C3 - Exercícios Lista 8

(1) $\vec{F} = x\vec{i} - y\vec{j}$, $S: \begin{cases} x^2+y^2+z^2 = a^2 \\ \text{No 1º octante} \end{cases}$

$\vec{n}$ apontando para a origem

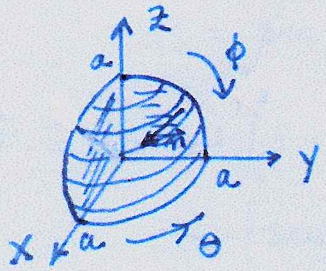
calcular $\int_S \vec{F}\cdot\vec{n}\, dS$

Solução:

Parametrizamos S com coord. esféricas

$$X(\phi,\theta) = (a\,\text{sen}\phi\cos\theta,\ a\,\text{sen}\phi\,\text{sen}\theta,\ a\cos\phi)$$

$$(\phi,\theta)\in D: \begin{cases} 0 \leq \phi \leq \pi/2 \\ 0 \leq \theta \leq \pi/2 \end{cases}$$



$$\vec{n} = (-\text{sen}\phi\cos\theta,\ -\text{sen}\phi\,\text{sen}\theta,\ -\cos\phi)$$

$$\vec{F}\cdot\vec{n} = -a\,\text{sen}^2\phi\cos^2\theta + a\,\text{sen}^2\phi\,\text{sen}^2\theta$$
$$= -a\,\text{sen}^2\phi\,(\cos^2\theta - \text{sen}^2\theta)$$
$$= -a\,\text{sen}^2\phi\cos 2\theta$$

$$dS = a^2\,\text{sen}\phi\, d\theta\, d\phi \quad (= \|X_\phi \wedge X_\theta\|\, d\theta\, d\phi)$$

$$\therefore \int_S \vec{F}\cdot\vec{n}\, dS = \int_0^{\pi/2}\int_0^{\pi/2} -a^3\,\text{sen}^2\phi\cdot\text{sen}\phi\cdot\cos 2\theta\, d\theta\, d\phi$$

$$= \int_0^{\pi/2} -a^3(1-\cos^2\phi)\,\text{sen}\phi\,\left(\frac{1}{2}\,\text{sen}\,2\theta\right)\Big|_0^{\pi/2} d\phi$$

$$= -a^3\int_0^{\pi/2}(\text{sen}\phi - \cos^2\phi\,\text{sen}\phi)\cdot\frac{1}{2}(0-0)\, d\phi$$

$$= 0$$

② Calcular $\int_S \vec{F}.\vec{n}\, dS$

$$\vec{F} = (x-y-4)\vec{i} + y\vec{j} + z\vec{k}, \quad S: \begin{cases} x^2+y^2+z^2 = 1 \\ z \geq 0 \end{cases}$$

$\vec{n}$ normal exterior

Solução

$$\vec{n} = (x,y,z), \quad \vec{F}.\vec{n} = (x-y-4)x + y^2 + z^2$$
$$= x^2+y^2+z^2 - x(y+4)$$
$$= 1 - x(y+4)$$

$$\begin{cases} x = \operatorname{sen}\phi \cos\theta \\ y = \operatorname{sen}\phi \operatorname{sen}\theta \\ z = \cos\phi \end{cases} \qquad (\phi,\theta) \in D: \begin{cases} 0 \leq \phi \leq \pi/2 \\ 0 \leq \theta \leq 2\pi \end{cases}$$

$$dS = \operatorname{sen}\phi \, d\phi \, d\theta$$

$$\vec{F}.\vec{n}\, dS = \left(1 - \operatorname{sen}^2\phi \operatorname{sen}\theta\cos\theta - 4\operatorname{sen}\phi\cos\theta\right)\operatorname{sen}\phi \, d\phi\, d\theta$$
$$= \left[\operatorname{sen}\phi - \operatorname{sen}^3\phi\operatorname{sen}\theta\cos\theta - 4\operatorname{sen}^2\phi\cos\theta\right] d\phi\, d\theta$$

$$\int_S \vec{F}.\vec{n}\, dS = \int_0^{\pi/2}\int_0^{2\pi} \left(\operatorname{sen}\phi - \operatorname{sen}^3\phi\operatorname{sen}\theta\cos\theta - 4\operatorname{sen}^2\phi\cos\theta\right) d\theta\, d\phi$$

$$= \int_0^{\pi/2} \left[2\pi\operatorname{sen}\phi - \operatorname{sen}^3\phi\left(\tfrac{1}{2}\operatorname{sen}^2\theta \Big|_0^{2\pi}\right) - 4\operatorname{sen}^2\phi\left(\operatorname{sen}\theta\Big|_0^{2\pi}\right)\right] d\phi$$

$$= 2\pi\int_0^{\pi/2} \operatorname{sen}\phi\, d\phi = 2\pi\left(-\cos\phi\Big|_0^{\pi/2}\right)$$

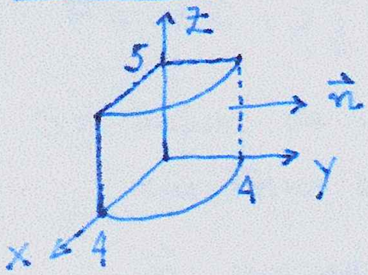
$$= -2\pi(0-1) = 2\pi$$

③ $\int_S \vec{F}\cdot\vec{n}\, dS = ?$

$\vec{F} = z\vec{i} + x\vec{j} - 3y^2 z\vec{k}$, $S: \begin{cases} x^2+y^2 = 16 \\ 0 \leq z \leq 5 \\ x \geq 0,\ y \geq 0 \end{cases}$

$\vec{n}$ exterior ao cilindro

Solução



$\vec{n} = \frac{1}{4}(x, y, 0)$

$\vec{F}\cdot\vec{n} = \frac{1}{4}(xz + xy)$

$$\int_S \vec{F}\cdot\vec{n}\, dS = \frac{1}{4}\int_S (xz+xy)\, dS$$

Parametrizamos S com coord. cilíndricas

$$\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \\ z = z \end{cases}$$

Em S, $r=4$, $0 \leq z \leq 5$, $0 \leq \theta \leq \pi/2$

ou seja:

$$X(\theta, z) = (4\cos\theta, 4\,\text{sen}\,\theta, z) \ , \ (\theta, z) \in D: \begin{cases} 0 \leq \theta \leq \pi/2 \\ 0 \leq z \leq 5 \end{cases}$$

$$dS = \| X_\theta \wedge X_z \|\, d\theta\, dz = 4\, d\theta\, dz$$

$$\therefore \int_S \vec{F}\cdot\vec{n}\, dS = \int_0^5 \int_0^{\pi/2} \frac{1}{\cancel{4}} \cdot 4\cos\theta\,(z + 4\,\text{sen}\,\theta)\cancel{4}\, d\theta\, dz$$

$$= 4\int_0^5 \int_0^{\pi/2} (z\cos\theta + 4\,\text{sen}\,\theta\cos\theta)\, d\theta\, dz$$

$$= 4\int_0^5 \left(z \left(\text{sen}\,\theta \Big|_0^{\pi/2} \right) + 2\,\text{sen}^2\theta \Big|_0^{\pi/2} \right) dz$$

$$\int_S \vec{F}\cdot\vec{n}\, dS = 4\int_0^5 (z+2)\, dz = 4\left(\frac{1}{2}z^2 + 2z\right)\Big|_0^5$$

$$= 2\cdot 25 + 8\cdot 5 = 50 + 40 = 90$$

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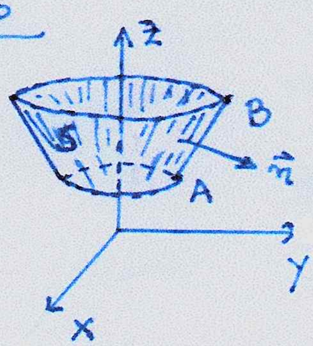
⑥ Calcular $\int_S \vec{F}\cdot\vec{n}\, dS$.

$\vec{F} = x\vec{\imath} + y\vec{\jmath} - \frac{z^2}{2}\vec{k}$

S: superf de revolução obtida ao girar o segmento AB, em torno do eixo Z. $A = (0,1,2)$, $B = (0,2,4)$

$\vec{n}$ normal exterior.

Solução



Uma parametrização para o segmento AB é

$$\vec{r}(t) = A + t(B-A)\ ,\ 0 \le t \le 1$$
$$= (0,\ 1+t,\ 2+2t)$$

Então, uma param. para S é:

$$X: \begin{cases} x = (1+t)\cos\theta \\ y = (1+t)\,\mathrm{sen}\,\theta \\ z = 2+2t \end{cases} \qquad (t,\theta)\in D: \begin{cases} 0 \le t \le 1 \\ 0 \le \theta \le 2\pi \end{cases}$$

$\vec{N} = X_t \wedge X_\theta$ é normal a S , $dS = \|\vec{N}\|\, dt\, d\theta$

Como $\vec{n}$ aponta para fora, $\vec{n} = \dfrac{-\vec{N}}{\|\vec{N}\|}$

$X_t = (\cos\theta, \mathrm{sen}\,\theta, 2)$

$X_\theta = (-(1+t)\,\mathrm{sen}\,\theta, (1+t)\cos\theta, 0)$

$\vec{N} = X_t \wedge X_\theta = (1+t)(-2\cos\theta, -2\,\mathrm{sen}\,\theta, 1)$

$$\vec{F}\cdot\vec{n}\,dS = \vec{F}\cdot(-\vec{N})\,d\theta\,dt$$

$$\vec{F}\cdot(-\vec{N}) = (1+t)^2\cdot 2\cos^2\theta + (1+t)^2\cdot 2\,\text{sen}^2\theta + (1+t)\cdot\frac{(2+2t)^2}{2}$$

$$= 2(1+t)^2 + 2(1+t)^3$$

$$\therefore \int_S \vec{F}\cdot\vec{n}\,dS = \int_D \left(2(1+t)^2 + 2(1+t)^3\right)dt\,d\theta$$

$$= \int_0^1\int_0^{2\pi}\left(2(1+t)^2+2(1+t)^3\right)d\theta\,dt$$

$$= 2\pi\left[\frac{2}{3}(1+t)^3 + \frac{1}{2}(1+t)^4\right]\Big|_0^1$$

$$= 2\pi\left[\frac{2}{3}(8-1) + \frac{1}{2}(16-1)\right]$$

$$\cancel{=}\ \cancel{2\pi\cdot 76}$$

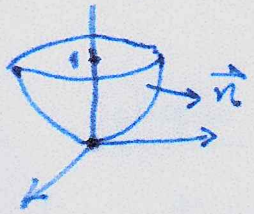
$$= 2\pi\cdot\frac{73}{6} = \frac{73}{3}\pi$$

7) $\int_S \vec{F}\cdot\vec{n}\,dS = ?$

$F = y\vec{i} - x\vec{j} + z^2\vec{k}$, $S: \begin{cases} z = x^2+y^2 \\ 0 \leq z \leq 1 \end{cases}$

$\vec{n}$ normal exterior.

Solução:



$\varphi(x,y) = (x, y, x^2+y^2)$, $(x,y) \in D$

$D: 0 \leq x^2+y^2 \leq 1$

é uma param. para S

$$\vec{n} = -\frac{\varphi_x \wedge \varphi_y}{\|\varphi_x \wedge \varphi_y\|}$$

$\varphi_x = (1, 0, 2x)$

$\varphi_y = (0, 1, 2y)$

$$\vec{n} = \frac{1}{\sqrt{1+4(x^2+y^2)}}(2x, 2y, -1)$$

$\varphi_x \wedge \varphi_y = (-2x, -2y, 1)$

$\|\varphi_x \wedge \varphi_y\| = \sqrt{1+4(x^2+y^2)}$

$dS = \|\varphi_x \wedge \varphi_y\|\,dx\,dy$

$$\vec{F}\cdot\vec{n}\,dS = (\cancel{2xy} - \cancel{2xy} - z^2)\,dx\,dy$$

$$= -(x^2+y^2)^2\,dx\,dy.$$

$$\therefore \int_S \vec{F}\cdot\vec{n}\,dS = -\int_D (x^2+y^2)^2\,dx\,dy$$

Em Coord. Polares: $D: \begin{cases} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{cases}$

$$\int_S \vec{F}\cdot\vec{n}\,dS = -\int_0^1 \int_0^{2\pi} (r^2)^2 \cdot r\,d\theta\,dr = -2\pi \int_0^1 r^5\,dr$$

$$= -2\pi \cdot \frac{1}{6} r^6 \Big|_0^1 = -\frac{2\pi}{6}$$

$$= -\pi/3$$