Lista de Exercícios $N^o 7$: Cálculo III

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1 *Determine uma parametrização para as seguintes superfícies:*

(a) *O plano de equação* $x + 2y - z = 5$;

(b) *O paraboloide* $z = 4x^2$;

(c) *O hiperboloide* $x^2 + y^2 - z^2 = 1$;

(d) *O cilindro elíptico* $4x^2 + 9y^2 = 36$;

(e) *O cone de revolução gerado pela rotação em torno do eixo Z da semi reta* $z = y,\ y \geq 0$;

Solução

$\vec{v} = (1, 2, -1)$ $P = (0, 0, -5)$

(a) $x + 2y - z = 5$

- Como gráfico da função $z = x + 2y - 5$, $z(x,y)$

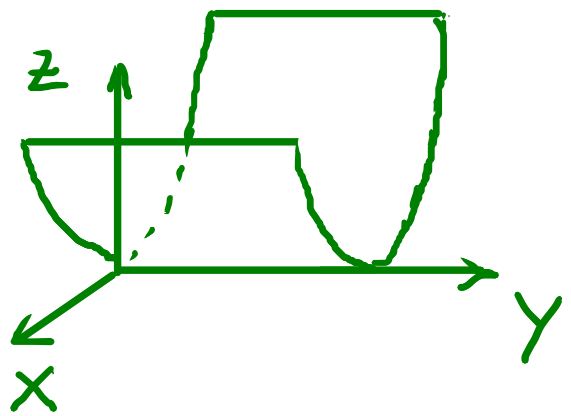
$\varphi: \mathbb{R}^2 \to \mathbb{R}^3$, $\varphi(x,y) = (x, y, x + 2y - 5)$

- Eq's paramétricas do plano

$(x,y,z) = (x, y, x+2y-5)$
$= (0,0,-5) + x(1,0,1) + y(0,1,2)$

$$\begin{cases} x = t \\ y = s \\ z = -5 + t + 2s \end{cases}, \quad s, t \in \mathbb{R}$$

(b) $z = 4x^2$

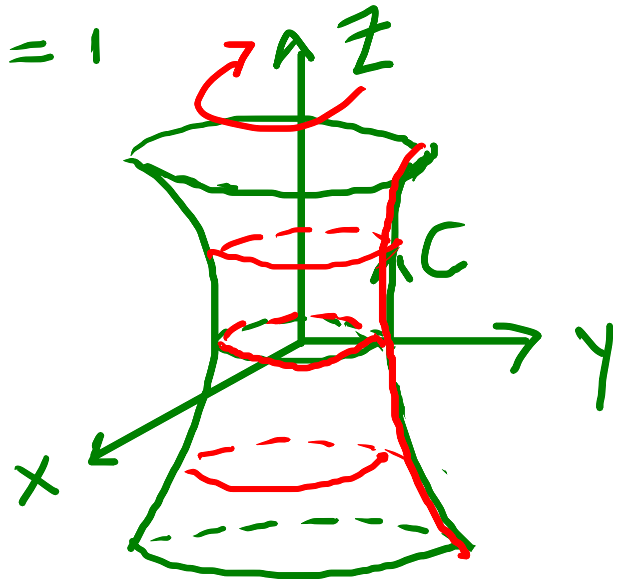


Como gráfico da função $z = 4x^2$

$\varphi: \mathbb{R}^2 \to \mathbb{R}^3$, $\varphi(x,y) = (x, y, 4x^2)$

(c) $x^2 + y^2 - z^2 = 1$

$y^2 - z^2 = 1$, $x = 0$



$z = 0$
$x^2 + y^2 = 1$
$z = cte = K$
$x^2 + y^2 = 1 + K^2 > 0$

Como superf. de revolução obtida ao girar a curva $C: \begin{cases} x = 0 \\ y^2 - z^2 = 1, \ y > 0 \end{cases}$ em torno do eixo Z

$ch^2 t - sh^2 t = 1$

Parametrização de C: $\begin{cases} x = 0 \\ y = \cosh t \\ z = \operatorname{senh} t \end{cases}, \quad t \in \mathbb{R}$

S: $\varphi(t,s) = (\cos s \cosh t, \operatorname{sen} s \cosh t, \operatorname{senh} t)$

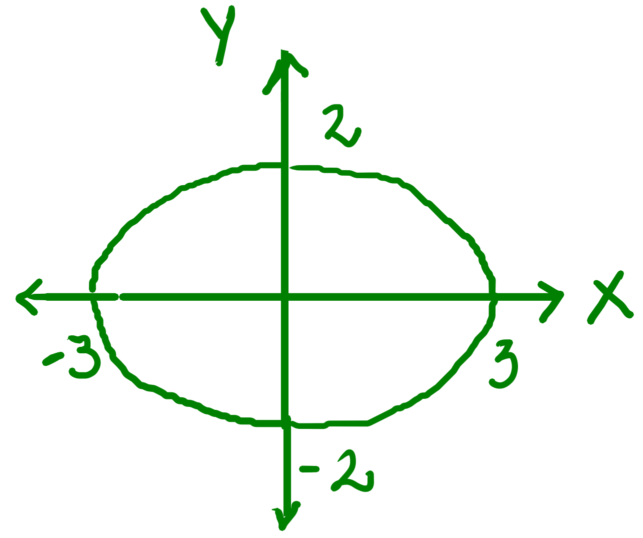
$t \in \mathbb{R}$, $s \in [0, 2\pi]$

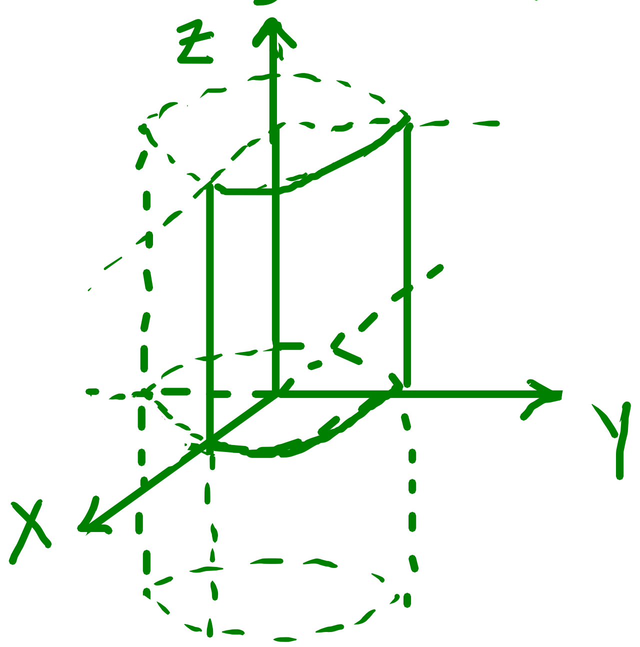
(d) *O cilindro elíptico* $4x^2 + 9y^2 = 36$;

(e) *O cone de revolução gerado pela rotação em torno do eixo* Z *da semi reta* $z = y,\ y \geq 0$;

(d) $4x^2+9y^2=36$ ou $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$

$\left(\dfrac{x}{3}\right)^2+\left(\dfrac{y}{2}\right)^2=1$





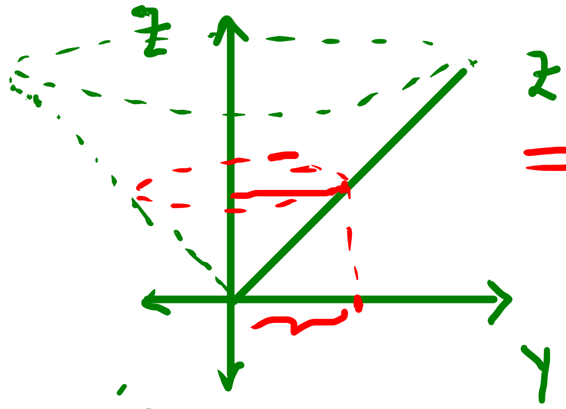
Coord cilíndricas:

$$\begin{cases} x = 3\cos\theta \\ y = 2\,\text{sen}\,\theta \\ z = z \end{cases}$$

$\varphi : [0,2\pi]\times\mathbb{R} \longrightarrow \mathbb{R}^3$

$\varphi(\theta, z) = (3\cos\theta,\ 2\,\text{sen}\,\theta,\ z)$

(e)



$z = y$

$y = u$

$z = y = u$

- Gráfico de $z = z(x,y) = y$

$\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3,\ \varphi(x,y) = (x, y, y)$

- Superf. de Revolução

$\varphi(u,v) = (u\cos v,\ u\,\text{sen}\,v,\ u)$

$u \geq 0,\quad 0 \leq v \leq 2\pi$

2 *Calcule* $\int_S f\,dS$ *onde a superfície* S *e o campo escalar* f *são dados por:*

(a) $f(x,y,z) = x^2 - xy^2 - z + 1,\qquad S: \begin{cases} X(u,v) = (u, v, u^2+1) \\ 0 \le u \le 1, \quad 0 \le v \le 2 \end{cases}$

(b) $f(x,y,z) = x^2 + y^2,\qquad S: x^2+y^2+z^2 = 4,\ z \ge 1$

(c) $f(x,y,z) = x^2 y,\qquad S: x^2+z^2 = a^2,\ 0 \le y \le 1$

(d) $f(x,y,z) = z^2,\qquad S: z = \sqrt{x^2+y^2}$, *entre os planos* $z = 1$ *e* $z = 4$

Solução

(a) $\int_S f\,ds = \int_D f(X(u,v))\,\|X_u \wedge X_v\|\,du\,dv$

$f(X(u,v)) = u^2 - uv^2 - (u^2+1) + 1 = -uv^2$

$X_u = (1,0,2u)$

$X_v = (0,1,0)$

$X_u \wedge X_v = (-2u, 0, 1)$

$\|X_u \wedge X_v\| = \sqrt{1+4u^2}$

$D: \begin{cases} 0 \le u \le 1 \\ 0 \le v \le 2 \end{cases}$

$$\int_S f\,ds = \int_0^1 \int_0^2 -uv^2 \sqrt{1+4u^2}\,dv\,du$$

$$= -\int_0^1 \frac{1}{3} v^3 \Big|_0^2 \cdot u\sqrt{1+4u^2}\,du\,dv$$

$$= -\int_0^1 \frac{8}{3} u\sqrt{1+4u^2}\,du$$

$t = 1+4u^2$

$dt = 8u\,du$

$u = 0 \Rightarrow t = 1$

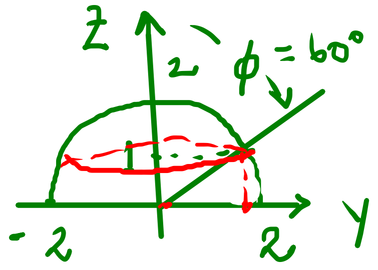
$u = 1 \Rightarrow t = 5$

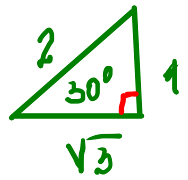
$$\int_S f\,ds = -\int_1^5 \frac{1}{3}\sqrt{t}\,dt = -\frac{1}{3}\cdot\frac{2}{3} t^{3/2}\Big|_1^5$$

$$= -\frac{2}{9}\left(5\sqrt{5} - 1\right)$$

$z = \sqrt{4 - x^2 - y^2}$

(b) $S: \begin{cases} x^2+y^2+z^2 = 4 \\ z \ge 1 \end{cases}$





$\cos\alpha = \frac{\sqrt{3}}{2}$

$\text{sen}\,\alpha = 1/2$

$\alpha = 30°$

Em coord. Esféricas $\quad S: \begin{cases} x = 2\,\text{sen}\,\phi\cos\theta \\ y = 2\,\text{sen}\,\phi\,\text{sen}\,\theta \\ z = 2\cos\phi \end{cases}$

$0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi/3 \quad (z \ge 1)$

$f(x,y) = x^2+y^2 = 4\,\text{sen}^2\phi\cos^2\theta + 4\,\text{sen}^2\phi\,\text{sen}^2\theta$

$= 4\,\text{sen}^2\phi$

$X_\phi = (2\cos\phi, \cos\theta, 2\cos\phi\,\text{sen}\,\theta, -2\,\text{sen}\,\phi)$

$X_\theta = (-2\,\text{sen}\,\phi\,\text{sen}\,\theta, 2\,\text{sen}\,\phi\cos\theta, 0)$

$X_\phi \wedge X_\theta = (4\,\text{sen}^2\phi\cos\theta, 4\,\text{sen}^2\phi\,\text{sen}\,\theta, 4\,\text{sen}\,\phi\cos\phi)$

$\|X_\phi \wedge X_\theta\| = \sqrt{16\,\text{sen}^4\phi + 16\,\text{sen}^2\phi\cos^2\phi} = 4\,\text{sen}\,\phi$

$x^2+y^2 = 3$

$x = r\cos\theta$

$y = r\,\text{sen}\,\theta$

$z = z = \sqrt{4-r^2}$

$\int (x^2+y^2)\,ds$

$\int r^3\sqrt{4-r^2}$

$$f(X(\phi,\theta)) = 4\,\text{sen}^2\phi \quad , \quad \|X_\phi \wedge X_\theta\| = 4\,\text{sen}\,\phi$$

$$D: \begin{cases} 0 \le \phi \le \pi/3 \\ 0 \le \theta \le 2\pi \end{cases}$$

$$\int_S f\,dS = \int_D f(X(\phi,\theta))\,\|X_\phi \wedge X_\theta\|\,d\phi\,d\theta$$

$$= \int_0^{\pi/3}\int_0^{2\pi} 4\,\text{sen}^2\phi \cdot 4\,\text{sen}\,\phi\;d\theta\,d\phi$$

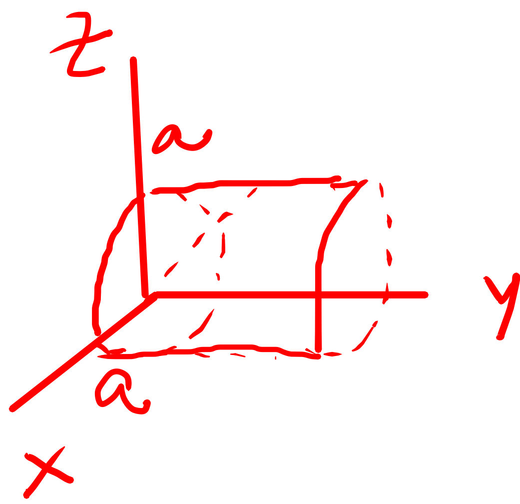
$$= 16\cdot 2\pi \int_0^{\pi/3} (1-\cos^2\theta)\,\text{sen}\,\theta\,d\theta$$

$$= 32\pi\left(-\cos\theta + \frac{1}{3}\cos^3\theta\right)\Big|_0^{\pi/3}$$

$$= -32\pi\left(\left[\frac{1}{2} - \frac{1}{3}\left(\frac{1}{2}\right)^3\right] - \left[1 - \frac{1}{3}\right]\right)$$

$$= -32\pi\left(\frac{1}{2} - \frac{1}{24} - \frac{2}{3}\right) = +\frac{20}{3}\pi$$

(c) $f(x,y,z) = x^2y, \qquad S: x^2+z^2 = a^2,\ 0 \le y \le 1$



$$S: \begin{cases} x = a\cos\theta \\ y = y \\ z = a\,\text{sen}\,\theta \end{cases} \qquad \begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le y \le 1 \end{array}$$

$$\begin{cases} \varphi_\theta = (-a\,\text{sen}\,\theta, 0, a\cos\theta) \ , \ \varphi_y = (0,1,0) \\ \varphi_\theta \wedge \varphi_y = (-a\cos\theta, 0, -a\,\text{sen}\,\theta) \\ \|\varphi_\theta \wedge \varphi_y\| = a \end{cases}$$

$$\int_S f\,dS = \int_0^{2\pi}\int_0^1 (a\cos\theta)^2\,y\,a\,dy\,d\theta$$

$$= a^3\int_0^{2\pi} \frac{1}{2}y\Big|_0^1 \cdot \cos^2\theta\,d\theta$$

$$= a^3/2\int_0^{2\pi} \cos^2\theta\,d\theta = \frac{a^3}{2}\int_0^{2\pi}\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right)d\theta$$

$$= \frac{a^3}{2}\cdot\frac{1}{2}\cdot 2\pi = \frac{1}{2}a^3\pi$$

(d) $f(x,y,z)=z^2$, $S: z=\sqrt{x^2+y^2}$, *entre os planos* $z=1$ *e* $z=4$

$\varphi(x,y)=(x,y,\sqrt{x^2+y^2})$ $\quad 1\le x^2+y^2\le 16$

$\varphi_x\wedge\varphi_y=\left(\frac{-x}{\sqrt{x^2+y^2}},\frac{-y}{\sqrt{x^2+y^2}},1\right)$

$\|\varphi_x\wedge\varphi_y\|^2=\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}+1=2$

$\int_S f\,dS=\int_D (x^2+y^2)\sqrt{2}\,dx\,dy$ $\quad D: 1\le x^2+y^2\le 16$

Em coord. polares: $D:\begin{cases}1\le r\le 4\\ 0\le\theta\le 2\pi\end{cases}$

$\int_S f\,dS=\sqrt{2}\int_1^4\int_0^{2\pi} r^2\,r\,d\theta\,dr$

$=2\sqrt{2}\pi\int_1^4 r^3\,dr=2\sqrt{2}\pi\,\frac{1}{4}r^4\Big|_1^4$

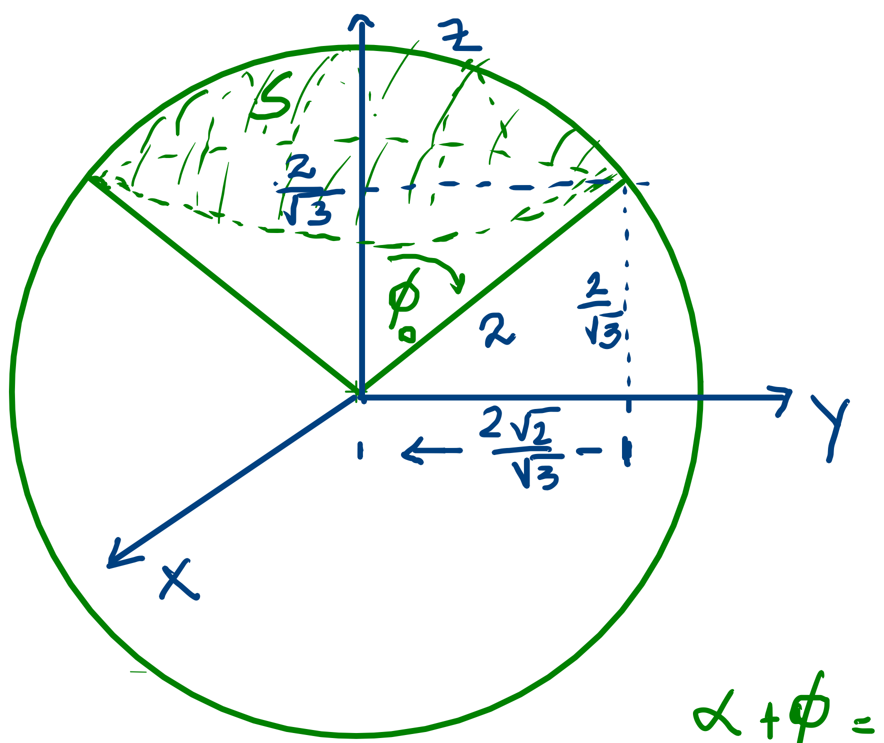
$=\frac{\sqrt{2}}{2}\pi\left(4^4-1\right)=\frac{255\sqrt{2}}{2}\pi$

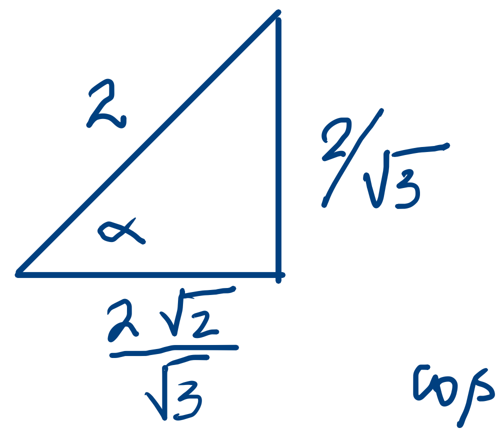
3 *Calcule a área da superfície* S, *parte da esfera* $x^2+y^2+z^2=4$ *que esta dentro do cone* $z=\sqrt{\frac{x^2+y^2}{2}}$.

$x^2+y^2+\frac{x^2+y^2}{2}=4$

$3x^2+3y^2=8$

$x^2+y^2=8/3$





$\frac{2\sqrt{2}}{\sqrt{3}}$

$\cos\alpha=\frac{\sqrt{2}}{\sqrt{3}}=\operatorname{sen}\phi_0$

$\operatorname{sen}\alpha=\frac{1}{\sqrt{3}}=\cos\phi_0$

$\alpha+\phi_0=\pi/2$

$S:\begin{cases}x=2\operatorname{sen}\phi\cos\theta\\ y=2\operatorname{sen}\phi\operatorname{sen}\theta\\ z=2\cos\phi\end{cases}$ $\quad \begin{matrix}0\le\theta\le 2\pi\\ 0\le\phi\le\phi_0\end{matrix}$

$dS=4\operatorname{sen}\phi\,d\phi\,d\theta$

$\text{área}(S)=\int_S dS=\int_D 4\operatorname{sen}\phi\,d\phi\,d\theta=\int_0^{2\pi}\int_0^{\phi_0}4\operatorname{sen}\phi\,d\phi\,d\theta$

$=8\pi\int_0^{\phi_0}\operatorname{sen}\phi\,d\phi=8\pi\left(-\cos\phi\Big|_0^{\phi_0}\right)=8\pi(1-\cos\phi_0)$

$=8\pi\left(1-\frac{1}{\sqrt{3}}\right)$

4 *Uma lâmina tem a forma de semi esfera $x^2 + y^2 + z^2 = a^2$, $z \geq 0$. Calcule o momento de inércia da lâmina, com relação ao eixo Z sabendo que a densidade em cada ponto é proporcional à distância desse ponto ao plano XY.*

Solução:

$$I_z = \int_S d^2 f \, ds$$

d = distância do pto ao eixo (no caso Z)

f = densidade

$$f = f(x,y,z) = kz$$

$$d = \sqrt{x^2+y^2}$$

$$I_z = \int_S (x^2+y^2) f \, ds$$

$$\therefore \quad I_z = \int_S (x^2+y^2) kz \, ds$$

Em coord. esféricas
$$\begin{cases} x = \rho \operatorname{sen}\phi \cos\theta \\ y = \rho \operatorname{sen}\phi \operatorname{sen}\theta \\ z = \rho \cos\phi \end{cases}$$

Temos $\rho = a$, $0 \leq \phi \leq \pi/2$, $0 \leq \theta \leq 2\pi$

$$x^2+y^2 = \rho^2 \operatorname{sen}^2\phi \underset{(\rho = a)}{=} a^2 \operatorname{sen}^2\phi$$

$$(x^2+y^2)z = a^3 \operatorname{sen}^2\phi \cos\phi$$

$$ds = \rho^2 \operatorname{sen}\phi \, d\phi \, d\theta = a^2 \operatorname{sen}\phi \, d\phi \, d\theta$$

$$\therefore \quad I_z = \int_0^{\pi/2} \int_0^{2\pi} k a^3 \operatorname{sen}^2\phi \cos\phi \cdot a^2 \operatorname{sen}\phi \, d\theta \, d\phi$$

$$= k a^5 \int_0^{\pi/2} \int_0^{2\pi} \operatorname{sen}^3\phi \cos\phi \, d\theta \, d\phi$$

$$= 2 k a^5 \pi \int_0^{\pi/2} \operatorname{sen}^3\phi \cos\phi \, d\phi$$

$$= \frac{k a^5 \pi}{2} \operatorname{sen}^4\phi \Big|_0^{\pi/2}$$

$$= \frac{1}{2} k a^5 \pi$$

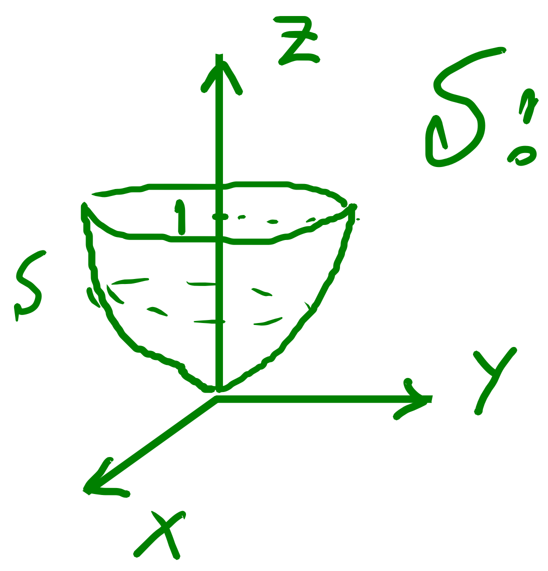
5 *Determine o centro de massa da superfície homogênea* $S : z = x^2 + y^2$ *com* $0 \leq z \leq 1$

Solução:

$$\bar{x} = \frac{1}{M}\int_S x f\, ds \;,\quad \bar{y} = \frac{1}{M}\int_S y f\, ds \;,\quad \bar{z} = \frac{1}{M}\int_S z f\, ds$$

f = densidade = k = cte (superf. homogênea)

$$M = \int_S f\, ds = k\int_S dS = k \text{ área}(S)$$



$$S: \begin{cases} \varphi(x,y) = (x, y, x^2+y^2) \\ 0 \leq x^2+y^2 \leq 1 \end{cases}$$

$\varphi_x = (1, 0, 2x)$

$\varphi_y = (0, 1, 2y)$

$$\varphi_x \wedge \varphi_y = (-2x, -2y, 1)$$

$$\|\varphi_x \wedge \varphi_y\| = \sqrt{1 + 4x^2 + 4y^2}$$

$$M = k\int_S ds = k\int_D \sqrt{1+4x^2+4y^2}\, dx\, dy$$

Em coord. polares: $D: \begin{cases} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{cases}$

$$\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \end{cases} \qquad x^2+y^2 = r^2$$

$$dx\,dy = r\,dr\,d\theta$$

$$M = k\int_0^1 \int_0^{2\pi} \sqrt{1+4r^2}\, r\, d\theta\, dr$$

$$= 2k\pi \int_0^1 \sqrt{1+4r^2}\, r\, dr$$

$$2k\pi \cdot \frac{2}{3} \cdot \frac{1}{8}(1+4r^2)^{3/2}\Big|_0^1 = \frac{1}{6}k\pi\left(5^{3/2} - 1\right)$$

$$\boxed{M = \frac{1}{6}k\pi\left(5\sqrt{5} - 1\right)}$$

$$\bar{x} = \frac{1}{M}\int_S x f \, ds .$$

$$\int_S x f \, ds = K\int_S x \, ds = K\int_D x\sqrt{1+4x^2+4y^2}\, dx\, dy$$

$$= K\int_0^1\int_0^{2\pi}\sqrt{1+4r^2}\, r\cos\theta \cdot r\, d\theta\, dr$$

$$= K\int_0^1\int_0^{2\pi} r^2\sqrt{1+r^2}\cos\theta \, d\theta\, dr$$

$$= K\int_0^1 r^2\sqrt{1+r^2}\left(\operatorname{sen}\theta \Big|_0^{2\pi}\right) dr$$

$$= 0$$

$$\therefore \boxed{\bar{x} = 0}$$

$$\bar{y} = \frac{1}{M}\int_S y f \, ds = \frac{1}{M} K\int_S y \, ds$$

$$\int_S y \, ds = \int_D y\sqrt{1+4(x^2+y^2)}\, dx\, dy$$

$$= \int_0^1\int_0^{2\pi} r\operatorname{sen}\theta \sqrt{1+4r^2}\, r\, d\theta\, dr$$

$$= \int_0^1\int_0^{2\pi} r^2\sqrt{1+4r^2}\operatorname{sen}\theta\, d\theta\, dr$$

$$= \int_0^1 r^2\sqrt{1+4r^2}\left(-\cos\theta\Big|_0^{2\pi}\right) dr$$

$$= 0$$

$$\therefore \boxed{\bar{y} = 0}$$

$$\bar{z} = \frac{1}{M}\iint_S z f\, dS = \frac{K}{M}\iint_D (x^2+y^2)\sqrt{1+4(x^2+y^2)}\, dx\,dy$$

$$\iint_S z f\, dS = K\int_0^1\int_0^{2\pi} r^2\sqrt{1+4r^2}\, r\, d\theta\, dr$$

$$= 2K\pi\int_0^1 r^3\sqrt{1+4r^2}\, dr$$

$$= 2K\pi\int_0^1 \frac{1}{8} r^2\sqrt{1+4r^2}\cdot 8r\, dr$$

$$u = 1+4r^2 \Rightarrow du = 8r\,dr$$

$$r^2 = \frac{u-1}{4} \qquad r=0 \Rightarrow u=1$$

$$r=1 \Rightarrow u=5$$

$$\therefore \iint_S z f\, dS = \frac{1}{4}K\pi\int_1^5 \frac{u-1}{4}\sqrt{u}\, du$$

$$= \frac{1}{16}K\pi\int_1^5 \left(u^{3/2} - u^{1/2}\right) du$$

$$\iint_S z f\, dS = \frac{1}{16}K\pi\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right)\Bigg|_1^5$$

$$= \frac{1}{16}K\pi\left[\left(\frac{2}{5}5^{5/2} - \frac{2}{3}5^{3/2}\right) - \left(\frac{2}{5} - \frac{2}{3}\right)\right]$$

etc