UNIVERSIDADE FEDERAL DA PARAÍBA
**CCEN - Departamento de Matemática**
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# Cálculo III - $1^a$ Prova

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Nome: ______________________________ Matrícula: ______________

**Questão 1** *(2.0 pts) Use a mudança de variáveis* $\begin{cases} u = xy \\ v = y \end{cases}$ *para calcular a integral dupla* $\int_D (x^2 + 2y^2)\,dA$ *sobre a região* $D \subset \mathbb{R}^2$ *limitada, no primeiro quadrante, pelas curvas* $xy = 1,\ xy = 2,\ y = x$ *e* $y = 2x$.

**Questão 2** *(3.0 pts.) Calcule as integrais abaixo:*

*(a)* $\int_W (x^2 + y^2 + z)^2\,dV$, $\quad W : \begin{cases} x^2 + y^2 \leq 1 \\ 0 \leq z \leq 1 \end{cases}$

*(b)* $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \sqrt{x^2 + y^2 + z^2}\,dz\,dy\,dx$

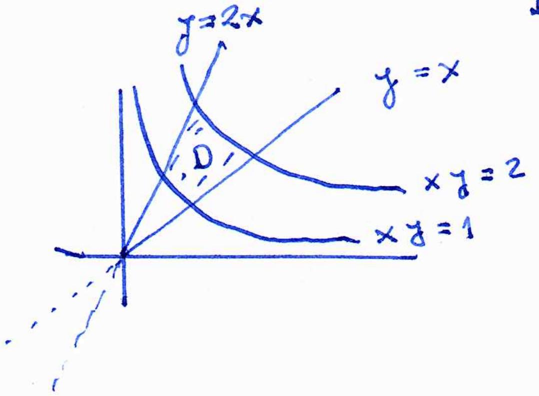
**Questão 3** *(2.0 pts) Calcule,* $\int_W \left( \frac{1}{\sqrt{x^2+y^2+z^2}} \right) dV$ *sendo* $W$ *a região interior ao cone* $z = \sqrt{x^2 + y^2}$ *limitada superiormente pela esfera* $x^2 + y^2 + z^2 = 4$ *e inferiormente pela esfera* $x^2 + y^2 + z^2 = 1$.

**Questão 4** *(3.0 pts.) Determine a massa do sólido* $W$ *limitado pelas superfícies* $x^2 + y^2 + z^2 = 9$ *e* $x^2 + y^2 + z^2 = 2y$ *sabendo que sua densidade é dada por* $f(x, y, z) = \frac{1}{x^2+y^2+z^2}$.

**Boa Prova !!**

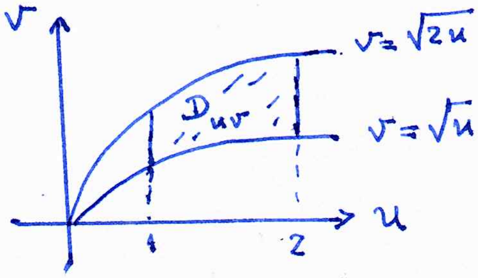
C3 P1 - 2023.1

(1) $\begin{cases} u = xy \\ v = y \end{cases}$ $\qquad \displaystyle\int_D (x^2 + 2y^2)\, dA$




$xy = 1 \Rightarrow u = 1$
$xy = 2 \Rightarrow u = 2$

$y = x \Rightarrow u = y^2 = v^2$
$y = 2x \Rightarrow u = \frac{1}{2} y^2 = \frac{1}{2} v^2$




$y = x \Rightarrow v = \sqrt{u}$
$y = 2x \Rightarrow v = \sqrt{2u}$

$D_{uv}: \begin{cases} 1 \le u \le 2 \\ \sqrt{u} \le v \le \sqrt{2u} \end{cases}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} = y$$

$$\boxed{\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{y} = \frac{1}{v}}$$

$$I = \int_D (x^2 + 2y^2)\, dA = \ ?$$

$$I = \int_1^2 \int_{\sqrt{u}}^{\sqrt{2u}} \left[\left(\frac{u}{v}\right)^2 + 2v^2\right] \cdot \frac{1}{v}\, dv\, du$$

$\left( x^2 + 2y^2 = \left(\frac{u}{v}\right)^2 + 2v \right.$

$$= \int_1^2 \int_{\sqrt{u}}^{\sqrt{2u}} \left(\frac{u^2}{v^3} + 2v\right) dv\, du$$

$$= \int_1^2 \left(\frac{-u^2}{2v^2} + v^2\right)\Bigg|_{\sqrt{u}}^{\sqrt{2u}} du$$

$$\int_1^2 \left[ \left( \frac{-u^2}{2(2u)} + 2u \right) - \left( \frac{-u^2}{2u} + u \right) \right] du$$

$$= \int_1^2 \left( \frac{-u}{4} + 2u + \frac{u}{2} - u \right) du$$

$$= \int_1^2 \frac{5u}{4}\, du = \frac{5u^2}{8} \Big|_1^2 = \frac{5}{8}(4-1)$$

$$= \frac{15}{8}$$

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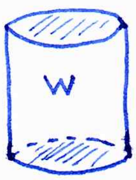
2 (a) $\displaystyle \int_W (x^2+y^2+z)^2\, dV$, $\quad W: \begin{cases} x^2+y^2 \leq 1 \\ 0 \leq z \leq 1 \end{cases}$




coord. cilíndricas

$$\begin{cases} x = r\cos\theta \\ y = r \operatorname{sen}\theta \\ z = z \end{cases}$$

$$W_{r\theta z}: \begin{cases} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq z \leq 1 \end{cases}$$

$$\int_W (x^2+y^2+z)^2\, dV = \int_0^1 \int_0^{2\pi} \int_0^1 (r^2+z)^2 \cdot r\, dr\, d\theta\, dz$$

$$\int_0^1 (r^2+z)^2 \cdot r\, dr = ?$$

$t = r^2 + z \Rightarrow dt = 2r\,dr$

$r = 0 \Rightarrow t = z$

$r = 1 \Rightarrow t = 1 + z$

$$\int_0^1 (r^2+z)^2 r\,dr = \int_z^{z+1} \frac{1}{2} t^2\,dt = \frac{1}{6} t^3 \Big|_z^{z+1}$$

$$= \frac{1}{6}(z+1)^3 - \frac{1}{6} z^3$$

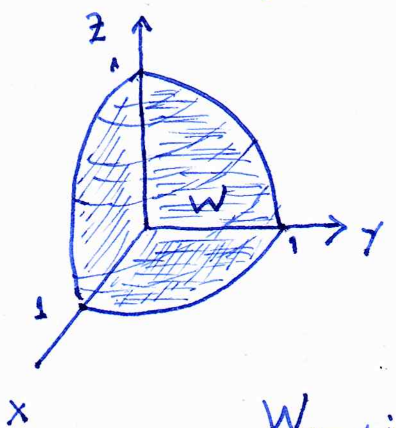
$$\therefore \int_0^1 \int_0^{2\pi} \int_0^1 (r^2+z)^2 r\,dr\,d\theta\,dz = \int_0^1 \int_0^{2\pi} \left(\frac{1}{6}(z+1)^3 - \frac{1}{6} z^3\right) d\theta\,dz$$

$$= \frac{2\pi}{6} \int_0^1 \left((z+1)^3 - z^3\right) dz = \frac{\pi}{3} \left(\frac{1}{4}(z+1)^4 - \frac{1}{4} z^4\right)\Big|_0^1$$

$$= \frac{\pi}{12}\left[\left(2^4 - 1\right) - \left(1 - 0\right)\right] = \frac{6\pi}{12} = \pi/2$$

$$\int_W (x^2+y^2+z)^2\,dV = \pi/2$$

(b) $$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$$




$$\begin{cases} x = \rho \cos\theta \,\mathrm{sen}\,\phi \\ y = \rho\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\phi \\ z = \rho\cos\phi \end{cases}$$

$$W_{\rho\theta\phi} : \begin{cases} 0 \leq \rho \leq 1 \\ 0 \leq \theta \leq \pi/2 \\ 0 \leq \phi \leq \pi/2 \end{cases}$$

$$I = \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} \sqrt{\rho^2}\, \rho^2 \operatorname{sen}\phi \, d\theta \, d\phi \, d\rho$$

$$= \pi/2 \int_0^1 \int_0^{\pi/2} \rho^3 \operatorname{sen}\phi \, d\phi \, d\rho$$

$$= \pi/2 \int_0^1 \rho^3 \left( -\cos\phi \Big|_0^{\pi/2} \right) d\rho$$

$$= \frac{\pi}{2} \int_0^1 \rho^3 \left( -(-1) \right) d\rho = \frac{\pi}{2} \int_0^1 \rho^3 \, d\rho$$

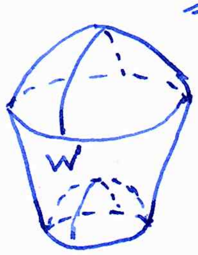
$$= \pi/8$$

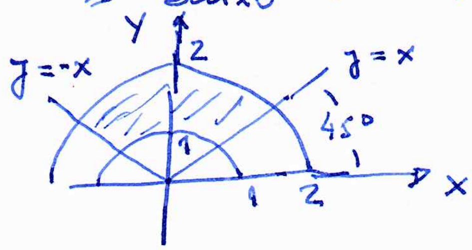
③ $$\int_W \frac{1}{\sqrt{x^2+y^2+z^2}} \, dV$$

$W$ interior ao cone $z = \sqrt{x^2+y^2}$

limitada por cima pela esfera $x^2+y^2+z^2 = 4$

" " baixo " " $x^2+y^2+z^2 = 1$




coord. esféricas

$$W_{\rho\theta\phi} : \begin{cases} 1 \leq \rho \leq 2 \\ 0 \leq \phi \leq \pi/4 \\ 0 \leq \theta \leq 2\pi \end{cases}$$

$$I = \int_W \frac{1}{\sqrt{x^2+y^2+z^2}} \, dV$$

$$I = \int_1^2 \int_0^{\pi/4} \int_0^{2\pi} \frac{1}{\sqrt{\rho^2}} \cdot \rho^2 \operatorname{sen}\phi \, d\theta \, d\phi \, d\rho$$

$$= 2\pi \int_1^2 \int_0^{\pi/4} \rho \operatorname{sen}\phi \, d\phi \, d\rho$$

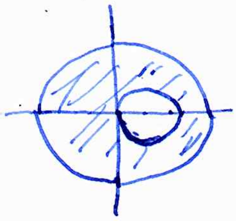
$$= 2\pi \int_1^2 \rho \left( -\cos\phi \Big|_0^{\pi/4} \right) d\rho$$

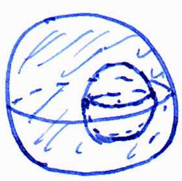
$$= -2\pi \int_1^2 \rho \left( \frac{\sqrt{2}}{2} - 1 \right) d\rho$$

$$= 2\left(1 - \frac{\sqrt{2}}{2}\right) \pi \int_1^2 \rho \, d\rho = \left(1 - \frac{\sqrt{2}}{2}\right) \pi \, (4 - 1)$$

$$= 3\left(1 - \frac{\sqrt{2}}{2}\right)\pi$$

4 W limitado por $x^2+y^2+z^2=9$ e $x^2+y^2+z^2=2y$





Massa de W = ?

densidade $f(x,y,z) = \dfrac{1}{x^2+y^2+z^2}$

Coord. esféricas

$$\begin{cases} x = \rho \cos\theta \operatorname{sen}\phi \\ y = \rho \operatorname{sen}\theta \operatorname{sen}\phi \\ z = \rho \cos\phi \end{cases}$$

$W_{\rho\theta\phi}$ : $0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq \pi$

$?? \leq \rho \leq 3$

$x^2+y^2+z^2 = 2y$

$\rho^2 = 2\rho \operatorname{sen}\theta \operatorname{sen}\phi$

$\boxed{\rho = 2 \operatorname{sen}\theta \operatorname{sen}\phi}$

$2\operatorname{sen}\theta\operatorname{sen}\phi \leq \rho \leq 3$

$$M(W) = \int_W \frac{1}{x^2+y^2+z^2}\, dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_{2\,\text{sen}\,\theta\,\text{sen}\,\phi}^{3} \frac{1}{\rho^2}\cdot\rho^2\,\text{sen}\,\phi \; d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \rho\,\text{sen}\,\phi \Big|_{2\,\text{sen}\,\theta\,\text{sen}\,\phi}^{3} d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}(3 - 2\,\text{sen}\,\theta\,\text{sen}\,\phi)\,\text{sen}\,\phi\; d\phi\, d\theta$$

$$= \int_0^{\pi}\int_0^{2\pi}\Big(\qquad\qquad\qquad\Big)\, d\theta\, d\phi$$

$$= \int_0^{\pi}\Big\{2\pi\cdot 3\,\text{sen}\,\phi + \Big(\xcancel{2\,\text{sen}^2\phi\,\cos\theta\Big|_0^{2\pi}}\Big)^{0}\Big\}\, d\phi$$

$$= 6\pi\int_0^{\pi}\text{sen}\,\phi\, d\phi = -6\pi\cos\phi\Big|_0^{\pi}$$

$$= -6\pi(-1-1) = \underline{\underline{12\pi}}$$