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SOLUTIONS OF THE CHEEGER PROBLEM VIA TORSION FUNCTIONS

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Let ϕ_p denote the *p-torsion function,* that is, the solution of the *torsional creep problem*:$-\Delta_p u = 1$ in Ω, $u = 0$ on $\partial\Omega$, where $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $p > 1$, and Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$. Let λ_p denote the first eigenvalue of the p-Laplacian with homogeneous Dirichlet data in Ω and $e_p > 0$ denote the associated first eigenfunction L^∞-normalized.

By using the bounds proved in [1]

$$\|\phi_p\|_\infty^{1-p} \leq \lambda_p \leq (|\Omega|^{-1}\|\phi_p\|_1)^{1-p} \tag{1}$$

together with the variational characterization of ϕ_p and an *a priori* estimate, we solve in [2] the *Cheeger problem*: minimize the quotient $(|\partial E|/|E|)$, where $|\partial E|$ and $|E|$ denote, respectively, the perimeter and the volume of a smooth subdomain $E \subseteq \Omega$. The minimum value $h(\Omega)$ is known as the *Cheeger constant* of Ω and a corresponding minimizing subdomain E is called a *Cheeger set of* Ω. The Cheeger problem is related to the eigenvalue problem for the 1-Laplacian. Both are equivalent (see [3,4]) to the following problem: minimize $H(v) := \int_\Omega |Dv|\,dx + \int_{\partial\Omega} |v|\,d\mathcal{H}^{N-1}$ on the set $\Lambda := \left\{v \in BV(\mathbb{R}^N) : \|v\|_1 = 1 \text{ and } v \equiv 0 \text{ in } R^N \backslash \overline{\Omega}\right\}$. Thus, in this BV-approach, if $\mu = \min_\Lambda H(v) = H(u)$, then $\mu = h(\Omega)$ and the set level $E_t = \{x \in \Omega : u(x) > t\}$ of the minimizing function u are Cheeger sets of Ω. Moreover, the pair (μ, u) is an eigenpair for the p-Laplacian.

In [2] we give two characterizations of the Cheeger constant:

$$\lim_{p\to 1^+} \|\phi_p\|_\infty^{1-p} = h(\Omega) = \lim_{p\to 1^+} \|\phi_p\|_1^{1-p}.$$

We also prove the convergence (up to subsequences) in $L^1(\Omega)$ of $\phi_p/\|\phi_p\|_1$, as $p \to 1^+$, to a bounded and nonnegative function $u \in \Lambda$ such that $h(\Omega) = H(u) = \min_\Lambda H(v)$. Therefore, the set levels E_t of u are Cheeger sets for almost all $0 \leq t \leq \|u\|_\infty$ and $(h(\Omega), u)$ is an eigenpair for the 1-Laplacian. Moreover, we derive an estimate for u implying that $N^N|B_1| \leq h(\Omega)^N|E_0| = |\partial E_0|^N/|E_0|^{N-1}$, where B_1 is the unit ball of $\mathbb{R}^N$. This relation is particularly interesting if Ω is convex, since the Cheeger set is known to be unique and we may write $u = \chi_{E_0}/|E_0|$. In this convex case, by using the concavity of $\phi_p^{1-\frac{1}{p}}$ and the Schwarz symmetrization, we also present in [2] an alternative proof of the characterizations of $h(\Omega)$ together with bounds for λ_p involving the Gamma function.

Kawohl and Fridman introduced in [4] the aforementioned BV-approach relating the Cheeger problem to the minimizing problem of the functional H on BV functions and the eigenvalue problem for the 1-Laplacian. They proved that $h(\Omega) = \lim_{p\to 1^+} \lambda_p$ and the L^1-convergence (up to subsequences) of e_p to a function w whose set levels are Cheeger sets of Ω. Our approach seems to be more appropriate to computational purposes since, in principle, torsion functions are easier to compute than the first eigenpair. Moreover, our results recover those given in [4].

References

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