Cálculo III   1ª Prova   2014.1

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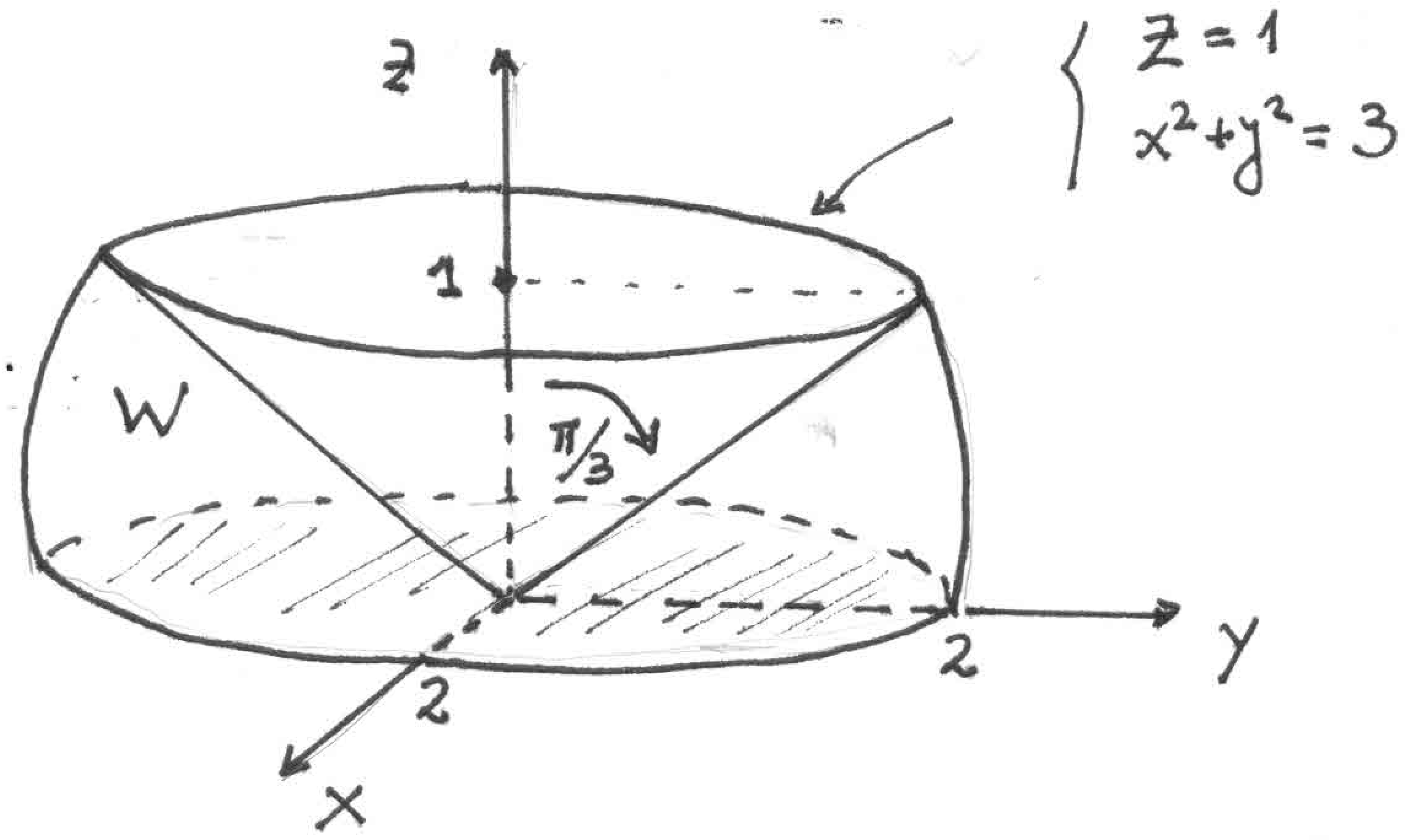
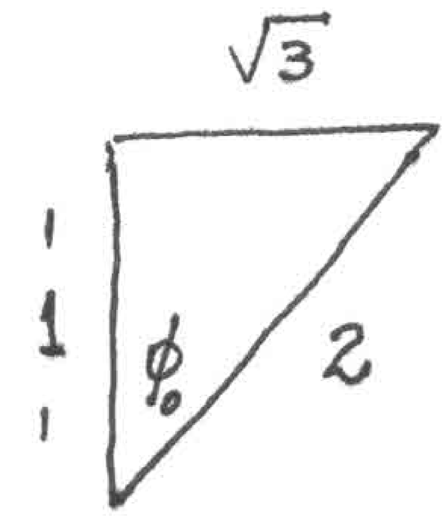
(1) Calcule o volume do sólido W acima do plano $z=0$, dentro da esfera $x^2+y^2+z^2=4$ e abaixo do cone $z=\sqrt{\dfrac{x^2+y^2}{3}}$.

Solução:



W em coord. esféricas

$$W: \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq \rho \leq 2 \\ \pi/3 \leq \phi \leq \pi/2 \end{cases}$$



$\operatorname{tg} \phi_0 = \sqrt{3}$

$\operatorname{sen} \phi_0 = \sqrt{3}/2$

$\cos \phi_0 = 1/2$

$\phi_0 = \pi/3$

Lembre que as coord. esféricas são dadas por:

$$\begin{cases} x = \rho \operatorname{sen}\phi \cos\theta \\ y = \rho \operatorname{sen}\phi \operatorname{sen}\theta \\ z = \rho \cos\phi \end{cases}$$

$$dx\,dy\,dz = \rho^2 \operatorname{sen}\phi \, d\phi \, d\theta \, d\rho$$

$$\operatorname{vol}(W) = \int_0^{2\pi}\int_0^2\int_{\pi/3}^{\pi/2} \rho^2 \operatorname{sen}\phi \, d\phi\, d\rho\, d\theta = \int_0^{2\pi}\int_0^2 -\rho^2\cos\phi \Big|_{\pi/3}^{\pi/2} d\rho\, d\theta$$

$$\text{vol}(W) = \int_0^{2\pi}\int_0^2 -\rho^2\left(0-\tfrac{1}{2}\right)d\rho\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 \frac{1}{2}\rho^2 d\rho\, d\theta = \int_0^{2\pi} \frac{1}{2}\cdot\frac{1}{3}\rho^3\Big|_0^2 d\theta$$

$$= \int_0^{2\pi} \frac{1}{6}(8-0)\,d\theta = \frac{8}{6}\cdot 2\pi = \frac{8}{3}\pi. \quad \text{u.v.}$$

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(2) Dada a integral Dupla

$$I = \int_D f\, dA = \int_{-1}^{1}\int_{1}^{1+\sqrt{1-x^2}} f(x,y)\,dy\,dx.$$

(a) Esboce a região D

(b) Inverta a ordem de integração

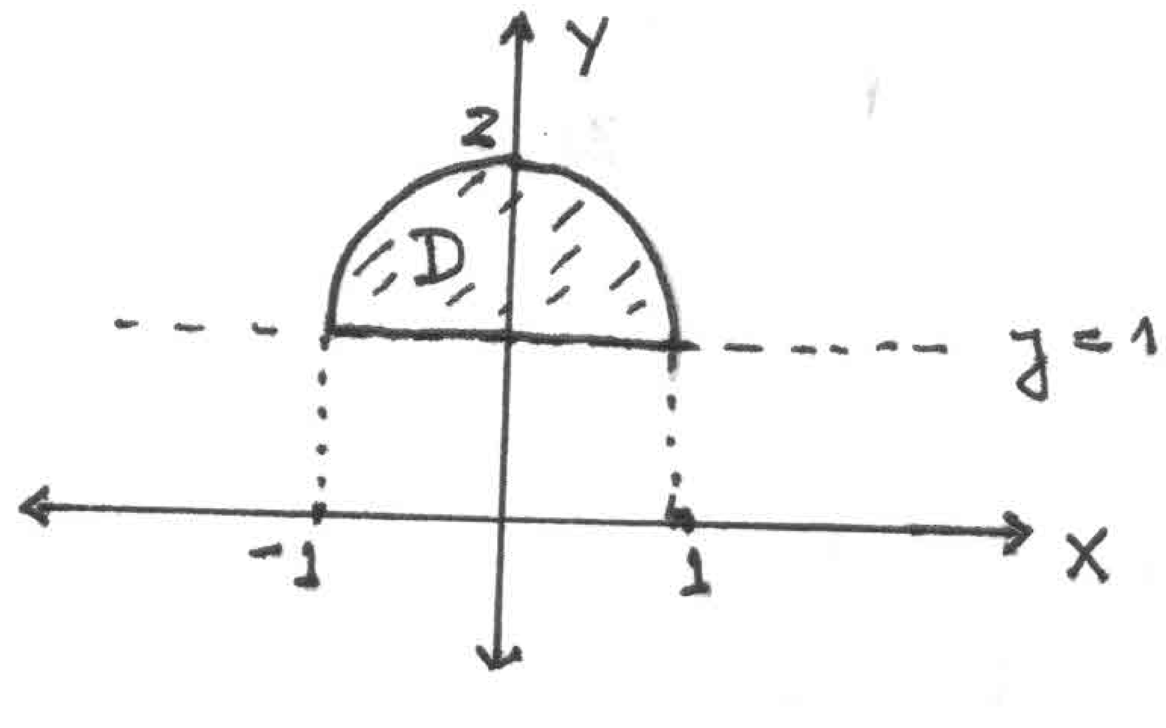
(c) Calcule I para $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2}}$

Solução

(a) observe que em coord. cartesianas D é descrito por

$$D: \begin{cases} -1 \le x \le 1 \\ 1 \le y \le 1+\sqrt{1-x^2} \end{cases}$$

$$y = 1+\sqrt{1-x^2} \Rightarrow (y-1)^2 + x^2 = 1, \quad y > 0$$



(b) A região D também pode ser descrita como:

$$\begin{cases} 1 \le y \le 2 \\ -\sqrt{1-(y-1)^2} \le x \le \sqrt{1-(y-1)^2} \end{cases} \quad \ldots \left(\text{"isto resulta de } x^2+(y-1)^2 = 1\text{"}\right)$$

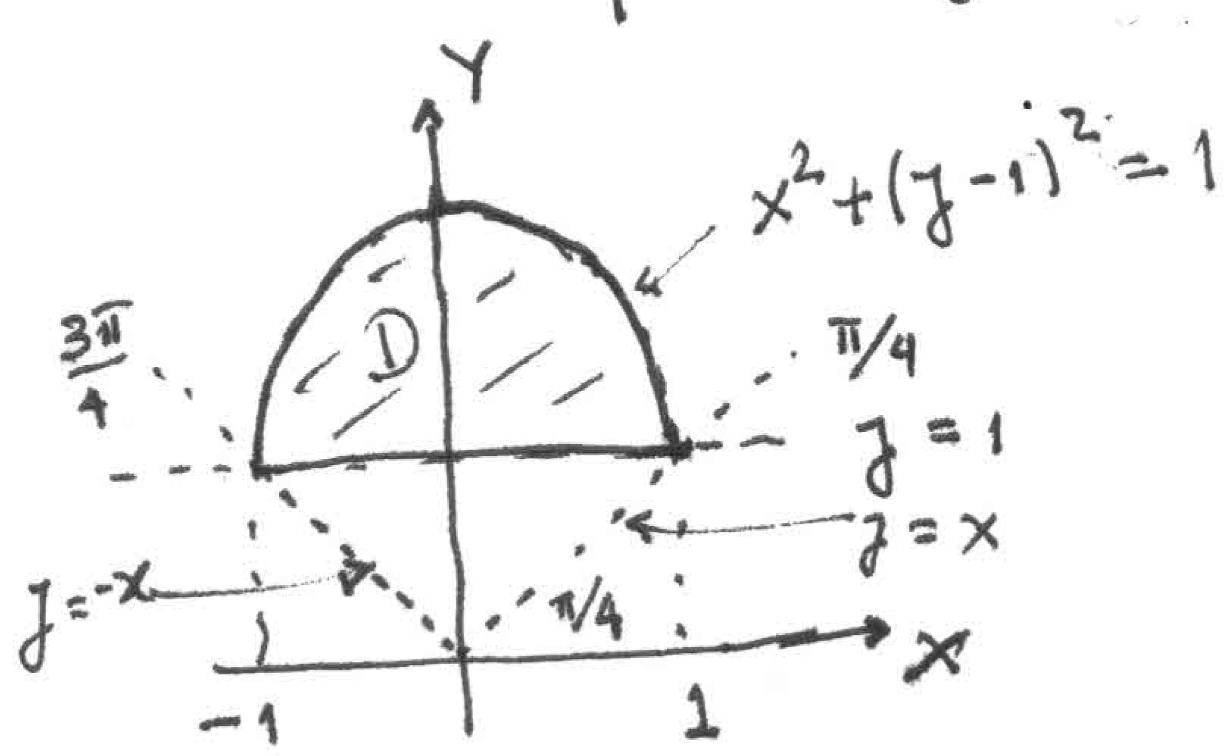
Assim,

$$I = \int_1^2 \int_{-\sqrt{1-(y-1)^2}}^{\sqrt{1-(y-1)^2}} f(x,y)\,dx\,dy$$

(c) Queremos calcular $I = \iint_D \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy$.

Em coord. polares temos

$$\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \\ x^2+y^2 = r^2 \end{cases}, \quad dx\,dy = r\,dr\,d\theta$$



$$x^2+(y-1)^2 = 1$$
$$\Rightarrow x^2+y^2-2y = 0$$
$$\Rightarrow r^2 - 2r\,\text{sen}\,\theta = 0$$
$$\Rightarrow \underbrace{r = 2\,\text{sen}\,\theta}$$

$$y = 1 \Rightarrow r\,\text{sen}\,\theta = 1$$
$$\Rightarrow r = \frac{1}{\text{sen}\,\theta}$$
$$r = \text{cosec}\,\theta$$

Então

$$I = \int_{\pi/4}^{3\pi/4} \int_{\text{cosec}\,\theta}^{2\,\text{sen}\,\theta} \frac{1}{\sqrt{r^2}} \cdot r\,dr\,d\theta = \int_{\pi/4}^{\frac{3\pi}{4}} \int_{\text{cosec}\,\theta}^{2\,\text{sen}\,\theta} dr\,d\theta$$

$$= \int_{\pi/4}^{3\pi/4} (2\,\text{sen}\,\theta - \text{cosec}\,\theta)\,d\theta$$

$$= \left(-2\cos\theta - \ln|\text{cosec}\,\theta - \text{cotg}\,\theta|\right)\Big|_{\pi/4}^{3\pi/4}$$

$$= \left(2\cos\theta + \ln|\text{cosec}\,\theta - \text{cotg}\,\theta|\right)\Big|_{3\pi/4}^{\pi/4}$$

$$= \left(2\frac{\sqrt{2}}{2} + \ln|\sqrt{2}+1|\right) - \left(-2\frac{\sqrt{2}}{2} + \ln(\sqrt{2}-1)\right) =$$

$$I = 2\sqrt{2} + \ln(\sqrt{2}+1) - \ln(\sqrt{2}-1)$$

$$= 2\sqrt{2} + \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) = 2\sqrt{2} + \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\cdot\frac{\sqrt{2}+1}{\sqrt{2}+1}\right)$$

$$= 2\sqrt{2} + \ln(\sqrt{2}+1)^2 = 2\sqrt{2} + 2\ln(\sqrt{2}+1)$$

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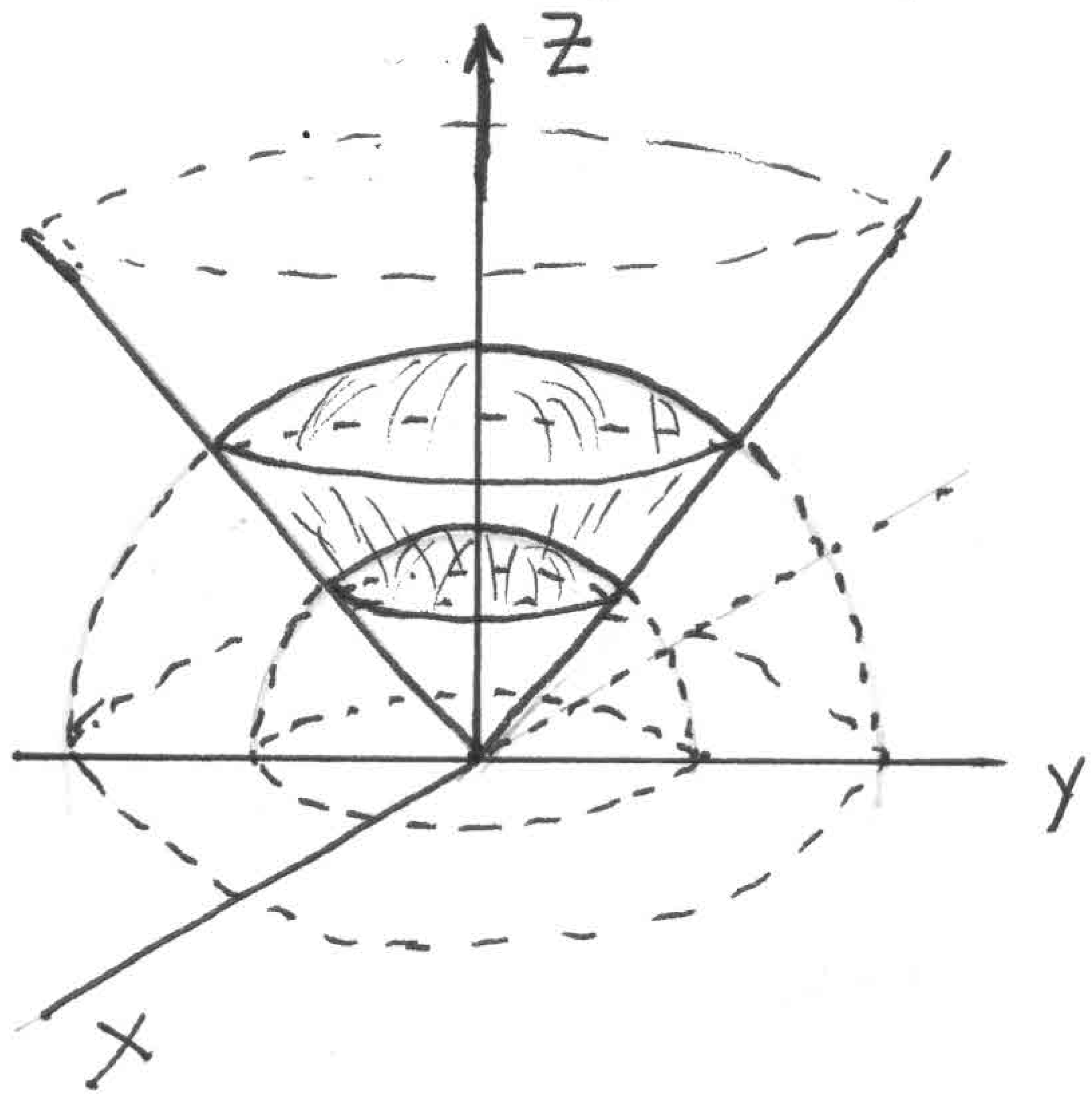
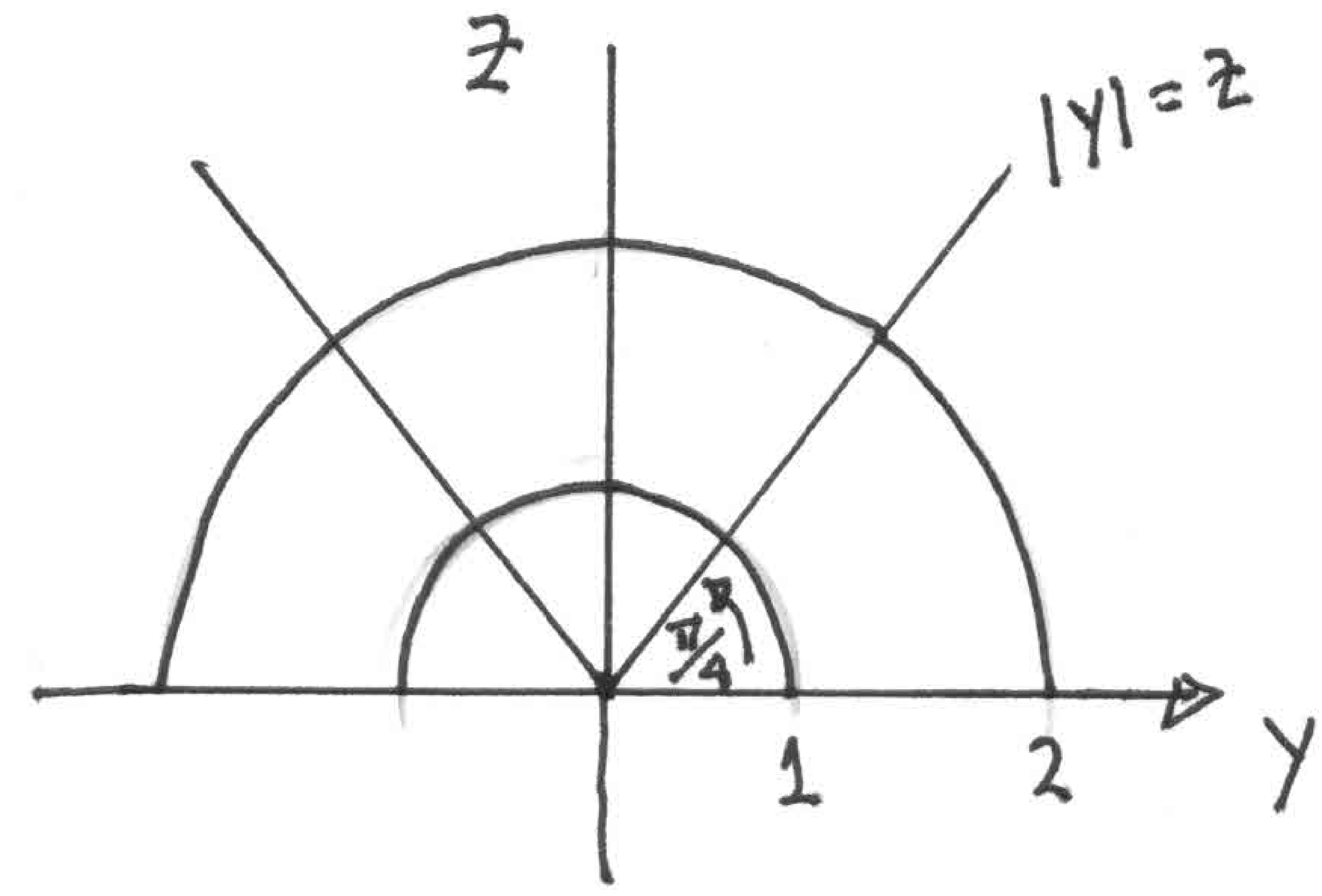
③ Calcule $\int_W \frac{1}{\sqrt{x^2+y^2+z^2}}\,dV$, sendo W a região interior ao cone $z=\sqrt{x^2+y^2}$ limitada superiormente pela esfera $x^2+y^2+z^2=4$ e inferiormente pela esfera $x^2+y^2+z^2=1$.

Solução:





Em coord. esféricas W é descrito como:

$$W: \begin{cases} 0 \leq \theta \leq 2\pi \\ 1 \leq \rho \leq 2 \\ 0 \leq \phi \leq \pi/4 \end{cases}$$

$$\begin{cases} x = \rho\,\text{sen}\,\phi\cos\theta \\ y = \rho\,\text{sen}\,\phi\,\text{sen}\,\theta \\ z = \rho\cos\phi \\ dV = \rho^2\,\text{sen}\,\phi\,d\phi\,d\theta\,d\rho \\ x^2+y^2+z^2 = \rho^2 \end{cases}$$

$$\int_W \frac{1}{\sqrt{x^2+y^2+z^2}}\,dV = \int_0^{2\pi}\int_1^2\int_0^{\pi/4} \frac{1}{\sqrt{\rho^2}}\cdot\rho^2\,\text{sen}\,\phi\,d\phi\,d\rho\,d\theta$$

$$= \int_0^{2\pi}\int_1^2\int_0^{\pi/4} \rho\,\text{sen}\,\phi\,d\phi\,d\rho\,d\theta$$

$$\int_W \frac{1}{\sqrt{x^2+y^2+z^2}}\,dV = \int_0^{2\pi}\int_1^2 -\rho\cos\phi \Big|_{\phi=0}^{\phi=\pi/4} d\rho\, d\theta$$

$$= \int_0^{2\pi}\int_1^2 \left(1-\frac{\sqrt{2}}{2}\right)\rho\, d\rho\, d\theta$$

$$= \left(1-\frac{\sqrt{2}}{2}\right)\cdot 2\pi \cdot \int_1^2 \rho\, d\rho$$

$$= 2\left(1-\frac{\sqrt{2}}{2}\right)\pi \; \frac{\rho^2}{2}\Big|_1^2$$

$$= 3\left(1-\frac{\sqrt{2}}{2}\right)\pi$$

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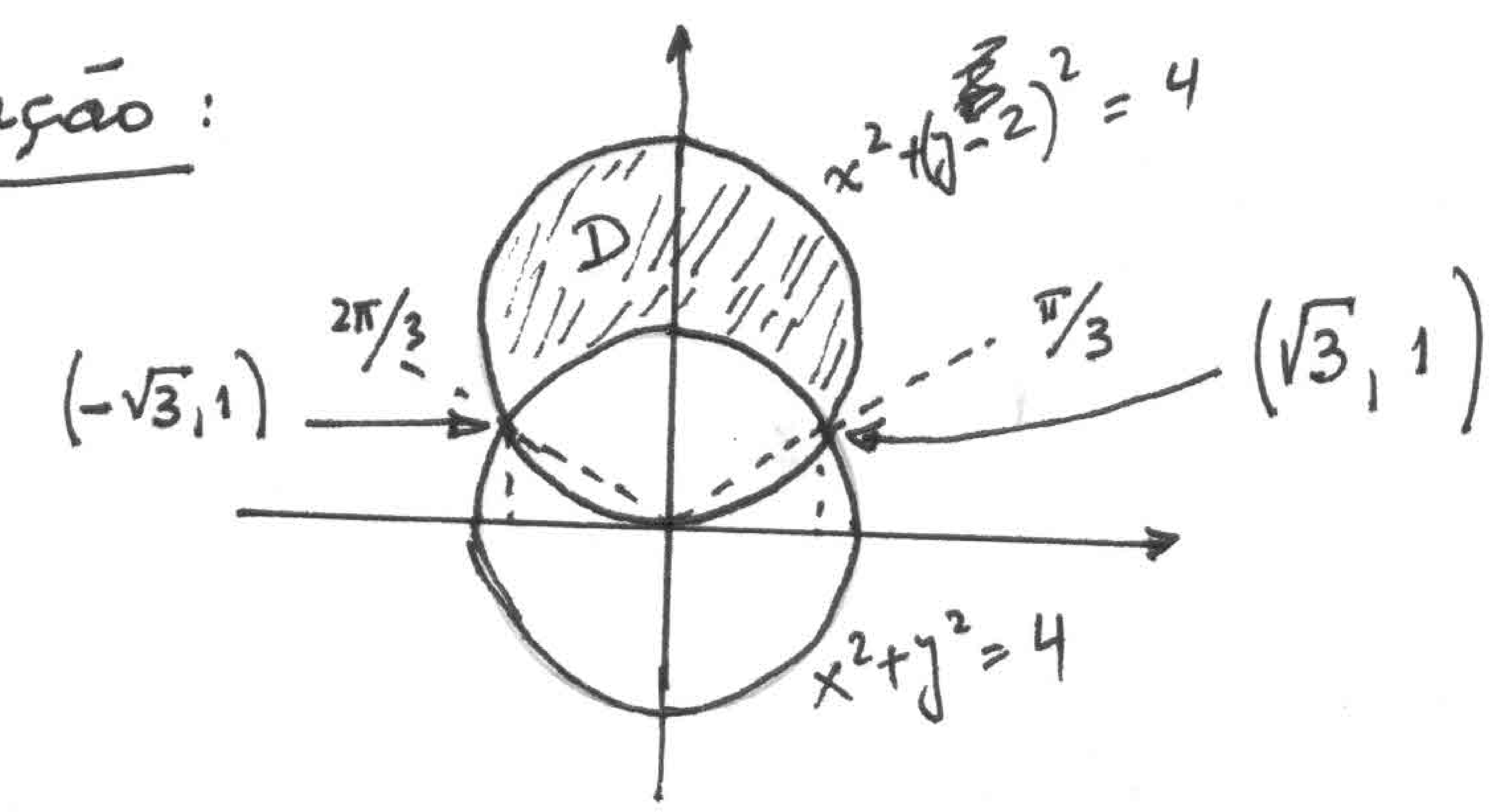
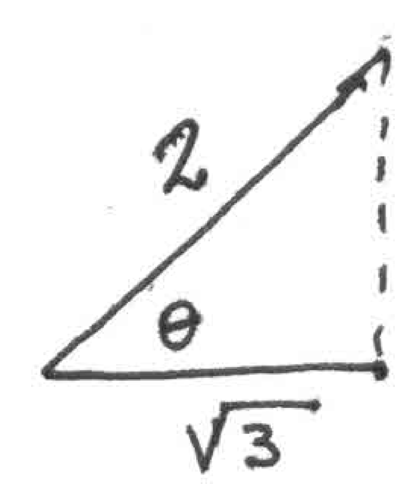
(4) Uma lâmina delgada tem a forma da região D que é interior à circunf. $x^2+(y-2)^2=4$ e interior à circunf. $x^2+y^2=4$. Calcule a massa da lâmina se a densidade é dada por $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2}}$

Solução :





$tg\,\theta = \dfrac{1}{\sqrt{3}}$ $\quad sen\,\theta = 1/2$

$\cos\theta = \sqrt{3}/2$

$\theta = 30° = \pi/6$

Em coord. polares D é descrito como

$$D: \begin{cases} \pi/6 \le \theta \le 5\pi/6 \\ \underbrace{2}_{(i)} \le r \le \underbrace{4\,sen\,\theta}_{(ii)} \end{cases}$$

(i) $x^2+y^2=4 \Rightarrow r^2=4$
$\Rightarrow r=2$

(ii) $x^2+(y-2)^2=4 \Rightarrow x^2+y^2-4y+4=4$
$\Rightarrow r^2-4r\,\text{sen}\,\theta=0$
$\Rightarrow r=4\,\text{sen}\,\theta$

Logo a massa M da lâmina é dada por

$$M=\int_{\pi/6}^{5\pi/6}\int_{2}^{4\,\text{sen}\,\theta}\frac{1}{\sqrt{r^2}}\cdot r\,dr\,d\theta$$

$$=\int_{\pi/6}^{5\pi/6}\int_{2}^{4\,\text{sen}\,\theta}dr\,d\theta=\int_{\pi/6}^{5\pi/6}(4\,\text{sen}\,\theta-2)\,d\theta$$

$$=-4\cos\theta\,\Big|_{\pi/6}^{5\pi/6}-2\theta\,\Big|_{\pi/6}^{5\pi/6}$$

$$=-4\left(\cos 5\pi/6-\cos\pi/6\right)-2\left(\frac{5\pi}{6}-\pi/6\right)$$

$$=-4\left(-\cos\pi/6-\cos\pi/6\right)-\frac{4}{3}\pi$$

$$=8\cos\pi/6-\frac{4}{3}\pi=8\frac{\sqrt{3}}{2}-\frac{4}{3}\pi$$

$$=4\left(\sqrt{3}-\pi/3\right)$$