UNIVERSIDADE FEDERAL DA PARAÍBA
CCEN - Departamento de Matemática
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Cálculo III - 2^a Prova

João Pessoa, 02 de outubro de 2023

Professor: Pedro A. Hinojosa

Nome: ______________________________ Matrícula: ________________

Questão 1 *(2.5 pts.) Calcule* $\int_C -2xydx + (x^2+y^2)dy$ *onde* C *é a curva dada por* $C : \begin{cases} x^2+4y^2 = 2x \\ y \geq 0 \end{cases}$

Questão 2 *(2.5 pts) Considere as integrais*

$$I_1 = \int_{\alpha_1} (2x+y)^2 dx - (x-2y)^2 dy \qquad e \qquad I_2 = \int_{\alpha_2} (2x+y)^2 dx - (x-2y)^2 dy$$

Onde $\alpha_1, \alpha_2 : [0,1] \to \mathbb{R}^2$, $\alpha_1(t) = (t, t^2)$, $\alpha_2(t) = (t^2, t)$. *Usando o teorema de Green calcule a diferença* $I_1 - I_2$

Questão 3 *(3.0 pts.) Seja* $\overrightarrow{F} = 6xy^3\overrightarrow{i} + 9x^2y^2\overrightarrow{j} + (4z+1)\overrightarrow{k}$

(a) *Verifique que o campo* $\overrightarrow{F}$ *é conservativo. Justifique sua resposta;*

(b) *Determine um potencial para* $\overrightarrow{F}$;

(c) *Calcule* $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$ *sendo* C *a curva parametrizada por:*

$$\alpha : [0, \pi] \to \mathbb{R}^3, \quad \alpha(t) = (cos(t), sen(t), t)\,.$$

Questão 4 *(2.0 pts) Calcule o trabalho realizado pelo campo*

$$\overrightarrow{F} = (x + e^{y^2})\,\overrightarrow{i} + (x^3 + 3xy^2 + 2xye^{y^2})\,\overrightarrow{j}$$

para deslocar uma partícula ao longo da semicircunferência $C : \begin{cases} x^2+y^2 = 4 \\ y \geq 0 \end{cases}$ *do ponto* $A = (2,0)$ *até o ponto* $B = (-2,0)$.

Boa Prova !!

C3 Prova 2 - 2023.1

① Calcular $\int_C -2xy\,dx + (x^2+y^2)\,dy$

$C: \begin{cases} x^2+4y^2 = 2x \\ y \geq 0 \end{cases}$

Solução

$x^2+4y^2 = 2x$

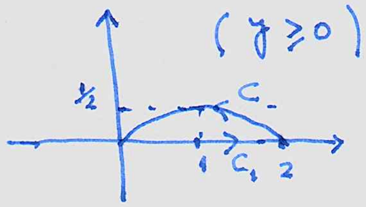
$x^2-2x+4y^2=0$

$(x-1)^2-1+4y^2=0$

$(x-1)^2+4y^2=1$

ou

$C: \begin{cases} \dfrac{(x-1)^2}{1^2} + \dfrac{y^2}{(1/2)^2} = 1 \\ y \geq 0 \end{cases}$



Seja C_1 a curva parametrizada por

$\alpha_1(t) = (t,0)$, $t \in [0,2]$

$C^+ \cup C_1$ é fronteira do domínio D

$4y^2 = 2x - x^2$

$y^2 = \dfrac{2x-x^2}{4}$

$y = \frac{1}{2}\sqrt{x(2-x)}$

$D_{xy}: \begin{cases} 0 \leq x \leq 2 \\ 0 \leq y \leq \frac{1}{2}\underbrace{\sqrt{x(2-x)}}_{\sqrt{1-(x-1)^2}} \end{cases}$

$4y^2 = 1-(x-1)^2$

$y = \frac{1}{2}\sqrt{1-(x-1)^2}$

Pelo Teor de Green,

$$\int_{C^+\cup C_1} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \quad \begin{cases} P = -2xy \\ Q = x^2+y^2 \end{cases}$$

$\dfrac{\partial Q}{\partial x} = 2x$, $\dfrac{\partial P}{\partial y} = -2x$, $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 4x$

$$\int_{C^+} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA - \int_{C_1} P\,dx + Q\,dy$$

$$\int_{C_1} P\,dx + Q\,dy = \int_0^2 (-2t \cdot 0)\,dt + (t^2 + 0)\cdot 0 = \underline{\underline{0}}$$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_0^2 \int_0^{\frac{1}{2}\sqrt{1-(x-1)^2}} 4x\,dy\,dx$$

$$= \int_0^2 2x\sqrt{1-(x-1)^2}\,dx$$

$$\begin{aligned} x - 1 &= \operatorname{sen}\theta \\ dx &= \cos\theta\,d\theta \end{aligned} \quad \left| \quad \begin{aligned} x = 0 &\Rightarrow \theta = -\pi/2 \\ x = 2 &\Rightarrow \theta = \pi/2 \end{aligned} \right.$$

$$\begin{aligned} 2x\sqrt{1-(x-1)^2}\,dx &= 2(1+\operatorname{sen}\theta)\sqrt{1+\operatorname{sen}^2\theta}\,.\cos\theta\,d\theta \\ &= 2(1+\operatorname{sen}\theta)\cdot\cos^2\theta\,d\theta \\ &= (2\cos^2\theta + 2\operatorname{sen}\theta\cos\theta)\,d\theta \\ &= \left[2\left(\frac{1+\cos 2\theta}{2}\right) + 2\operatorname{sen}\theta\cos\theta\right] d\theta \\ &= (1 + \cos 2\theta + 2\operatorname{sen}\theta\cos\theta)\,d\theta \end{aligned}$$

$$\int_0^2 2x\sqrt{1-(x-1)^2}\,dx = \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta + 2\operatorname{sen}\theta\cos\theta)\,d\theta$$

$$= \pi - \frac{1}{2}\underbrace{\cancel{\operatorname{sen} 2\theta\Big|_{-\pi/2}^{\pi/2}}}_{0} + \underbrace{\cancel{\operatorname{sen}^2\theta}\Big|_{-\pi/2}^{\pi/2}}_{0}$$

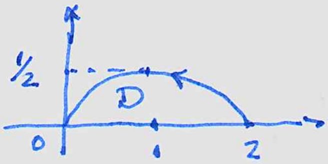
$$\therefore \quad \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \pi$$

$$\int_{C^+} P\,dx + Q\,dy = \pi - \cancelto{0}{\int_{C_1} P\,dx + Q\,dy} = \pi$$

Outra solução para $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

Fazendo a mudança de coord.

$$\begin{cases} x-1 = r\cos\theta \\ 2y = r\,\mathrm{sen}\,\theta \end{cases} \qquad \left((x-1)^2 + 4y^2 = r^2\right)$$



$$D_{r\theta}: \begin{cases} 0 \leq r \leq 1 \\ 0 \leq \theta \leq \pi \end{cases}$$

$$J = \det \begin{pmatrix} \cos\theta & -r\,\mathrm{sen}\,\theta \\ \frac{1}{2}\mathrm{sen}\,\theta & \frac{1}{2} r\cos\theta \end{pmatrix} = \frac{1}{2} r$$

$$\iint_D 4x\,dx\,dy = \int_0^1 \int_0^\pi 4(1 + r\cos\theta) \cdot \frac{1}{2} r \, d\theta\, dr$$

$$= \int_0^1 \int_0^\pi (2r + 2r^2\cos\theta)\, d\theta\, dr$$

$$= \int_0^1 \left(2\pi r + 2r^2 \cancelto{0}{\mathrm{sen}\,\theta \Big|_0^\pi}\right) dr$$

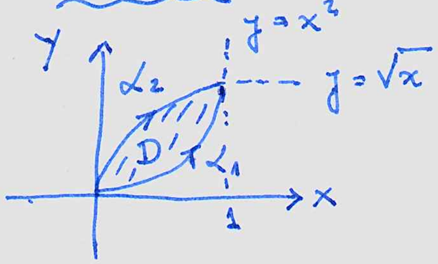
$$= \int_0^1 2\pi r\, dr = \pi r^2 \Big|_0^1 = \pi$$

② $$I_j = \int_{\alpha_j} (2x+y)^2\,dx - (x-2y)^2\,dy \qquad (j=1,2)$$

$$\alpha_j : [0,1] \longrightarrow \mathbb{R}^2, \quad \begin{cases} \alpha_1(t) = (t, t^2) \\ \alpha_2(t) = (t^2, t) \end{cases}$$

Usar Green para calcular $I_1 - I_2$

Solução:



Seja $D \subseteq \mathbb{R}^2$ o domínio t.q.

$$\partial D = \alpha_1 \cup \alpha_2^-$$

$$P = (2x+y)^2$$
$$Q = -(x-2y)^2$$

$$I_1 - I_2 = \int_{\alpha_1} P\,dx + Q\,dy - \int_{\alpha_2} P\,dx + Q\,dy$$

$$= \int_{\alpha_1} \cdots + \int_{\alpha_2^-} \cdots = \int_{\alpha_1 \cup \alpha_2^-} P\,dx + Q\,dy.$$

$$= \int_{\partial D} P\,dx + Q\,dy \overset{\text{Green}}{=} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$P = (2x+y)^2 \Rightarrow \frac{\partial P}{\partial y} = 2(2x+y)$$

$$Q = -(x-2y)^2 \Rightarrow \cancel{\frac{\partial Q}{\partial y} = -2(x-2y)} \; \frac{\partial Q}{\partial x} = -2(x-2y)$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -2(x-2y) - 2(2x+y) = -6x + 2y$$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_0^1 \int_{x^2}^{\sqrt{x}} (-6x+2y)\,dy\,dx$$

$$\int_0^1 \int_{x^2}^{\sqrt{x}} (-6x+2y)\,dy\,dx = \int_0^1 \left(-6x(\sqrt{x}-x^2) + y^2\Big|_{x^2}^{\sqrt{x}}\right) dx$$

$$= \int_0^1 \left(-6x^{3/2} + 6x^3 + x - x^4\right) dx$$

$$= \left(-6\cdot\frac{2}{5}x^{5/2} + \frac{6}{4}x^4 + \frac{1}{2}x^2 - \frac{1}{5}x^5\right)\Big|_0^1$$

$$= -\frac{12}{5} + \frac{3}{2} + \frac{1}{2} - \frac{1}{5} = 2 - \frac{13}{5} = -3/5$$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = -3/5$$

$$\therefore \quad I_1 - I_2 = -3/5$$

(3) $\vec{F} = 6xy^3\,\vec{\imath} + 9x^2y^2\,\vec{\jmath} + (4z+1)\,\vec{k}$

(a) $\vec{F}$ é conservativo.

Como $\mathrm{Dom}(\vec{F}) = \mathbb{R}^3$ que é simplesmente conexo (sem buracos) basta verificar que $\mathrm{rot}(\vec{F}) = \vec{0}$.

$$\mathrm{rot}(\vec{F}) = \det\begin{pmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ 6xy^3 & 9x^2y^2 & 4z+1 \end{pmatrix}$$

$$= (0-0)\,\vec{\imath} - (0-0)\,\vec{\jmath} + (18xy^2 - 18xy^2)\,\vec{k}$$

$$= \vec{0}$$

$\therefore$ $\vec{F}$ é conservativo

(b) um potencial para $\vec{F}$..

Queremos encontrar $f = f(x,y,z)$ t.q. $\nabla f = \vec{F}$

$$\left(\nabla f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k} \right)$$

Então, queremos f de modo que

$$f_x = P = 6xy^3 \;,\; f_y = Q = 9x^2y^2 \;\text{ e }\; f_z = R = 4z+1$$

$$f_x = 6xy^3 \Rightarrow f = 3x^2y^3 + C(y,z)$$

derivando c/r a y temos

$$f_y = 9x^2y^2 + \frac{\partial C}{\partial y} = Q = 9x^2y^2$$

Daí, $\dfrac{\partial C}{\partial y} = 0$ $\therefore$ $C = C(z)$

Assim, $f = 3x^2y^3 + C(z)$

derivando c/r a z temos

$$f_z = 0 + \frac{dC}{dz} = R = 4z+1$$

$\therefore$ $\dfrac{dC}{dz} = 4z+1$. Daí $C = 2z^2 + z + K$

$(K = \text{cte})$

Obtemos:

$$f = 3x^2y^3 + 2z^2 + z + K$$

(c) Calcular $\int_C \vec{F} \cdot d\vec{r}$, C parametrizada por

$\alpha : [0,\pi] \longrightarrow \mathbb{R}^3$,

$\alpha(t) = (\cos t, \operatorname{sen} t, t)$

Como $\vec{F}$ é conservativo $\int_C \vec{F} \cdot d\vec{r}$ não depende da curva C (só dos ptos final e inicial)

$$\int_C \vec{F} \cdot d\vec{r} = f(\alpha(\pi)) - f(\alpha(0))$$

$$\alpha(\pi) = (\cos\pi, \operatorname{sen}\pi, \pi) = (-1, 0, \pi)$$

$$\alpha(0) = (1, 0, 0)$$

$f = 3x^2y^2 + 2z^2 + z + K$

$$f(\alpha(\pi)) = f(-1, 0, \pi) = 0 + 2\pi^2 + \pi + K$$

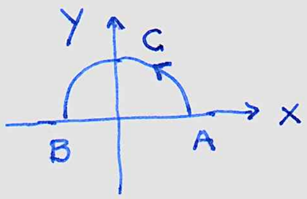
$$f(\alpha(0)) = f(1, 0, 0) = 0 + 0 + 0 + K$$

$$f(\alpha(\pi)) - f(\alpha(0)) = 2\pi^2 + \pi$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = 2\pi^2 + \pi$$

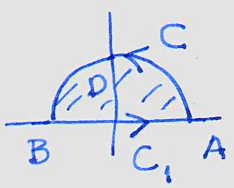
4 $\quad \vec{F} = (x + e^{y^2})\vec{i} + (x^3 + 3xy^2 + 2xy\,e^{y^2})\vec{j}$

$$C: \begin{cases} x^2 + y^2 = 4 \\ y \geq 0 \end{cases}$$



$A = (2, 0)$
$B = (-2, 0)$

$$W = \int_C \vec{F} \cdot d\vec{r}$$



seja C_1 o segmento de reta do pto B até o pto A

$$C_1: \begin{cases} y = 0 \\ -2 \leq x \leq 2 \end{cases} \quad \alpha_1(t) = (t, 0), \ t \in [2, 2]$$

$D \subseteq \mathbb{R}^2$ t.q. $\partial D = C \cup C_1$

Green $\displaystyle\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$

$\partial D = C \cup C_1$

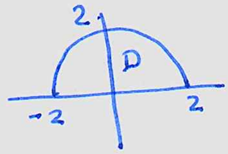
$\displaystyle\therefore \int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy = -\int_{C_1} P\,dx + Q\,dy + \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$

$P = x + e^{y^2} \Rightarrow \dfrac{\partial P}{\partial y} = 2y e^{y^2}$

$Q = x^3 + 3xy^2 + 2xy e^{y^2} \Longrightarrow \dfrac{\partial Q}{\partial x} = 3x^2 + 3y^2 + 2y e^{y^2}$

$\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 3x^2 + 3y^2$

$D_{xy} : \begin{cases} x^2 + y^2 \leq 4 \\ y \geq 0 \end{cases}$



Em coord. Polares $\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \end{cases}$ $D_{r\theta} : \begin{cases} 0 \leq r \leq 2 \\ 0 \leq \theta \leq \pi \end{cases}$

$\displaystyle\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_0^2 \int_0^\pi 3r^2 \cdot r \, d\theta \, dr$

$\displaystyle = \int_0^2 3\pi r^3 \, dr = \frac{3}{4} \pi r^4 \Big|_0^2$

$= 12\pi$

Em C_1: $\displaystyle\int_{C_1} P\,dx + Q\,dy = \int_{-2}^{2} (t+1)\,dt = \left(\frac{1}{2} t^2 + t \right) \Big|_{-2}^{2}$

$= 4$

$\begin{cases} \alpha_1(t) = (t, 0) \\ t \in [-2, 2] \\ x = t \quad dx = dt \\ y = 0 \quad dy = 0 \end{cases}$

$\displaystyle\therefore W = \int_C \vec{F} \cdot d\vec{r} = 12\pi - 4$