1}}), \quad \forall (x,u) \in \Omega \times \mathbb{R}. \tag{4}$$

Consequently the functional $\Psi : W_0^{1,N}(\Omega) \to \mathbb{R}$ given by

$$\Psi(u) = \int F(x,u)dx \tag{5}$$

is well defined and belongs to $C^1(W_0^{1,N}(\Omega), \mathbb{R})$ with

$$\Psi'(u)v = \int f(x,u)v dx, \quad \forall v \in W_0^{1,N}(\Omega). \tag{6}$$

To prove these statements we notice that from (4) we also have

$$\mid F(x,u) \mid \leq C_1 \exp(\beta \mid u \mid^{\frac{N}{N-1}}), \qquad \forall (x,u) \in \Omega \times \mathbb{R}. \tag{7}$$

Thus, since $\exp(\beta \mid u \mid^{\frac{N}{N-1}}) \in L^1$ for all $u \in W_0^{1,N}(\Omega)$, we see that the expressions in (5) and (6) are well defined. Finally, using standard arguments (cf. [7]) and the fact that given any strong convergent sequence (u_n) in $W_0^{1,N}(\Omega)$ there exist a subsequence (u_{n_k}) and $h \in W_0^{1,N}(\Omega)$ such that $|u_{n_k}(x)| \leq h(x)$ a.e. $x \in \Omega$ (to see this we use the same argument used in the proof of Fischer-Riesz theorem), we have that Ψ belongs to $C^1(W_0^{1,N}(\Omega), \mathbb{R})$.

It follows from the assumptions on the function a that for all $u \in \mathbb{R}$

$$\frac{1}{N}A(|u|^N) \geq \frac{b_1}{N}|u|^N + \frac{c_1}{p}|u|^p \tag{8}$$

$$\frac{1}{N}A(|u|^N) \leq \frac{b_2}{N}|u|^N + \frac{c_2}{p}|u|^p. \tag{9}$$

where $A(t) = \int_0^t a(\tau)d\tau$, and furthermore the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(u) = A(|u|^N)$ is strictly convex. Consequently, the functional $\Phi : W_0^{1,N}(\Omega) \to \mathbb{R}$ given by

$$\Phi(u) = \frac{1}{N}\int A(|\nabla u|^N)dx \tag{10}$$

is well defined, weakly lower semicontinuous, Fréchet differentiable, and the derivative of Φ is continuous. Moreover Φ' belongs to the class $(S)_+$, that is, for any sequence (u_n) in $W_0^{1,N}(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u, \quad \text{and} \\ \limsup_{n\to\infty}\langle \Phi'(u_n), u_n - u\rangle \leq 0 \end{cases} \tag{11}$$

it follows that $u_n \to u$ in $W_0^{1,N}(\Omega)$ (here $\rightharpoonup$ denotes weak convergence and $\to$ strong convergence). This is a special case of a more general class of operators studied by Browder [3].

Therefore, if function a satisfies conditions $(a_1)-(a_3)$ and the nonlinearity f is continuous and satisfies (4), we conclude that the functional $I : W_0^{1,N}(\Omega) \to \mathbb{R}$ given by

$$I(u) = \frac{1}{N}\int A(|\nabla u|^N)dx - \int F(x, u)dx,$$

is well defined and belongs to $C^1(W_0^{1,N}(\Omega), \mathbb{R})$ with

$$I'(u)v = \int_\Omega a(|\nabla u|^N)|\nabla u|^{N-2}\nabla u.\nabla v dx - \int_\Omega f(x, u)v dx, \forall v \in W_0^{1,N}(\Omega).$$

Consequently, equation (1) is precisely the Euler equation of the functional I and the weak solutions of (1) are critical points of I and conversely. This allows us to use Critical Point Theory to obtain weak solutions of problem (1).

To find the critical points of functional I we shall consider the Palais-Smale (PS) compactness condition. Let $(X, \|\cdot\|)$ be a real Banach space and $I \in C^1(X, \mathbb{R})$. We recall that I satisfies condition (PS) if any sequence $(u_n) \subset X$ for which

$$(i)\ I(u_n) \to c, \qquad (ii)\ I'(u_n) \to 0, \quad \text{as } n \to \infty. \tag{12}$$

has a convergent subsequence.

Lemma 1 *Assume that f is continuous and has subcritical growth. Let (u_n) be a sequence in $W_0^{1,N}(\Omega)$ such that $u_n \rightharpoonup u$. Then $\lim_{n\to+\infty} \int f(x,u_n)(u_n-u)=0$.*

Proof. Let (u_n) be a sequence converging weakly to some u in $W_0^{1,N}(\Omega)$. Thus, we can take a subsequence, denoted again by (u_n), such that $u_n \to u$ in L^p, $\forall p \geq 1$. Since (u_n) is a bounded sequence, we may choose $\beta > 0$ sufficiently small such that $\beta \| u_n \|_{W_0^{1,N}}^{\frac{N}{N-1}} < \alpha_N$, $\forall n$. Since f has subcritical growth, we obtain

$$\int | f(x,u_n(x)) |^q \, dx \leq C \int \exp(q\beta \mid u_n \mid^{\frac{N}{N-1}}) dx$$
$$\leq C \int \exp\left[q\beta \| u_n \|_{W_0^{1,N}}^{\frac{N}{N-1}} \left(\frac{u_n}{\|u_n\|_{W_0^{1,N}}} \right)^{\frac{N}{N-1}} \right] dx \leq C.$$

for large n, if we choose $q > 1$ sufficiently close to 1. Using Hölder's inequality, the last estimate implies

$$\int f(x,u_n)(u_n-u) \leq \left[\int | f(x,u_n(x)) |^q \right]^{\frac{1}{q}} \left[\int | u_n - u |^p \right]^{\frac{1}{p}} \leq C \left[\int | u_n - u |^p \right]^{\frac{1}{p}}.$$

where $1/p + 1/q = 1$. Thus the proof is completed, since $u_n \to u$ in L^p. □

Lemma 2 *Let Φ: $W_0^{1,N}(\Omega) \to \mathbb{R}$ be a C^1 functional such that Φ' belongs to the class $(S)_+$ and, in addition, suppose that the function f is continuous and has subcritical growth. Then the functional $I(u) = \Phi(u) - \int F(x,u)dx$ satisfies condition (PS), provided that every sequence (u_n) in $W_0^{1,N}(\Omega)$ satisfying (12), is bounded.*

Proof. Let $(u_n) \subset W_0^{1,N}(\Omega)$ be a sequence satisfying (12). Then

$$| I'(u_n)v | = | \Phi'(u_n)v - \int f(x,u_n)v | \leq \epsilon_n \| v \|, \ \forall \ v \in W_0^{1,N}(\Omega).$$

where $\epsilon_n \to 0$ as $n \to \infty$. Since (u_n) is a bounded sequence in $W_0^{1,N}(\Omega)$, we can take a subsequence, denoted again by (u_n), such that $u_n \rightharpoonup u$ and $u_n \to u$ in L^q, $\forall q \geq 1$. Then considering $v = u_n - u$ in the inequality above and using the fact that $\lim_{n\to+\infty} \int f(x,u_n)(u_n - u) = 0$ (see Lemma 1 above), we obtain

$$\Phi'(u_n)(u_n-u) \to 0.$$

Since $u_n \rightharpoonup u$ and $\Phi' \in (S)_+$, the result is proved. □

Lemma 3 *Assume that the function a satisfies $(a_1)-(a_3)$, with $Nb_2 < \mu b_1$, and that the nonlinearity f is continuous, has subcritical growth and satisfies (H_1). Then the functional I satisfies condition (PS).*

Proof. Using $(a_1)-(a_3)$ with $Nb_2 < \mu b_1$ we obtain positive constants c, d such that

$$\frac{\mu}{N}A(u)-a(u)u \geq cu-d \quad \forall u \in \mathbb{R}^+. \tag{13}$$

Now, let (u_n) be a sequence in $W_0^{1,N}(\Omega)$ satisfying (12). Thus,

$$\frac{1}{N}\int A(|\nabla u_n|^N)-\int F(x,u_n)\to c \tag{14}$$

$$|\int a(|\nabla u_n|^N)|\nabla u_n|^N-\int f(x,u_n)u_n| \leq \epsilon_n \| u_n \|_{W_0^{1,N}}. \tag{15}$$

where $\epsilon_n \to 0$, as $n\to\infty$. Multiplying (14) by μ, subtracting (15) from the expression obtained and using (13) we come to the conclusion that

$$\int |\nabla u_n|^N-\int(\mu F(x,u_n)-f(x,u_n)u_n) \leq C+\epsilon_n \| u_n \|_{W_0^{1,N}}.$$

From this inequality and using condition (H_1), we easily find that (u_n) is a bounded sequence in $W_0^{1,N}(\Omega)$. Finally, to conclude the proof it suffices to use Lemma 2. □

3 Proofs of the existence results

3.1 Proof of Theorem 1

Lemma 4 *Assume that the hypotheses of Theorem 1 hold. Then there exist $\delta, \rho > 0$ such that $I(u)\geq\delta$ if $\| u \|_{W_0^{1,N}}=\rho$. Moreover, $I(tu)\to-\infty$ as $t\to+\infty$, for all $u\in W_0^{1,N}(\Omega)\backslash\{0\}$.*

Proof. Using (H_2) and (4) we can choose $\eta < c_1+b_1\delta_p(N)$ such that for $q > N$

$$F(x,u)\leq \eta\lambda_1(p)\frac{|u|^p}{p}+C\exp(\beta|u|^{\frac{N}{N-1}})|u|^q, \quad \forall(x,u)\in\Omega\times\mathbb{R}.$$

On the other hand, from Hölder's inequality and Trudinger-Moser inequality we obtain

$$\begin{aligned}\int\exp(\beta|u|^{\frac{N}{N-1}})|u|^q &\leq \left\{\int\exp\left[\beta r\| u \|_{W_0^{1,N}}^{\frac{N}{N-1}}\left(\frac{|u|}{\|u\|_{W_0^{1,N}}}\right)^{\frac{N}{N-1}}\right]\right\}^{1/r}\left\{\int|u|^{sq}\right\}^{1/s}\\ &\leq C(N)\left\{\int|u|^{sq}\right\}^{1/s}\end{aligned}$$

if $\| u \| \leq\sigma$, where $\beta r\sigma^{\frac{N}{N-1}} < \alpha_N$ and $1/r+1/s=1$. Using the variational characterization of the first eigenvalue and the Sobolev embedding, these two estimates and inequality (8 imply

$$I(u)\geq\frac{b_1}{N}\| u \|_{W_0^{1,N}}^N+\frac{c_1}{p}\| u \|_{W_0^{1,p}}^p-\frac{\eta}{p}\| u \|_{W_0^{1,p}}^p-C_1\| u \|_{W_0^{1,N}}^q$$

Since $\eta < c_1 + b_1\delta_p(N)$ and $p \leq N < q$ we can choose $\rho > 0$ such that $I(u) \geq \delta$ if $\| u \|_{W_0^{1,N}} = \rho$.

Now we shall prove the second assertion. Choosing any $u \in W_0^{1,N}(\Omega)\backslash\{0\}$, the second inequality in (9) and (2) lead to

$$I(tu) \leq \frac{b_2 t^N}{N} \int | \nabla u |^N + \frac{c_2 t^p}{p} \int | \nabla u |^p - dt^\mu \int | u |^\mu + C.$$

Therefore, $I(tu) \to -\infty$ as $t \to +\infty$, since $d > 0$ and $\mu > N \geq p$. □

It follows from Lemma 3 that the functional I satisfies (PS). Now to obtain a nontrivial solution to problem (1) we use Lemma 4 and apply the Mountain-Pass Theorem (Theorem 2.2 of [7]). Assuming that the function $f(x,u)$ is odd in u, we can apply a $\mathbb{Z}_2$ version of the Mountain-Pass Theorem (Theorem 9.12 of [7]), in order to conclude that functional I has an unbounded sequence of critical values $c_n = I(u_n)$.

Finally (as Theorem 9.38 in [7] or Theorem 1 in [9]) we prove that (u_n) is an unbounded sequence. Using condition (a_3) we obtain

$$a(u)u \geq \frac{1}{N} A(u), \quad \forall u \in \mathbb{R}^+. \tag{16}$$

On the other hand, since $c_n = I(u_n)$ and $I'(u_n)u_n = 0$, we obtain respectively

$$\int a(| \nabla u_n |^N) | \nabla u_n |^N = \int f(x, u_n) u_n \tag{17}$$

$$\frac{1}{N} \int A(| \nabla u_n |^N) - \int F(x, u_n) = c_n \tag{18}$$

Multiplying (17) by $1/N$, subtracting (18) from the expression obtained and using (16) we conclude that

$$(1 - \frac{1}{N}) \int a(| \nabla u_n |^N) | \nabla u_n |^N + \int \left[\frac{1}{N} f(x, u_n) u_n - F(x, u_n) \right] \geq c_n.$$

From the above estimate it follows that (u_n) is unbounded, since $c_n \to +\infty$.

3.2 Proof of Theorem 2

Lemma 5 *Suppose that the hypotheses of Theorem 3 hold. Denote by H_k the finite dimensional subspace of $H_0^1(\Omega)$ generated by the eigenfunctions of $(-\Delta, H_0^1(\Omega))$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_k$ and $W_k = H_k^\perp$ where $H_k^\perp$ denotes the orthogonal subspace of H_k in $H_0^1(\Omega)$. Then, there are constants $\alpha, \rho > 0$ such that*

$$I(u) \geq \alpha \textit{ if } \| u \| = \rho \textit{ and } u \in W_k.$$

Moreover, if we consider φ_{k+1} as an eigenfunction of $(-\Delta, H_0^1(\Omega))$ corresponding to the eigenvalue λ_{k+1}, the set

$$Q = \{v + s\varphi_{k+1} : v \in H_k, \quad \| v \|_{H_0^1} \leq R, \quad 0 \leq s \leq R\}$$

and ∂Q as its relative boundary in $H_k \dotplus span\{\varphi_{k+1}\}$, then there exists $R > \rho$ such that $I(u) \leq 0$, for all $u \in \partial Q$.

Proof. From condition (H_3) and inequality (4) we have

$$F(x, u) \leq \frac{b_1}{2}\gamma u^2 + C\exp(\beta u^2) \mid u \mid^q,$$

for $q > 2$. Now, as in the proof of Lemma 4, using Trudinger-Moser inequality, the variational characterization of the eigenvalue λ_{k+1} and the first inequality in (3), we obtain

$$I(u) \geq \frac{b_1}{2}(1 - \frac{\gamma}{\lambda_{k+1}}) \| u \|_{H_0^1}^2 - C \| u \|_{H_0^1}^q, \quad \forall u \in W_k.$$

Since $\gamma < \lambda_{k+1}$ and $2 < q$, the first assertion follows.

Condition (H_4) and the second inequality in (3) yield

$$I(u) \leq \frac{b_2}{2}\int \mid \nabla u \mid^2 - \frac{b_2}{2}\lambda_k \int \mid u \mid^2 \leq 0, \quad \forall u \in H_k.$$

On the other hand, using the first assertion of Lemma 5, we can choose $R > 0$ sufficiently large such that

$$I(u) \leq 0. \quad \forall u \in \partial Q \text{ with } \| u \|_{H_0^1} \geq R.$$

Consequently we obtain that $I(u) \leq 0$, for all $u \in \partial Q$. □

In view of Lemma 3 and Lemma 6, to obtain a nontrivial solution to problem (1) we apply the Generalized Mountain Pass Theorem (Theorem 5.3 of [7]). Finally, to obtain an unbounded sequence of weak solutions we proceed as in the proof of Theorem 1.

4 Some examples

Example 1 *Let $F \in C^1(\mathbb{R}, \mathbb{R})$ such that $F(u) \sim (1 + \delta_p(N))\lambda\frac{|u|^p}{p}$ as $\mid u \mid \to 0$ and $F(u) \sim \mid u \mid^{\frac{N}{N-1}} \exp(\beta \mid u \mid^{\frac{N}{N-1}} / \ln \mid u \mid)$ as $\mid u \mid \to \infty$, where $1 < p \leq N$ and $\beta > 0$. Thus, as a consequence of Theorem 1, the problem*

$$-\Delta_N u - \Delta_p u = F'(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

has a nontrivial weak solution provided that $\lambda < \lambda_1(p)$. Moreover, this problem has an unbounded sequence of weak solutions if we assume that F is even. Notice that in this case $a(u) = 1 + u^{\frac{p-N}{N}}$.

Example 2 *Consider problem (1) with* $a(u) = 1 + \epsilon(1+u)^{-2}$ *and* $F(u) = \frac{\gamma}{2}u^2 + (1 - \chi(u))\exp(\eta \mid u \mid^2 / \ln(\mid u \mid +2))$ *where* $\chi \in C^1(\mathbb{R}, [0,1])$, $\chi(u) = 1$, $\forall u \in (-\delta, \delta)$, $\chi(u) = 0$, $\forall u \notin (-2\delta, 2\delta)$, $\delta > 0$ *and* $\varepsilon \geq 0$. *That is*

$$-div\left\{\left(1+\epsilon\left(1+\mid \nabla u \mid^2\right)^{-2}\right)\nabla u\right\} = F'(u) \ in \ \Omega, \quad u = 0 \ on \ \partial\Omega.$$

Thus, using Theorem 2, this problem has a nontrivial weak solution provided that $(1+\varepsilon)\lambda_k < \gamma < \lambda_{k+1}$.

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