C3 — LISTA de EXERCÍCIOS Nº 9

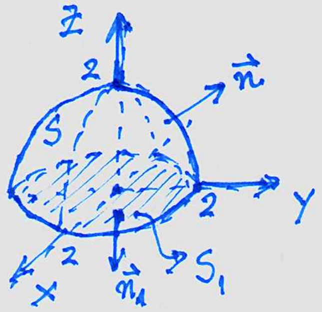
(1) $\vec{F} = (xy^2 + e^y)\vec{i} + (yz^2 + \operatorname{sen}^2 x)\vec{j} + (5 + x^2 z)\vec{k}$

$S: \begin{cases} x^2+y^2+z^2 = 4 \\ z \geq 0 \end{cases}$ $\vec{n}$ normal exterior

Calcular o fluxo de $\vec{F}$ através de S

$\left(\int_S \vec{F} \cdot \vec{n} \, dS \right)$

Solução



Seja $S_1: \begin{cases} z = 0 \\ x^2+y^2 \leq 4 \end{cases}$ $\vec{n}_1 = \vec{k}$ normal a S_1

e seja W o sólido t.q. $\partial W = S \cup S_1$

Pelo Teor. de Gauss (Teor. da Divergência) temos:

$$\int_W \operatorname{div}(\vec{F})\, dV = \int_S \vec{F} \cdot \vec{n}\, dS + \int_{S_1} \vec{F} \cdot \vec{n}_1 \, dS_1$$

Em S_1, $z = 0$ e $\vec{F} \cdot \vec{n}_1 = 5 + x^2 \cancel{z}^{\,0} = 5$

$$\therefore \int_{S_1} \vec{F} \cdot \vec{n}_1 \, dS_1 = 5 \int_{S_1} dS_1 = 5 \text{ área}(S_1) = 5 \cdot \pi \cdot 2^2$$

$$= 20\pi$$

$\operatorname{div}(\vec{F}) = y^2 + z^2 + x^2$ $\quad W_{\rho\phi\theta}: \begin{cases} 0 \leq \rho \leq 2 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi/2 \end{cases}$

$$\int_W \operatorname{div}(\vec{F})\, dV = \int_0^2 \int_0^{2\pi} \int_0^{\pi/2} \rho^2 \cdot \rho^2 \operatorname{sen}\phi \, d\phi \, d\theta \, d\rho$$

(coord. Esféricas)

$$\int_W \operatorname{div}(\vec{F})\,dV = 2\pi \int_0^2 \int_0^{\pi/2} \rho^4 \operatorname{sen}\phi \, d\phi \, d\rho$$

$$= 2\pi \int_0^2 \rho^4 \underbrace{\left(-\cos\phi \Big|_0^{\pi/2}\right)}_{-(0-1)=1} d\rho$$

$$= 2\pi \int_0^2 \rho^4 \, d\rho = 2\pi \cdot \frac{1}{5} \rho^5 \Big|_0^2$$

$$= \frac{2^6}{5}\pi = \frac{64}{5}\pi$$

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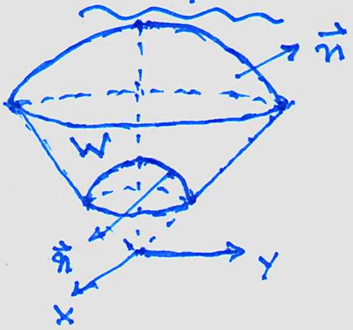
(2) $W: \begin{cases} x^2+y^2+z^2 \geq 1 \\ x^2+y^2+(z-2)^2 \leq 4 \\ z \geq \sqrt{x^2+y^2} \end{cases}$ $\quad S = \partial W$

$\vec{n}$ normal exterior

$$\vec{F} = \left(\frac{x^3}{3}+y\right)\vec{i} + \frac{y^3}{3}\vec{j} + \left(\frac{z^3}{3}+z\right)\vec{k}$$

Calcular $\int_S \vec{F}\cdot\vec{n}\,dS$

Solução



$\operatorname{div}(\vec{F}) = x^2+y^2+z^2$

Coord. Esféricas $\quad W_{\rho\phi\theta}: \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi/4 \\ 1 \leq \rho \leq 4\cos\phi \end{cases}$

$x^2+y^2+(z-2)^2 = 4$

$\underbrace{x^2+y^2+z^2}-4z+4 = 4$

$\rho^2 - 4\rho\cos\phi = 0$

$\{\rho = 4\cos\phi\}$

$\begin{cases} x = \rho \operatorname{sen}\phi\cos\theta \\ y = \rho\operatorname{sen}\phi\operatorname{sen}\theta \\ z = \rho\cos\phi \end{cases}$

$$\int_{S=\partial W} \vec{F}\cdot\vec{n}\,dS \overset{\text{Gauss}}{=} \int_W \operatorname{div}(\vec{F})\,dV$$

$$\int_W div(\vec{F})\,dV = \int_0^{2\pi}\int_0^{\pi/4}\int_1^{4\cos\phi} \rho^2\cdot\rho^2 sen\phi \, d\rho\, d\phi\, d\theta$$

$$= 2\pi \int_0^{\pi/4} \frac{1}{5}\rho^5 \Big|_1^{4\cos\phi} sen\phi\, d\phi$$

$$= \frac{2\pi}{5}\int_0^{\pi/4}\left(4^5\cos^5\phi\, sen\phi - sen\phi\right) d\phi$$

$$= \frac{2\cdot 4^5}{5}\pi\left(-\frac{1}{6}\cos^6\phi\Big|_0^{\pi/4}\right) + \frac{2\pi}{5}\cos\phi\Big|_0^{\pi/4}$$

$$= \frac{4^5}{5\cdot 3}\pi\left(-\left(\frac{\sqrt{2}}{2}\right)^6 + 1\right) + \frac{2\pi}{5}\left(\frac{\sqrt{2}}{2} - 1\right)$$

$$= \dots = \frac{1}{15}(890 - 3\sqrt{2})\pi$$

(3) $f: \mathbb{R}^3 \to \mathbb{R}$, C^2

$\nabla^2 f = x^2 + y^2$

$S: x^2+y^2+z^2 = 1$

$\vec{n}$ exterior

Calcular $\int_S \nabla f\cdot\vec{n}\, dS$

Solução

Lembre: $\nabla^2 f = \Delta f = div(\nabla f)$

Gauss: $$\int_{\partial W = S} \nabla f\cdot\vec{n}\, dS = \int_W div(\nabla f)\, dV = \int_W \nabla^2 f\, dV$$

$$\int_W \nabla^2 f\, dV = \int_0^1\int_0^{\pi}\int_0^{2\pi} \rho^2 sen^2\phi\cdot\rho^2 sen\phi\, d\theta\, d\phi\, d\rho$$

$$\begin{cases} x = \rho \operatorname{sen}\phi \cos\theta \\ y = \rho \operatorname{sen}\phi \operatorname{sen}\theta \\ z = \rho \cos\phi \end{cases} \Rightarrow \quad x^2+y^2 = \rho^2 \operatorname{sen}^2\phi$$

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$$\int_W \nabla^2 f \, dV = 2\pi \int_0^{\pi} \frac{1}{5}\rho^5 \Big|_0^1 \operatorname{sen}^3\phi \, d\phi$$

$$= \frac{2\pi}{5}\int_0^{\pi} \operatorname{sen}^3\phi \, d\phi = \frac{2\pi}{5}\int_0^{\pi} (1-\cos^2\phi)\operatorname{sen}\phi \, d\phi$$

$$= \frac{2\pi}{5}\left(-\cos\phi \Big|_0^{\pi}\right) + \frac{2\pi}{5}\cdot\frac{1}{3}\cos^3\phi \Big|_0^{\pi}$$

$$= \frac{2\pi}{5}\big(-(-1-1)\big) + \frac{2\pi}{5\cdot 3}(-1-1)$$

$$= \frac{4\pi}{5} - \frac{4\pi}{5\cdot 3} = \frac{4\pi}{5}\left(1-\frac{1}{3}\right) = \frac{8\pi}{15}$$

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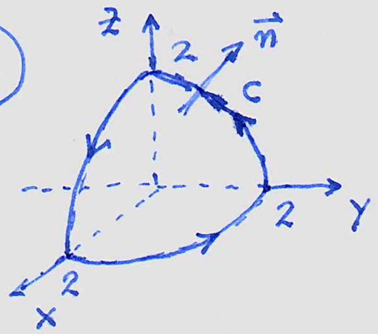
(4)



$$\oint S: \begin{cases} x^2+y^2+z^2 = 4 \\ x \geq 0,\ y \geq 0,\ z \geq 0 \end{cases}$$

$C = \partial S$ $\quad \vec{n}$ normal exterior a S

$$\vec{F} = (x^x + z^2)\vec{i} + (y^y + x^2)\vec{j} + (z^z + y^2)\vec{k}$$

calcular $\quad W = \oint_{\partial S = C} \vec{F}\cdot d\vec{r}$

Solução

stokes $\quad \displaystyle\int_S \operatorname{rot}(\vec{F})\cdot\vec{n}\, dS = \oint_{\partial S} \vec{F}\cdot d\vec{r}$

$$\operatorname{rot}(\vec{F}) = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ x^x+z^2 & y^y+x^2 & z^z+y^2 \end{pmatrix}$$

$$= (0+2y)\vec{i} - (0-2z)\vec{j} + (2x-0)\vec{k}$$

$$= +2y\,\vec{i} + 2z\,\vec{j} + 2x\,\vec{k}$$

$$\vec{n} = \frac{1}{2}\left(x\,\vec{i} + y\,\vec{j} + z\,\vec{k}\right)$$

$$\text{rot}(\vec{F})\cdot\vec{n} = xy + yz + xz$$

$$\begin{cases} x = 2\,\text{sen}\,\phi\cos\theta \\ y = 2\,\text{sen}\,\phi\,\text{sen}\,\theta \\ z = 2\cos\phi \end{cases}$$

$$S: \begin{cases} \rho = 2 \\ 0 \leq \theta \leq \pi/2 \\ 0 \leq \phi \leq \pi/2 \end{cases}$$

$$dS = 4\,\text{sen}\,\phi\, d\phi\, d\theta$$

$$\text{rot}(\vec{F})\cdot\vec{n} = 4\,\text{sen}^2\phi\,\text{sen}\,\theta\cos\theta + 4\,\text{sen}\,\theta\,\text{sen}\,\phi\cos\phi + 4\cos\theta\,\text{sen}\,\phi\cos\phi$$

$$\text{rot}(\vec{F})\cdot\vec{n}\,dS = 16\left(\text{sen}^3\phi\,\text{sen}\,\theta\cos\theta + \text{sen}^2\phi\cos\phi\,\text{sen}\,\theta + \text{sen}^2\phi\cos\phi\cos\theta\right)d\phi\,d\theta$$

$$\int_0^{\pi/2}\int_0^{\pi/2} \text{sen}^3\phi\,\text{sen}\,\theta\cos\theta\,d\theta\,d\phi = \int_0^{\pi/2}\text{sen}^3\phi\cdot\frac{1}{2}\text{sen}^2\theta\Big|_0^{\pi/2} d\phi$$

$$= \frac{1}{2}\int_0^{\pi/2}\text{sen}^3\phi\,d\phi$$

$$= \dots = \frac{1}{2}\cdot\frac{2}{3} = 1/3$$

$$\int_0^{\pi/2}\int_0^{\pi/2}\text{sen}^2\phi\cos\phi\,\text{sen}\,\theta\,d\phi\,d\theta = \int_0^{\pi/2}\frac{1}{3}\text{sen}^3\phi\Big|_0^{\pi/2}\cdot\text{sen}\,\theta\,d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2}\text{sen}\,\theta\,d\theta = \frac{1}{3}$$

$$\int_0^{\pi/2}\int_0^{\pi/2}\text{sen}^2\phi\cos\phi\cos\theta\,d\phi\,d\theta = \int_0^{\pi/2}\frac{1}{3}\text{sen}^3\phi\Big|_0^{\pi/2}\cos\theta\,d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2}\cos\theta\,d\theta = 1/3$$

$$\therefore \int_S \text{rot}(\vec{F})\cdot\vec{n}\,dS = 16\cdot\left(1/3 + 1/3 + 1/3\right) = 16$$

$$\therefore \int_{C=\partial S}\vec{F}\cdot d\vec{r} = 16$$

(7) $\vec{F} = x\vec{j} + xy\,\vec{k}$

$S: \begin{cases} x^2+y^2+z^2/4 = 1 \\ z \leq 1 \end{cases}$ $\qquad \vec{n}$ exterior

usar stokes para calcular $\int_S \text{rot}(\vec{F}) \cdot \vec{n}\, dS$

<u>Solução</u>

$\partial S: \begin{cases} z=1 \\ x^2+y^2 = 3/4 \end{cases}$ percorrida no sentido anti horário

$\partial S: \quad \gamma(t) = \left(\frac{\sqrt{3}}{2}\cos t, \frac{\sqrt{3}}{2}\,\text{sen}\, t, 1\right), \qquad t \in [0, 2\pi]$

$\gamma'(t) = \left(-\frac{\sqrt{3}}{2}\,\text{sen}\, t, \frac{\sqrt{3}}{2}\cos t, 0\right)$

$$\int_S \text{rot}(F) \cdot \vec{n}\, dS = \int_{\partial S} \vec{F} \cdot d\vec{r}$$

$$= \int_0^{2\pi} \frac{3}{4}\cos^2 t\, dt = \frac{3}{4}\pi$$

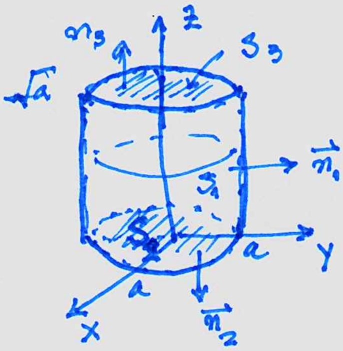
(9) $\vec{F} = (x + z\cos y)\vec{i} + (x-y+z)\vec{j} + (z^4 - 3a^2)\vec{k}$

$S_1: \begin{cases} x^2+y^2 = a^2 \\ 0 \leq z \leq \sqrt{a} \end{cases} \qquad S_2: \begin{cases} x^2+y^2 \leq a^2 \\ z = 0 \end{cases}$

$S = S_1 \cup S_2$

$\int_S \vec{F} \cdot \vec{n}\, dS = a^3\pi$, $\quad \vec{n}$ "apontando para fora"

calcular o valor de a.



Sejam $S_3 : \begin{cases} x^2 + y^2 \leq a^2 \\ z = \sqrt{a} \end{cases}$

e W o sólido t.q. $\partial W = S \cup S_3$

$$\int_W div(\vec{F})\,dV = \int_S \vec{F}\cdot\vec{n}\,dS + \int_{S_3} \vec{F}\cdot\vec{n}_3\,dS$$

$$= a^3\pi + \int_{S_3} \vec{F}\cdot\vec{n}_3\,dS$$

$\vec{n}_3 = \vec{k}$

$\vec{F}\cdot\vec{n}_3 = z^4 - 3a^2$ em S_3, $z = \sqrt{a}$

$= a^2 - 3a^2 = -2a^2$

$$\int_{S_3} -2a^2\,dS = -2a^2\,\text{área}(S_3) = -2a^2\cdot\pi a^2 = -2a^4\pi$$

$div(\vec{F}) = 1 - 1 + 4z^3 = 4z^3$

$$\int_W div(\vec{F})\,dV = \int_W 4z^3\,dV \qquad W_{r\theta z} : \begin{cases} 0 \leq r \leq a \\ 0 \leq \theta \leq 2\pi \\ 0 \leq z \leq \sqrt{a} \end{cases}$$

$$= \int_0^{\sqrt{a}} \int_0^a \int_0^{2\pi} 4z^3 \cdot r\,d\theta\,dr\,dz \qquad \begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \\ z = z \end{cases}$$

$$= 2\pi \cdot \int_0^{\sqrt{a}} \frac{1}{2} r^2 \Big|_0^a \cdot 4z^3\,dz$$

$$= 4\pi a^2 \int_0^{\sqrt{a}} z^3\,dz = 4\pi a^2 \cdot \frac{1}{4} z^4 \Big|_0^{\sqrt{a}} = \pi a^4$$

Daí, $\pi a^4 = -2\pi a^4 + \pi a^3$

$3\pi a^4 = \pi a^3$

$3a = 1$

$\therefore$ $a = 1/3$