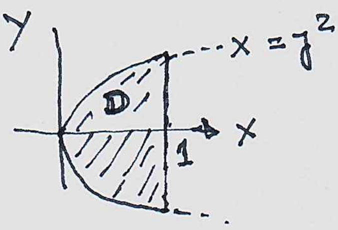
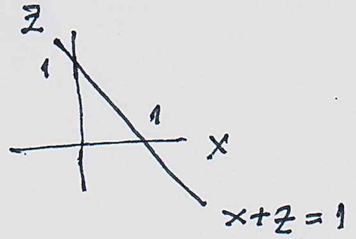
C3 **LISTA 3** Prof. Pedro A. Hinojosa

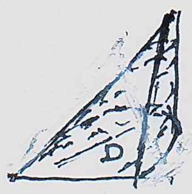
(1) Determinar o volume do sólido $W \subseteq \mathbb{R}^3$.

(a) W limitado pelo cilindro $x = y^2$ e os planos $z = 0$ e $x + z = 1$

Solução







$$W: \begin{cases} (x,y) \in D \\ 0 \le z \le 1-x \end{cases} \qquad D: \begin{cases} -1 \le x \le 1 \\ y^2 \le x \le 1 \end{cases}$$

$$\text{vol}(W) = \int_D \int_0^{1-x} dz\, dA = \int_D (1-x)\, dA$$

$$= \int_{-1}^{1} \int_{y^2}^{1} (1-x)\, dx\, dy = \int_{-1}^{1} \left(x - \frac{1}{2}x^2\right)\Big|_{y^2}^{1} dy$$

$$= \int_{-1}^{1} \left[\left(1 - \tfrac{1}{2}\right) - \left(y^2 - \frac{1}{2}y^4\right)\right] dy$$

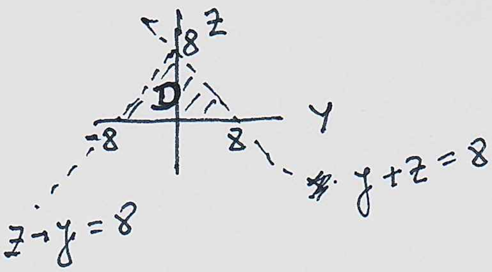
$$= \int_{-1}^{1} \left(\tfrac{1}{2} - y^2 + \frac{1}{2}y^4\right) dy$$

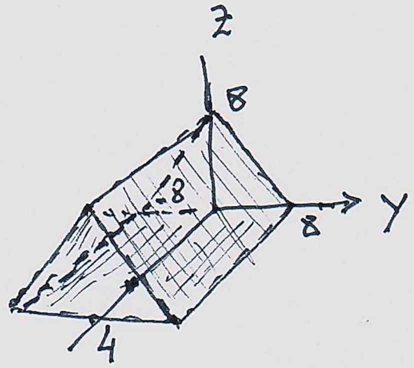
$$= \left(\frac{1}{2}y - \frac{1}{3}y^3 + \frac{1}{10}y^5\right)\Big|_{-1}^{1} = 2\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{10}\right)$$

$$= \tfrac{8}{15} \text{ unidades de volume.}$$

b) W limitado por $z-y=8$, $z+y=8$, $x=0$, $x=4$ e $z=0$

Solução





$$\text{vol}(W) = \int_D \int_0^4 dx\, dA \qquad W: \begin{cases} (y,z) \in D \\ 0 \leq x \leq 4 \end{cases}$$

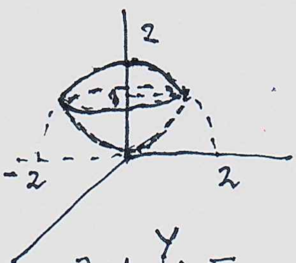
$$= 4\int_D dA = 4\, \text{área}(D) \qquad D: \begin{cases} 0 \leq z \leq 8 \\ z-8 \leq y \leq 8-z \end{cases}$$

$$= 4 \cdot \frac{1}{2}\, 16 \cdot 8$$

$$= 256$$

(c) W limitado por $x^2+y^2+z^2=4$ e por $x^2+y^2=3z$

Solução

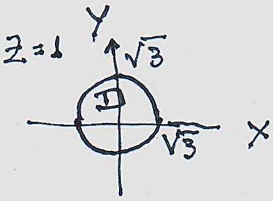


$$\left.\begin{matrix} x^2+y^2 = 3z \\ x^2+y^2+z^2=4 \end{matrix}\right\} \Rightarrow z^2+3z-4=0$$

$$\Rightarrow (z-1)(z+4)=0$$

$$\Rightarrow z=1 \text{ ou } \underbrace{z=-4}_{\text{Não !!}}$$

(z>0)



$z=1$

$\Downarrow$

$x^2+y^2=3$

$$D_{r\theta}: \begin{cases} 0 \leq r \leq \sqrt{3} \\ 0 \leq \theta \leq 2\pi \end{cases}$$

$$W: \begin{cases} (x,y) \in D \\ \frac{1}{3}(x^2+y^2) \leq z \leq \sqrt{4-x^2-y^2} \end{cases}$$

$$\mathrm{vol}(W) = \int_D \int_{\frac{1}{3}(x^2+y^2)}^{\sqrt{4-x^2-y^2}} dz\, dA$$

$$\begin{cases} x = r\cos\theta \\ y = r\,\mathrm{sen}\,\theta \end{cases} \qquad D_{r\theta} : \begin{cases} 0 \leq r \leq \sqrt{3} \\ 0 \leq \theta \leq 2\pi \end{cases}$$

$$\mathrm{vol}(W) = \int_0^{\sqrt{3}} \int_0^{2\pi} \left(\sqrt{4-r^2} - \frac{1}{3} r^2 \right) r \, d\theta \, dr$$

$$= 2\pi \int_0^{\sqrt{3}} \left(r\sqrt{4-r^2} - \frac{1}{3} r^3 \right) dr$$

$$= 2\pi \left(-\frac{1}{3}(4-r^2)^{3/2} - \frac{1}{12} r^4 \right) \Big|_0^{\sqrt{3}}$$

$$= 2\pi \left(-\frac{1}{3} - \frac{9}{12} - \left(-\frac{4^{3/2}}{3} \right) \right)$$

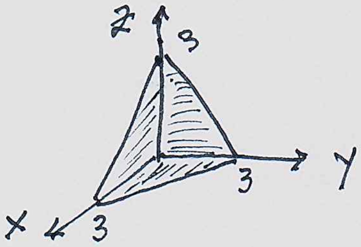
$$= 2\pi \left(\frac{8}{3} - \frac{1}{3} - \frac{3}{4} \right) = 2\pi \left(\frac{7}{3} - \frac{3}{4} \right)$$

$$= 2\pi \cdot \frac{19}{12} = \frac{19}{6}\pi$$

(2) Calcular $\int_W f\,dV$

(a) $f(x,y,z) = x-y$ W limitado pelos planos coord. e pelo plano $x+y+z=3$

Solução



$$W: \begin{cases} 0 \le x \le 3 \\ 0 \le y \le 3 \\ 0 \le z \le 3-x-y \end{cases}$$

$$\int_W f\,dV = \int_0^3 \int_0^3 \int_0^{3-x-y} (x-y)\,dz\,dy\,dx$$

$$= \int_0^3 \int_0^3 (x-y)(3-x-y)\,dy\,dx$$

$$= \int_0^3 \int_0^3 \left[3(x-y) - (x^2-y^2)\right] dy\,dx$$

$$= \int_0^3 \int_0^3 \left(-x^2+3x+y^2-3y\right) dy\,dx$$

$$= \int_0^3 \left(-3x^2+9x+\left(\frac{1}{3}y^3-\frac{3}{2}y^2\right)\Big|_0^3\right) dx$$

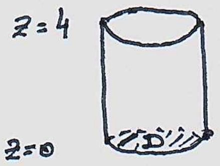
$$= \int_0^3 \left[-3x^2+9x+\left(9-\frac{27}{2}\right)\right] dx$$

$$= \left(-x^3+\frac{9x^2}{2}-\frac{9}{2}x\right)\Big|_0^3 = -27+\frac{81}{2}-\frac{27}{2}$$

$$= -27+27 = \underline{\underline{0}}$$

b) $f(x,y,z) = x^2+y^2$, $W: \begin{cases} x^2+y^2 \leq 1 \\ 0 \leq z \leq 4 \end{cases}$

<u>Solução</u>:



$D: \; x^2+y^2 \leq 1$

$D_{r\theta}: \begin{cases} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{cases}$

$\begin{cases} x = r\cos\theta \\ y = r \operatorname{sen}\theta \\ z = z \end{cases}$

$$\int_W f \, dV = \int_D \int_0^4 (x^2+y^2) \, dz \, dA$$

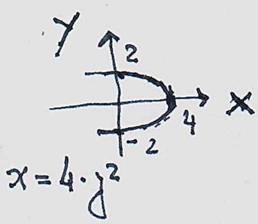
$$= \int_0^{2\pi} \int_0^1 4 r^2 r \, dr \, d\theta$$

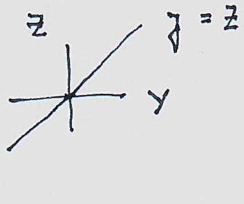
$$= 2\pi \int_0^1 4r^3 \, dr = 2\pi \, r^4 \Big|_0^1$$

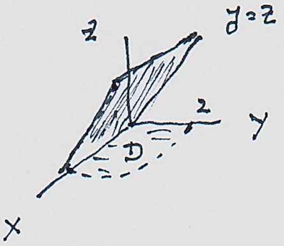
$$= 2\pi$$

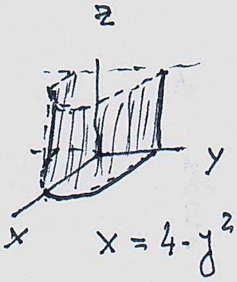
(c) $f(x,y,z) = 1$, W limitada por $\begin{cases} x = 4-y^2 \\ y = z \\ x = 0 \\ z = 0 \end{cases}$

<u>Solução</u>









$D: \begin{cases} 0 \leq x \leq 4 \\ 0 \leq y \leq \sqrt{4-x} \end{cases}$ $\quad W: \begin{cases} (x,y) \in D \\ 0 \leq z \leq y \end{cases}$

$$\int_W f \, dV = \int_D \int_0^y dz \, dA = \int_D y \, dA$$

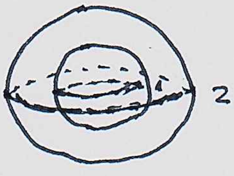
$$= \int_0^4 \int_0^{\sqrt{4-x}} y \, dy \, dx = \int_0^4 \frac{1}{2}(4-x) \, dx$$

$$= \left(2x - \frac{1}{4}x^2\right)\Big|_0^4 = 8 - \frac{1}{4} \cdot 4^2 = 4$$

(d) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$, $W: \begin{cases} x^2+y^2+z^2 \geq 1 \\ x^2+y^2+z^4 \leq 4 \end{cases}$

Solução

$$\int_W f\,dV = ?$$



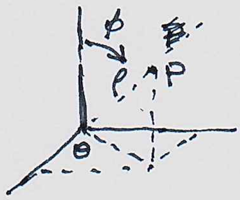
coord. esféricas

$$\begin{cases} x = \rho \,\mathrm{sen}\,\phi \cos\theta \\ y = \rho\,\mathrm{sen}\,\phi\,\mathrm{sen}\,\theta \\ z = \rho\cos\phi \end{cases}$$

$$x^2+y^2+z^2 = \rho^2$$

$$\text{“} dx\,dy\,dz = \rho^2 \mathrm{sen}\,\phi \, d\rho\, d\phi\, d\theta \text{”}$$

$$W_{\rho\theta\phi} : \begin{cases} 1 \leq \rho \leq 2 \\ 0 \leq \phi \leq \pi \\ 0 \leq \theta \leq 2\pi \end{cases}$$



$$\int_W f\,dV = \int_1^2 \int_0^\pi \int_0^{2\pi} \sqrt{\rho^2} \cdot \rho^2 \mathrm{sen}\,\phi \, d\theta\, d\phi\, d\rho$$

$$= 2\pi \int_1^2 \int_0^\pi \rho^3 \mathrm{sen}\,\phi\, d\phi\, d\rho$$

$$= 2\pi \int_1^2 \rho^3 \left(-\cos\phi \Big|_0^\pi \right) d\rho$$

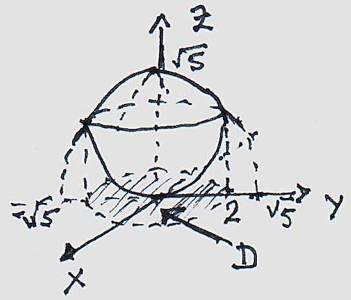
$$= 2\pi \int_1^2 \rho^3 \left(-(-1-1) \right) d\rho$$

$$= 4\pi \int_0^1 \rho^3\, d\rho = \pi \rho^4 \Big|_0^1 = \pi$$

(e) $f(x,y,z) = z$ W limitado por $z = \frac{1}{4}(x^2+y^2)$ e $x^2+y^2+z^2 = 5$

$$\int_W f\,dV = ?$$

Solução



$$4z + z^2 = 5$$
$$z^2 + 4z - 5 = 0$$
$$(z+5)(z-1) = 0$$
$$\cancel{z = -5} \qquad \underline{z = 1}$$

$D: \ x^2+y^2 \leq 4$ $\qquad z = 1 \Rightarrow \underline{x^2+y^2 = 4}$

coord. cilíndricas $\begin{cases} x = r\cos\theta \\ y = r \operatorname{sen}\theta \\ z = z \end{cases}$ $\qquad D_{r\theta}: \begin{cases} 0 \leq r \leq 2 \\ 0 \leq \theta \leq 2\pi \end{cases}$

$W_{r\theta z}: \begin{cases} 0 \leq r \leq 2 \\ 0 \leq \theta \leq 2\pi \\ \frac{1}{4}r^2 \leq z \leq \sqrt{5-r^2} \end{cases}$ $\qquad \underline{J = r}$

$$\int_W f\,dV = \int_0^2 \int_0^{2\pi} \int_{\frac{1}{4}r^2}^{\sqrt{5-r^2}} z \cdot r\,dz\,d\theta\,dr$$

$$= 2\pi \int_0^2 \frac{1}{2} z^2 \Big|_{\frac{1}{4}r^2}^{\sqrt{5-r^2}} \cdot r\,dr$$

$$= \pi \int_0^2 \left[(5-r^2) - \frac{1}{16}r^4\right] dr$$

$$= \pi \int_0^2 \left(-\frac{1}{16}r^4 - r^2 + 5\right) dr$$

$$= \pi \left(\frac{-1}{16} \cdot \frac{r^5}{5} - \frac{1}{3}r^3 + 5r\right)\Big|_0^2$$

$$= \pi \left(-\frac{2}{5} - \frac{8}{3} + 10\right) = \frac{104}{15}\pi$$

③ Massa de W

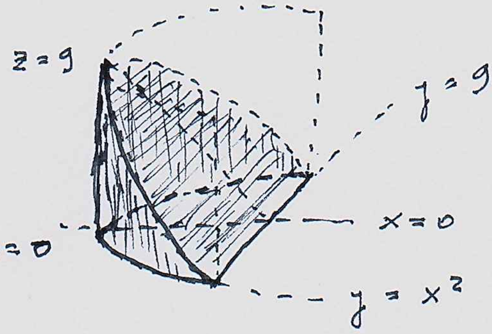
W no 1º octante limitado por $\begin{cases} y = x^2 \\ y = 9 \\ z = 0 \\ x = 0 \\ y + z = 9 \end{cases}$

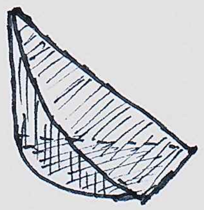
densidade de W:

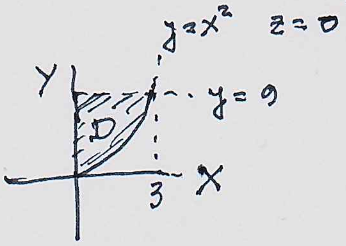
$f(x,y,z) = x + y$

<u>Solução</u>:

$$M(W) = \int_W f \, dV$$







$D: \begin{cases} 0 \leq x \leq 3 \\ x^2 \leq y \leq 9 \end{cases}$ $\qquad W: \begin{cases} (x,y) \in D \\ 0 \leq z \leq 9 - y \end{cases}$

$$M(W) = \int_W f \, dV = \int_D \int_0^{9-y} (x+y) \, dz \, dA$$

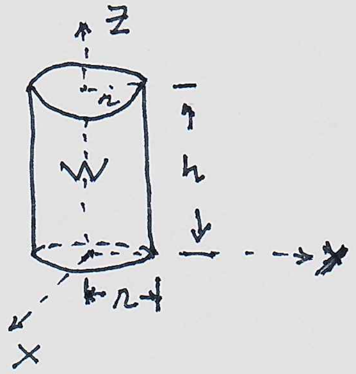
$$= \int_D (x+y)(9-y) \, dA = \int_0^3 \int_{x^2}^{9} (x+y)(9-y) \, dy \, dx$$

$$= \int_0^3 \int_{x^2}^{9} (9x + 9y - xy - y^2) \, dy \, dx$$

$$= \int_0^3 \left(9x(9-x^2) + \frac{9}{2}(9^2 - x^4) - \frac{1}{2}x(9^2 - x^4) - \frac{1}{3}(9^3 - x^6) \right)$$

$$= \int_0^3 \quad \ldots \quad \text{etc}$$

④



$$W: \begin{cases} x^2+y^2 \le r \\ 0 \le z \le h \end{cases}$$

$f(x,y,z) = kz$ densidade proporcional à distância do pto $(x,y,z) \in W$ à base.

Determinar o momento de inércia em relação ao eixo de simetria de W

Solução

$$I_\ell = \int_W d^2 f \, dV$$

$d = d(x,y,z)$ é a distância do pto $(x,y,z) \in W$ à reta ℓ.
(no caso ℓ é o eixo z)

$\therefore \quad d = \sqrt{x^2+y^2}$

$$I_\ell = \int_W (x^2+y^2) f \, dV$$

$$W_{R\theta z}: \begin{cases} x = R\cos\theta \\ y = R\operatorname{sen}\theta \\ z = z \end{cases}$$

$$I_\ell = \int_0^{2\pi}\int_0^{r}\int_0^{h} R^2 \cdot kz\, R\, dz\, dR\, d\theta \qquad \begin{cases} 0 \le R \le r \\ 0 \le \theta \le 2\pi \\ 0 \le z \le h \end{cases}$$

$$= 2\pi \int_0^{r}\int_0^{h} kR^3 z\, dz\, dR$$

$$= 2k\pi \int_0^{r} R^3 \cdot h\, dR = 2hk\pi \int_0^{r} R^3 dR$$

$$= \frac{1}{2} h k r^4 \pi$$

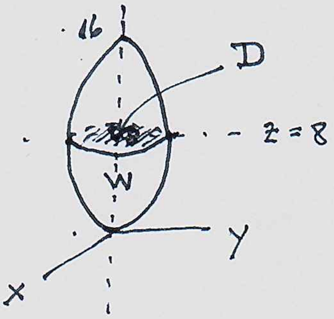
(5) W limitado por $\begin{cases} z = 16 - 2x^2 - 2y^2 \\ \quad e \\ z = 2x^2 + 2y^2 \end{cases}$

densidade de W : $f(x,y,z) = \sqrt{x^2+y^2}$

$\underline{M(W) = ?}$

Solução :

$$M(W) = \int_W f \, dV$$



$$z = 16 - (2x^2+2y^2)$$
$$z = 16 - z$$
$$2z = 16 \qquad \boxed{z = 8}$$

$$D : \{ \; x^2+y^2 \leq 4$$

$$D_{r\theta} : \begin{cases} 0 \leq r \leq 2 \\ 0 \leq \theta \leq 2\pi \end{cases}$$

$$W : \begin{cases} (x,y) \in D \\ 2x^2+2y^2 \leq z \leq 16 - 2x^2 - 2y^2 \end{cases}$$

$$\cancel{M(W) = \int_0^{2\pi}\int_0^2\int_{2x^2+y^2}^{16-x^2-2y^2}} \qquad M(W) = \int_0^{2\pi}\int_0^2\int_{2r^2}^{16-2r^2} \sqrt{r^2} \cdot r \, dz \, dr \, d\theta$$

$$M(W) = 2\pi \int_0^2 \int_{2r^2}^{16-2r^2} r^2 \, dz \, dr = 2\pi \int_0^2 \left[(16-2r^2) - 2r^2\right] r^2 \, dr$$

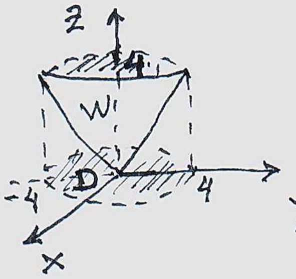
$$= 2\pi \int_0^2 (16r^2 - 4r^4) \, dr = 2\pi \left(\frac{16}{3} r^3 - \frac{4r^5}{5} \right) \Big|_0^2$$

$$= 2\pi \left(\frac{16 \cdot 8}{3} - \frac{4 \cdot 32}{5} \right) = \frac{512}{15} \pi$$

(6) $\underline{\underline{I_z = ?}}$ W limitado por $\begin{cases} z = \sqrt{x^2+y^2} \\ e \quad z = 4 \end{cases}$

densidade de W: $f(x,y,z) = z$

Solução:



$z = 4 \Rightarrow x^2+y^2 = 16$ W: $\begin{cases} (x,y) \in D \\ r \leq z \leq 4 \end{cases}$

D: $\begin{cases} 0 \leq r \leq 4 \\ 0 \leq \theta \leq 2\pi \end{cases}$

$$I_z = \int_W (x^2+y^2) \cdot f \, dV = \int_0^4 \int_0^{2\pi} \int_r^4 r^2 \cdot z \cdot r \, dz \, d\theta \, dr$$

$$= 2\pi \int_0^4 \int_r^4 r^3 z \, dz \, dr = 2\pi \int_0^4 \frac{1}{2} z^2 \Big|_r^4 \cdot r^3 \, dr$$

$$= \pi \int_0^4 (4^2 - r^2) r^3 \, dr = \pi \int_0^4 (4^2 r^3 - r^5) \, dr$$

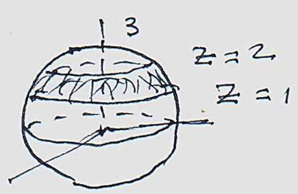
$$= \pi \left(4^2 \frac{r^4}{4} - \frac{r^6}{6} \right) \Big|_0^4$$

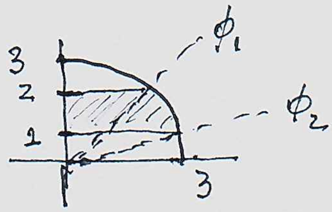
$$= \pi \cdot \left(4^5 - \frac{4^6}{6} \right) = \underline{\underline{\frac{4^5}{3} \pi}}$$

$$\boxed{I_z = \frac{4^5}{3} \pi}$$

(7) Volume da parte da esfera $x^2+y^2+z^2 = 9$ entre os planos $z=1$ e $z=2$

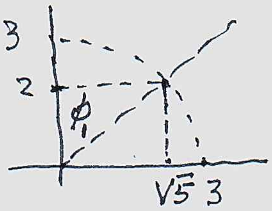
Solução



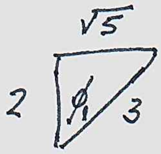


$$\begin{cases} x = \rho \operatorname{sen}\phi \cos\theta \\ y = \rho \operatorname{sen}\theta \operatorname{sen}\theta \\ z = \rho \cos\phi \end{cases}$$

$$W_{\rho\phi\theta} : \begin{cases} 0 \leq \rho \leq 3 \\ 0 \leq \theta \leq 2\pi \\ \phi_1 \leq \phi \leq \phi_2 \end{cases}$$

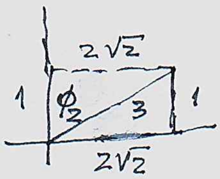


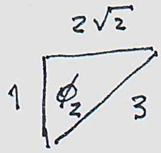
$$\left.\begin{array}{l} z = 2 \\ x^2+y^2+z^2 = 9 \end{array}\right\} \Rightarrow x^2+y^2 = 5$$



$$\operatorname{sen}\phi_1 = \frac{\sqrt{5}}{3}$$

$$\cos\phi_1 = 2/3$$





$$\operatorname{sen}\phi_2 = \frac{2\sqrt{2}}{3}$$

$$\cos\phi_2 = 1/3$$

$$\text{Volume} = V = \int_0^3 \int_0^{2\pi} \int_{\phi_1}^{\phi_2} \rho^2 \operatorname{sen}\phi \, d\phi \, d\theta \, d\rho$$

$$= 2\pi \int_0^3 \rho^2 \left(-\cos\phi \Big|_{\phi_1}^{\phi_2} \right) d\rho$$

$$= 2\pi \int_0^3 \rho^2 (\cos\phi_1 - \cos\phi_2) \, d\rho$$

$$= 2\pi \left(\frac{2}{3} - \frac{1}{3} \right) \int_0^3 \rho^2 \, d\rho$$

$$= \frac{2\pi}{3} \int_0^3 \rho^2 d\rho = \frac{2\pi}{3} \cdot \frac{1}{3} \rho^3 \Big|_0^3$$

$$= \frac{2\pi}{3^2} \cdot 3^3 = 6\pi$$