# Universidade Federal da Paraíba <br> Universidade Federal de Campina Grande <br> Programa Associado de Pós-Graduação em Matemática <br> Doutorado em Matemática 

# On Milnor classes of constructible functions 

por

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Tese apresentada ao Corpo Docente do Programa Associado de Pós-Graduação em Matemática UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

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## Universidade Federal da Paraíba <br> Universidade Federal de Campina Grande <br> Programa Associado de Pós-Graduaçāo em Matemática Doutorado em Matemática

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## Resumo

O principal objetivo deste trabalho é apresentar uma generalização do importante invariante da Teoria das Singularidades, chamado número de Milnor. Tal generalização é o que chamamos de número de Milnor logarítmico. Bem como explanar sobre definições um pouco mais gerais no contexto de funções construtíveis, apresentando observações, exemplos e propriedades. Dentre os conceitos que trabalhamos estão também a classe de Fulton-Johnson, a classe de Schwartz-MacPherson, a classe de Milnor e a classe de Segre.

Palavras-chave: Número de Milnor; Classe de Milnor; Classe de Segre; Classes características; variedades singulares.

## Abstract

The main goal of this thesis is to present a generalization of the important invariant of the singularity theory, called the Milnor number. Such generalization is what we call the logarithmic Milnor number. As well as to discuss about more general definitions in the context of constructible functions, presenting observations, examples and properties. Among the concepts we work on are also the Fulton-Johnson class, the SchwartzMacPherson class, the Milnor class and the Segre class.

Keywords: Milnor number; Milnor class; Segre class; Characteristic classes; singular varieties.

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## Introdução

The Milnor's fibration theorem is one of the main tools for the topological study of the fibers of holomorphic functions near their critical points. Among the information obtained is a well-known invariant called the Milnor number. The classic definition of this number is purely topological. However, there is an algebraic version of Milnor number which is given as follows. Consider $f:\left(\mathbb{C}^{n+1}, 0\right) \longrightarrow(\mathbb{C}, 0)$ a germ of holomorphic function, with isolated critical point at 0 . The Milnor number, denoted by $\mu(f)$, coincides with the number $\operatorname{dim}_{\mathbb{C}} \mathcal{O}_{n+1} /\left(\partial f / \partial z_{0}, \ldots, \partial f / \partial z_{n}\right)$, where $\mathcal{O}_{n+1}$ is the ring of germs of analytic functions at the origin. Among the generalizations of Milnor number we highlight Parusiński-Milnor number which is defined in a context not necessarily with isolated singularity. Let $M$ be a $n$-dimensional compact complex manifold and let $L$ be a holomorphic line bundle over $M$. Consider $X:=v^{-1}(0)$ a divisor in $M$, where $v$ is a regular holomorphic section of $L$. The Parusiński-Milnor number is defined by

$$
\mu(X)=(-1)^{n-1}(\chi(M \mid L)-\chi(X)),
$$

where for a vector bundle $E$ over $M$,

$$
\chi(M \mid E):=\int_{M} c(E)^{-1} c_{\text {top }}(E) c(M) \cap[M] .
$$

In collaboration with P. Pragacz, Parusiński presented some interesting properties of the above number.

Characteristic classes are certain kinds of cohomology classes associated to vector bundles over spaces. The classic case is when we have a smooth manifold and their tangent bundle. This theory began in the year 1935 with the works of H. Whitney and E. Stiefel, arising then Stiefel-Whitney classes. We also have the Pontrjagin and

Euler classes in the real case. In 1946, S.S. Chern defined characteristic classes for complex vector bundles, which are called Chern classes. In a topological approach, obstruction theory can be used to define Chern classes. However, for a differentialgeometric approach, Chern-Weil theory can be used to define Chern classes.

The first notions of Chern class for singular varieties appeared in the works of W.T. Wu and J. Mather. The obstruction approach of Chern classes is due to M.H. Schwartz, and R. MacPherson presented an axiomatic theory about Chern classes. Moreover, Macpherson gave a positive answer to the Deligne and Grothendieck's Conjecture on the existence of Chern classes seen as a natural transformation from the covariant functor of the constructible functions to the homology functor with a good behavior regarding pushforward. The MacPherson's original work was in context of complex algebraic varieties and on homology groups. For a context completely algebraic, by extending to varieties over an arbitrary field of characteristic zero, see [Ken]; for a context on Chow groups, see [Ful, 19.1.7]. Using the Alexander duality, one has that the Schwartz class and the MacPherson class coincide, being so-called SchwartzMacPherson classes. The Fulton and Fulton-Johnson classes are other known generalizations. In the definition of these classes another class appears, called the Segre class. It is known that for complete intersections the Fulton classes and Fulton-Johnson classes coincide with the Chern class of the virtual bundle.

We can find relationship between the Schwartz-MacPherson class and the FultonJohnson class in works of A. Parusiński, P. Pragacz, J.P. Brasselet, D. Lehmann, J. Seade, T. Suwa, S. Yokura and P. Aluffi, see [PP3], [BLSS], [Y], [A4]. Motivating to define the notion of a Milnor class as being the difference, up to sign, between the Schwartz-MacPherson class and the Fulton-Johnson class. Explicitly, let $X$ be a $n$ dimensional irreducible analytic (or algebraic) variety embedded in a smooth manifold $M$. The Milnor class of $X$ is defined by

$$
\mathcal{M}(X)=(-1)^{n-1}\left(c^{F J}(X)-c^{S M}(X)\right)
$$

where $c^{F J}(X)$ denotes the Fulton-Johnson class of $X$ and $c^{S M}(X)$ denotes the SchwartzMacPherson class of $X$. An interesting fact is that the degree of Milnor class is equals the Parusiński-Milnor number, that is,

$$
\mu(X)=\int_{X} \mathcal{M}(X) .
$$

We will present some definitions that are generalizations of some elements that we have explained above, as well as some properties and examples. To this end, this thesis was divided into five chapters, which we will describe briefly.

The Chapter 1 is devoted to some preliminary facts and definitions. Among these facts and definitions are the notions of vector fields with isolated singularities, Poincaré-Hopf theorem, holomorphic vector bundles, hermitian metric, logarithmic forms, free divisors, constructible functions and Milnor number. Moreover, we organized a summary about intersection theory, with definitions and results used throughout subsequent chapters.

In the Chapter 2, we begin with a brief explanation of the Schwartz-MacPherson class and generalize a formula due to Parusiński and Pragacz involving the Euler characteristic. Then we present the definition of a logarithmic connection due to P. Deligne (Definition 2.2.2). Such a connection is used to extend the definition of $\mu$-number due to Parusiński (Definition 2.2.4). The Theorem 2.2.3 due to Parusiński motivated us to Proposition 2.2.6. Let $M$ be a $n$-dimensional compact connected complex manifold and let $X:=v^{-1}(0)$ be a divisor in $M$, where $v$ is a regular holomorphic section of $L$. Given $D$ another divisor in $M$, the logarithmic Milnor number of $X$ with poles along $D$ or, simply, logarithmic Milnor number of $X$ is defined by

$$
\mu_{D}(X)=(-1)^{n}\left(\chi\left(X ; \mathbf{1}_{X \backslash D}\right)-\int_{M} c(L)^{-1} c_{1}(L) c\left(\operatorname{Der}_{M}(-\log D)\right) \cap[M]\right)
$$

In the Proposition 2.2.13 we present the following relationship between logarithmic Milnor number and Parusiński-Milnor number:

$$
\mu_{D}(X)=\mu(X)-(-1)^{n} \chi\left(X ; \mathbf{1}_{X \cap D}\right)+(-1)^{n} \int_{M} c(L)^{-1} c_{1}(L) c_{*}(D)
$$

Now we generalize our definition of logarithmic Milnor number to an arbitrary constructible function. We define the Milnor number relative to $\alpha$ as

$$
\mu(Y ; \alpha)=(-1)^{\operatorname{dim} Y}\left(\int_{M} c(E)^{-1} c_{t o p}(E) c_{*}(\alpha)-\chi\left(Y ;\left.\alpha\right|_{Y}\right)\right)
$$

where $Y$ is a closed subvariety of $M$ of pure dimension given as zero set of a regular holomorphic section of a holomorphic vector bundle $E$ over $M$ and $\alpha$ is a constructible function on $M$.

Lastly we present a generalization of Milnor class due to J. Schürmann (Definition 2.3.4). For a regular embedding $\iota: X \hookrightarrow Z$ and $\alpha$ constructible function on $Z$. Schürmann defined the Milnor class of the pair $X \subset Z$ relative to $\alpha$ as

$$
\mathcal{M}(X \subset Z ; \alpha)=(-1)^{\operatorname{dim} X}\left(c\left(N_{X} Z\right)^{-1} \cap \iota^{*}\left(c_{*}(\alpha)\right)-c_{*}\left(\iota^{*}(\alpha)\right)\right) \in H_{*}(X) .
$$

Assuming that $Z$ is a smooth variety and $X$ is the zero-scheme of a regular section of a vector bundle $E$ on $Z$. We show that

$$
\mu(X ; \alpha)=\int_{X} \mathcal{M}(X ; \alpha) .
$$

In a natural way, from the Schürmann's definition, one defines the Fulton-Johnson class of the pair $X \subset Z$ relative to $\alpha$ and the Schwartz-MacPherson class of the pair $X \subset Z$ relative to $\alpha$.

The Chapter 3 starts with the result due to M. Kwieciński (Theorem 3.1.1), which states the following: Let $X$ and $Y$ be manifolds and let $\alpha$ and $\beta$ be constructible functions on $X$ and $Y$, respectively. Then

$$
c_{*}(\alpha \otimes \beta)=c_{*}(\alpha) \times c_{*}(\beta) .
$$

The Theorem 3.1.1 will be of great use throughout this text. We use the same to show Proposition 3.1.2 and Proposition 3.1.3. As a consequence of these propositions we get the Theorem 3.1.4, which generalizes a product formula due to T. Ohmoto and S. Yokura.

Let $M$ be an $n$-dimensional compact complex analytic manifold. Define $M^{(r)}:=$ $M \times \cdots \times M$. And let $Z(t)$ be the zero set of a regular holomorphic section $t$ of a holomorphic $d$-vector bundle $E$ over $M^{(r)}$. Hence, $Z(t)$ is a closed subvariety of $M^{(r)}$ of dimension $n r-d$. Consider $\Delta: M \longrightarrow M^{(r)}$ the diagonal morphism, which is a regular embedding of codimension $n r-n$. The morphism $\Delta$ induces the refined Gysin homomorphism

$$
\Delta^{!}: H_{2 k}(Z(t)) \longrightarrow H_{2(k-n r+n)}\left(Z\left(\Delta^{*}(t)\right)\right) .
$$

The Lemma 3.2.1, due to R. Callejas-Bedregal, M. F. Z. Morgado and J.Seade, is used in Propositions 3.2.2 and 3.2.3, which are generalizations for constructible functions of results involving the refined Gysin homomorphism above and the Fulton-Johnson
and Schwartz-MacPherson classes, found in [B-M-S]. And thus, the Proposition 3.2.4, which states that

$$
\Delta^{!}(\mathcal{M}(Z(t) ; \alpha))=(-1)^{n r-n} c\left(\left(\left.T M\right|_{Z\left(\Delta^{*} t\right)}\right)^{\oplus r-1}\right) \cap \mathcal{M}\left(Z\left(\Delta^{*} t\right) ; \Delta^{*} \alpha\right)
$$

follows as a consequence of these propositions, where $\alpha$ is a constructible function on $M^{(r)}$.

Now, for each $i=1, \cdots, r$, let $E_{i}$ be a holomorphic vector bundle of rank $d_{i}$ over $M$ and let $X_{i}:=s_{i}^{-1}(0)$ be a $\left(n-d_{i}\right)$-dimensional local complete intersection, where $s_{i}$ is a regular holomorphic section on $E_{i}$. Set $X:=X_{1} \cap \cdots \cap X_{r}$. Then we generalize, for constructible functions, some intersection formulas due to R. Callejas-Bedregal, M. F. Z. Morgado and J.Seade, involving the Fulton-Johnson class, the Schwartz-MacPherson class and the Milnor class. Explicitly, for $\alpha_{1}, \cdots, \alpha_{r}$ constructible functions on $M$, we have that

$$
\begin{aligned}
c^{F J}(X ; \alpha) & =c\left(\left(\left.T M\right|_{X}\right)^{\oplus r-1}\right)^{-1} \cap c^{F J}\left(X_{1} ; \alpha_{1}\right) \cdot \ldots \cdot c^{F J}\left(X_{r} ; \alpha_{r}\right), \\
c^{S M}(X ; \alpha) & =c\left(\left(\left.T M\right|_{X}\right)^{\oplus r-1}\right)^{-1} \cap c^{S M}\left(X_{1} ; \alpha_{1}\right) \cdot \ldots \cdot c^{S M}\left(X_{r} ; \alpha_{r}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{M}(X ; \alpha)= & (-1)^{\operatorname{dim} X} c\left(\left(\left.T M\right|_{X}\right)^{\oplus r-1}\right)^{-1} \cap\left(c^{F J}\left(X_{1} ; \alpha_{1}\right) \cdots c^{F J}\left(X_{r} ; \alpha_{r}\right)-\right. \\
& \left.-c_{*}\left(X_{1} ; \alpha_{1}\right) \cdots c_{*}\left(X_{r} ; \alpha_{r}\right)\right)
\end{aligned}
$$

where $\alpha$ denotes the constructible function $\alpha_{1} \otimes \cdots \otimes \alpha_{r}$.
In the Chapter 4, we look for a notion for Segre class relative to constructible functions that had "a good behavior"with the definition of Fulton-Johnson class relative to a constructible function due to Schürmann. This reason is for the following fact. Let $X$ be a proper closed subscheme of a variety $Y$. Consider $\widetilde{Y}$ the blow-up of $Y$ along $X, \widetilde{X}=P\left(N_{X} Y\right)$ the exceptional divisor and $\eta: \widetilde{X} \longrightarrow X$ the projection, where $N_{X} Y$ is the normal bundle. The Segre class of $X$ in $Y$ is characterized by

$$
s(X, Y)=\sum_{i \geq 0} \eta_{*}\left(c_{1}(\mathcal{O}(1))^{i} \cap\left[P\left(N_{X} Y\right)\right]\right)
$$

where $\mathcal{O}(1)$ is the canonical line bundle on $P\left(N_{X} Y\right)$. Suppose that $Y=M$ is nonsingular. The Fulton class of $X$ is defined by

$$
c^{F}(X)=c\left(\left.T M\right|_{X}\right) \cap s(X, M) .
$$

In this way, motivated by Schürmann's definition, we define the Segre class of $X$ relative to a constructible function $\alpha$ on $M$ as being

$$
s(X \subset M ; \alpha)=\eta_{*}\left(\sum_{i \geq 0} c_{1}\left(\mathcal{O}_{\mathbb{P}\left(N_{X} M\right)}(1)\right)^{i} \cap \eta^{*}\left(c\left(\left.T M\right|_{X}\right)^{-1} \cap \iota^{*} c_{*}(\alpha)\right)\right)
$$

where $\iota: X \hookrightarrow M$ is the regular embedding. And we define the Fulton class of $X$ relative to $\alpha$ as

$$
c^{F}(X \subset M ; \alpha)=c\left(\left.T M\right|_{X}\right) \cap s(X \subset M ; \alpha) .
$$

Note that, this definition coincides with the definition of Fulton-Johnson class relative to a construtible function due to Schürmann. At long last, we show two results. Proposition 4.0.3 is about the Segre class of a product of schemes and Proposition 4.0.5 is about a pullback of a Segre class.

Finally, the Chapter 5 is dedicated to some auxiliary definitions and properties.

## Capítulo 1

## Preliminary

### 1.1 Index of vector fields

As references we cite [B-S-S] and [M-T].
Let $v=\sum_{i=1}^{m} f_{i} \partial / \partial x_{i}$ be a vector field on an open $U \subset \mathbb{R}^{m}$. One says that a point $p \in U$ is a singularity of $v$ if $f_{i}(p)=0$ for all $i=1, \ldots, m$. The singularity is isolated if at every point $x$ near $p$ there is at least one component of $v$ which is not zero. Consider $v$ a continuous vector field on $U$ with an isolated singularity at $p$, and consider $\mathbb{S}_{\epsilon}$ a small sphere in $U$ around $p$. The Poincaré-Hopf index of $v$ at $p$, denoted by $\operatorname{Ind}(v, p)$, is defined to be the degree of the Gauss map $v /\|v\|$ from $\mathbb{S}_{\epsilon}$ to the unit sphere in $\mathbb{R}^{m}$.

Now let $M$ be an $m$-dimensional smooth manifold. A vector field $v$ on $M$ is locally expressed as above and one defines of the local index at an isolated singularity extending in the natural way. Note that, this definition not depend on the local chart. The total index of $v$, denoted by $\operatorname{Ind}_{M} v$, is the sum of all its local indices at the singular points.

Theorem 1.1.1 (Poincaré-Hopf) Let $M$ be a closed oriented manifold and let $v$ be a continuous vector field on $M$ with finitely many isolated singularities. Then,

$$
\operatorname{Ind}_{M} v=\chi(M)
$$

independently of $v$, where $\chi(M)$ denotes the Euler characteristic of $M$.

These notions can be extended to sections of a oriented vector bundle as follows. Let $E$ be an oriented vector bundle over a compact oriented smooth manifold $M$, and let $\pi: E \longrightarrow M$ be its projection. Suppose that rank of $E$ is equals to $\operatorname{dim} M=m$. Consider $s_{0}$ the zero section of $E$ and $s$ another arbitrary smooth section of $E$.

Definition 1.1.2 Let $p \in M$ be a zero for $s$, with $s(p)=s_{0}(p)$. One says that $s$ is transversal to $s_{0}$ at $p$ if

$$
\begin{equation*}
D_{p} s\left(T_{p} M\right) \cap D_{p} s_{0}\left(T_{p} M\right)=0 \tag{1.1}
\end{equation*}
$$

and $s$ is called transversal to $s_{0}$ if this holds for all zeros of $s$.

Say that $s$ is transversal to $s_{0}$ at $p$ is equivalent to statement that $D_{p} s\left(T_{p} M\right)$ is the graph of a linear isomorphism $A$ from $T_{p} M$ to the fiber $E_{p}$. By assumption, both vector spaces are oriented. In this way, one defines the local index $\iota(s ; p)$ to be +1 if $A$ preserves the orientations, and -1 if not. In particular, (1.1) forces $p$ to be an isolated zero of $s$. Moreover, if $s$ is transversal to the zero section then the number of zeros of $s$ is finite, since $M$ was assumed compact.

Theorem 1.1.3 If $s$ is transverse to the zero section, then

$$
I(e(E))=\sum_{p} \iota(s ; p)
$$

where the sum runs over the zeros of $s, e(E)$ is the Euler class of $E$ and $I: H^{m}(M) \longrightarrow$ $\mathbb{R}$.

Proof. See [M-T, Theorem 21.9 and Theorem 21.11].
As a consequence, for any oriented compact smooth manifold $M$, one has $I(e(T M))=$ $\chi(M)$.

### 1.2 Holomorphic vector bundles

Let $M$ be a differentiable manifold. A $C^{\infty}$ complex vector bundle on $M$ consist of a family $\left\{E_{x}\right\}_{x \in M}$ of complex vector spaces, together with a $C^{\infty}$ manifold structure on $E=\cup_{x \in M} E_{x}$ such that the projection map $\pi: E \longrightarrow M$ taking $E_{x}$ to $x$ is $C^{\infty}$; and for every $x_{0} \in M$ there is as open set $U$ in $M$ containing $x_{0}$ and a diffeomorphism $\varphi_{U}: \pi^{-1}(U) \longrightarrow U \times \mathbb{C}^{r}$, called trivialization, taking the vector space $E_{x}$ isomorphically onto $\{x\} \times \mathbb{C}^{r}$ for each $x \in U$. The dimension of the fibers $E_{x}$ of $E$ is called the rank
of $E$. A vector bundle of rank 1 is called a line bundle. A section $s$ of the vector bundle $E$ over $U \subset M$ is a $C^{\infty}$ map $s: U \longrightarrow E$ such that $s(x) \in E_{x}$ for all $x \in U$. A frame for $E$ over $U \subset M$ is a collection $s_{1}, \ldots, s_{r}$ of sections of $M$ over $U$ such that $\left\{s_{1}(x), \ldots, s_{r}(x)\right\}$ is a basis for $E_{x}$ for all $x \in U$.

Now assume that $M$ is a complex manifold. A holomorphic vector bundle $\pi$ : $E \longrightarrow M$ is a complex vector bundle together with the structure of a complex manifold on $E$, such that for any $x \in M$ there is an open set $U$ in $M$ with $x \in U$ and a trivialization $\varphi_{U}: \pi^{-1}(U) \longrightarrow U \times \mathbb{C}^{r}$ which is a biholomorphism map of complex manifolds. A section $s$ of the holomorphic bundle $E$ over $U \subset M$ is said to be holomorphic if $s: U \longrightarrow E$ is a holomorphic map. A frame $\underline{s}=s_{1}, \ldots, s_{r}$ is called holomorphic if each $s_{i}$ is a holomorphic map. A hermitian metric on $M$ is defined as a positive definite hermitian inner product

$$
(,)_{z}: T_{z}^{\prime} M \otimes \overline{T_{z}^{\prime} M} \longrightarrow \mathbb{C}
$$

where $T_{z}^{\prime} M$ denotes the holomorphic tangent space at $z$ for each $z \in M$, depending smoothly on $z$, that is, for local coordinates $z$ on $M$ the functions $h_{i j}(z)=\left(\partial / \partial z_{i}, \partial / \partial z_{j}\right)$ are $C^{\infty}$.

### 1.3 Logarithmic forms and free divisors

Let $U$ be a domain of $\mathbb{C}^{n}$, and let $D \subseteq U$ be a divisor of $U$ defined by an equation $h(z)=0$, where $h$ is holomorphic on $U$. A meromorphic $q$-form $\omega$ on $U$ is called a $q$-form with logarithmic pole along $D$ or logarithmic $q$-form (along $D$ ) if it satisfies the following equivalent conditions:
(a) $h \omega$ and $h d \omega$ are holomorphic on $U$.
(b) $h \omega$ and $d h \wedge \omega$ are holomorphic on $U$.
(c) There exists a holomorphic function $g(z)$, a holomorphic $(q-1)$-form $\xi$ and a holomorphic $q$-form $\eta$ on $U$, such that

- $\operatorname{dim}_{\mathbb{C}} D \cap\{z \in U: g(z)=0\} \leq n-2$,
- $g \omega=\frac{d h}{h} \wedge \xi+\eta$.
(d) There exists an $(n-2)$-dimensional analytic set $A \subset D$ such that the germ of $\omega$ at any point $p \in D \backslash A$ belongs to $\frac{d h}{h} \wedge \Omega_{U, p}^{q-1}+\Omega_{U, p}^{q}$, where $\Omega_{U, p}^{q}$ denotes the module of germs of holomorphic $q$-forms on $U$ at $p$.

Let $S$ an $n$-dimensional complex manifold and $D$ be a divisor of $D$. Consider $h_{p}=0$ a reduced equation for $D$, locally at $p \in D$. A meromorphic $q$-form $\omega$ is $\operatorname{logarithmic}$ along $D$ at $p$ if $h_{p} \omega$ and $h_{p} d \omega$ are holomorphic. We denote $\Omega_{S, p}^{q}(\log D):=$ $\{$ germ of $\operatorname{logarithmic} q$-form at $p\}$ and $\Omega_{S}^{q}(\log D):=\bigcup_{p \in S} \Omega_{S, p}^{q}(\log D)$.

Definition 1.3.1 Let $S$ an n-dimensional complex manifold and $D$ be a divisor of $D$. Consider $h_{p}=0$ a reduced equation for $D$, locally at $p \in D$. A holomorphic vector field $\delta$ on $S$ is logarithmic if it satisfies the following equivalent conditions:
(a) For any smooth point $p \in D$, the tangent vector $\delta(p)$ of $p$ is tangent to $D$,
(b) For any point $p \in D$, the derivation $\delta h_{p}$ of the local equation for $D$ belongs to the ideal $\left(h_{p}\right) \mathcal{O}_{S, p}$.

We denote $\operatorname{Der}_{S, p}(\log D):=\{\delta:$ germ of a holomorphic vector field on $S$ at $p$ such that $\left.\delta\left(h_{p}\right) \in\left(h_{p}\right)\right\}$ and $\operatorname{Der}_{S}(\log D):=\bigcup_{p \in S} \operatorname{Der}_{S, p}(\log D)$. Note that, $\operatorname{Der}_{S}(\log D)$ is a coherent $\mathcal{O}_{S}$-submodule of $\operatorname{Der}_{S}$, where $\operatorname{Der}_{S}$ is a sheaf of holomorphic vector fields on $S$.

Definition 1.3.2 Let $D$ be a divisor in $S$ and let $p \in D$, we say that $D$ is a free divisor in $p$ if $\Omega_{S, p}^{1}(\log D)$ (or its dual $\operatorname{Der}_{S, p}(\log D)$ ) is a $\mathcal{O}_{S, p}$-free module. Moreover, we say that $D$ is a free divisor if $\Omega_{S, p^{\prime}}^{1}(\log D)$ (or its dual $\operatorname{Der}_{S, p^{\prime}}(\log D)$ ) is a $\mathcal{O}_{S, p^{\prime}}$ free module for all $p^{\prime} \in D$.

### 1.4 Constructible functions

The references are [Scha] and [K-S].
Let $X$ be a real analytic manifold. A function $\alpha: X \longrightarrow \mathbb{Z}$ is called constructible if for each $m \in \mathbb{Z}$, the set $\alpha^{-1}(m)$ is subanalytic and the family $\left\{\alpha^{-1}(m)\right\}_{m \in \mathbb{Z}}$ is locally finite, or equivalent, by triangulation theorem, if there exists a locally finite covering $X=\bigcup_{i \in I} X_{i}$ and

$$
\alpha=\sum m_{i} \mathbf{1}_{X_{i}}
$$

where $m_{i}$ are integers, $X_{i}$ are (closed) analytic subset of $X$ and $\mathbf{1}_{X_{i}}$ is the characteristic functions of $X_{i}$. The set of all constructible functions on $X$, denoted by $C F(X)$,
is endowed with a structure of algebra. If $\alpha$ has support compact, then all $X_{i}$ are compacts. Thus one defines the Weighted Euler characteristic of $\alpha$ as been

$$
\chi(X ; \alpha)=\sum_{i} m_{i} \chi\left(X_{i}\right),
$$

where $\chi$ is the topological Euler characteristic. Now, let $f: X \longrightarrow Y$ be a morphism of analytic manifolds. Given $\beta$ a constructible function on $Y$, one defines the inverse image, or pullback, of $\beta$ by $f$ as been the constructible function $f^{*} \beta$ on $X$ defined by $f^{*} \beta(x)=\beta(f(x))$, for all $x \in X$. Assume that $f: X \longrightarrow Y$ is proper morphism on the support of a constructible function $\alpha$ on $X$, one defines the direct image, or pushfoward, as been the constructible function $f_{*} \alpha$ on $Y$ defined by

$$
f_{*}(\alpha)(y)=\chi\left(f^{-1}(y) ;\left.\alpha\right|_{f^{-1}(y)}\right),
$$

for all $y \in Y$. Note that, given a analytic subset $W$ of $X$, the characteristic function $\mathbf{1}_{W}$ is constructible.

### 1.5 Intersection theory

The references are [Ful] and $[\mathrm{H}]$.

### 1.5.1 Algebraic Schemes

We say that a scheme $X$ is algebraic over a field $K$ if there is a morphism of finite type from $X$ to $\operatorname{Spec}(\mathrm{K})$. This means that, $X$ has a finite covering by affine sets whose coordinate rings are finitely generated $K$-algebras. We denote the coordinate ring of an affine open $U$ by $A(U)$. However, the word scheme means an algebraic scheme over some field. A closed subscheme $Y$ of a scheme $X$ is defined by an ideal sheaf $\mathcal{I}(Y)$ in the structure sheaf $\mathcal{O}_{X}$ of $X$.

A variety is a reduced and irreducible (integral) algebraic scheme. A subvariety $V$ of a scheme $X$ is a reduced and irreducible closed subscheme of $X$. One has that a subvariety $V$ corresponds to a prime ideal in the coordinate ring of any affine open set meeting $V$. The local ring of $X$ along $V$, denoted by $\mathcal{O}_{V, X}$, is the localization of such a coordinate ring at the corresponding prime ideal; its maximal ideal is denoted by $\mathcal{M}_{V, X}$. The function field of $V$, denoted by $R(V)$, is the residue field $\mathcal{O}_{V, X} / \mathcal{M}_{V, X}$.

The dimension of a scheme $X$, denoted by $\operatorname{dim} X$, is the maximum length $n$ of a chain

$$
\varnothing \varsubsetneqq V_{0} \varsubsetneqq V_{1} \varsubsetneqq \ldots \varsubsetneqq V_{n} \subset X
$$

of subvarieties of $X$. A scheme $X$ is pure dimensional if all irreducible components of $X$ have the same dimension. A point on a scheme $X$ is a 0 -dimensional subvariety of $X$. We say that a point $p$ of $X$ is regular if $\mathcal{O}_{p, X}$ is a regular local ring. The open set of regular points in $X$ is denoted by $X_{\text {reg. }}$.

The affine $n$-space, denoted by $\mathbb{A}^{n}$, is the affine variety whose coordinate ring is the polynomial ring $K\left[x_{1}, \cdots, x_{n}\right]$. The subscheme of $\mathbb{A}^{n}$ defined by an ideal $I=$ $\left(f_{1}, \cdots, f_{n}\right)$ in $K\left[x_{1}, \cdots, x_{n}\right]$ is denoted by $V(I)$.

A morphism $f: X \longrightarrow Y$ of algebraic schemes is assumed to be compatible with the structure morphism to $\operatorname{Spec}(\mathrm{K})$, where $K$ is the ground field. If $f$ maps an affine open subset $U^{\prime}$ of $X$ into an affine open subset $U$ of $Y$, then $f$ corresponds to a homomorphism $f^{*}: A(U) \longrightarrow A\left(U^{\prime}\right)$ of $K$-algebras.

Let $f: X \longrightarrow S$ and $g: Y \longrightarrow S$ be morphisms. The fibre product of $X$ and $Y$ over $S$, denoted by $X \times_{S} Y$, comes equipped with projections $p: X \times_{S} Y \longrightarrow X$ and $q: X \times{ }_{S} Y \longrightarrow Y$ which satisfy the following universal property: for any scheme $Z$ with morphisms $u: Z \longrightarrow X$ and $v: Z \longrightarrow Y$ such that $f \circ u=g \circ v$, there is a unique morphism $(u, v): Z \longrightarrow X \times_{S} Y$ such that $p \circ(u, v)=u$ and $q \circ(u, v)=v$. A commutative square of morphism

is called a fibre square. In a fibre diagram, all squares appearing in the diagram are required to be fibre squares. When $S=\operatorname{Spec}(\mathrm{K})$, where $K$ is the ground field, it is called Cartesian product of $X$ and $Y$, denotes $X \times Y$ in place of $X \times{ }_{S} Y$.

A morphism $f: X \longrightarrow Y$ is separated if the diagonal morphism from $X$ to $X \times_{Y} X$ is a closed imbedding. We say that a morphism $f: X \longrightarrow Y$ is proper if it is separated, and universally closed, i.e., for all $Y^{\prime} \longrightarrow Y$, the induced morphism from $X \times_{Y} Y^{\prime}$ to $Y^{\prime}$ takes closed sets to closed sets. A scheme is complete if the structural morphism to $\operatorname{Spec}(\mathrm{K})$ is proper.

A morphism $f: X \longrightarrow Y$ is flat if for $U \subset Y, U^{\prime} \subset X$ affine open sets with $f\left(U^{\prime}\right) \subset U$, the induced map $f^{*}: A(U) \longrightarrow A\left(U^{\prime}\right)$ makes $A\left(U^{\prime}\right)$ a flat $A(U)$-module. A morphism $f: X \longrightarrow Y$ has relative dimension $n$ if for all subvarieties $V$ of $Y$, and all irreducible components $V^{\prime}$ of $f^{-1}(V)$, $\operatorname{dim} V^{\prime}=\operatorname{dim} V+n$. A morphism $f: X \longrightarrow Y$ is called smooth if $f$ is flat of some relative dimension $n$, and the sheaf of relative differentials $\Omega_{X / Y}^{1}$ is a locally free sheaf of rank $n$. We say that a scheme $X$ is nonsingular, or smooth, if it is smooth over $\operatorname{Spec}(\mathrm{K})$. If $f: X \longrightarrow Y$ is smooth of relative dimension $n$, the relative tangent bundle, denoted by $T_{X / Y}$, is the vector whose sheaf of sections is the dual bundle to $\Omega_{X / Y}^{1}$. When $Y=\operatorname{Spec}(\mathrm{K})$, denotes $T_{X}$ in place of $T_{X / Y}$.

Let $X$ be a scheme. A scheme $E$ equipped with a morphism $\pi: E \longrightarrow X$ is called a vector bundle of rank $r$ on $X$ if there are an open covering $\left\{U_{i}\right\}$ of $X$ and isomorphisms $\Phi_{i}$ of $\pi^{-1}\left(U_{i}\right)$ with $U_{i} \times \mathbb{A}^{r}$ over $U_{i}$, such that over $U_{i} \cap U_{j}$ the composites $\varphi_{i} \circ \varphi_{j}^{-1}$ are linear. A section of $E$ is a morphism $s: X \longrightarrow E$ such that $\pi \circ s=i d_{X}$. Several basic operations are defined for vector bundles, compatibly with the corresponding notions for sheaves: direct sum $E \oplus F$, tensor product $E \otimes F$, exterior product $\bigwedge^{i} E$, symmetric product $\operatorname{Sym}^{i} E$, dual bundle $E^{\vee}$, pull-back $f^{*} E$ for a morphism $f: X^{\prime} \longrightarrow X$. If $E$ is a vector bundle of rank $n$ on $X$, then its sheaf of sections of $E$ is a locally free sheaf $\mathcal{E}$ of $\mathcal{O}_{X}$-modules of rank $r$. Conversely, for any locally free coherent sheaf $\mathcal{E}$ of rank $n$ on $X$, one can produce a vector bundle $E=\operatorname{Spec}\left(\operatorname{Sym}\left(\mathcal{E}^{\vee}\right)\right)$. The trivial bundle of rank one on $X$ is often denoted by 1. Lastly, a line bundle is a vector bundle $L$ of rank one.

Let $X$ be a closed subscheme of a scheme $Y$, defined by an ideal sheaf $\mathcal{I}$. The blow-up of $Y$ along $X$, denoted by $B l_{X} Y$, is the projective cone over $Y$ of the sheaf of $\mathcal{O}_{Y}$-algebras $\bigoplus_{n \geq 0} \mathcal{I}^{n}$, that is, $B l_{X} Y=\operatorname{Proj}\left(\bigoplus_{n \geq 0} \mathcal{I}^{n}\right)$. Let us denote $\widetilde{Y}=B l_{X} Y$ and $\pi$ the projection from $\widetilde{Y}$ to $Y$. The canonical invertible sheaf $\mathcal{O}(1)$ on the projective cone $\widetilde{Y}$ is the ideal sheaf of $\pi^{-1}(X)$, which is a Cartier divisor on $\widetilde{Y}$, called the exceptional divisor. Let $\widetilde{X}=\pi^{-1}(X)$. Note that, one has $\widetilde{X}=\operatorname{Proj}\left(\left(\bigoplus_{n \geq 0} \mathcal{I}^{n}\right) \otimes_{\mathcal{O}_{Y}} \mathcal{O}_{X}\right)=\operatorname{Proj}\left(\bigoplus_{n \geq 0} \mathcal{I}^{n} / \mathcal{I}^{n+1}\right)$, which is the projective normal cone to $X$ in $Y$, denoted by $P\left(C_{X} Y\right)$.

### 1.5.2 Rational Equivalence

Let $X$ be an algebraic scheme. A $k$-cycle on $X$ is a finite sum $\sum n_{i}\left[V_{i}\right]$, where the $V_{i}$ are $k$-dimensional subvarieties of $X$ and the $n_{i}$ are integers. Tho free abelian broup on the $k$-dimensional subvarieties of $X$ is called group of $k$-cycles on $X$, which we denote by $Z_{k} X$. Consider a $(k+1)$-dimensional subvariety $W$ of $X$ and $r \in R(W)^{*}$. Choose $f$ and $g$ in $\mathcal{O}_{V, W}$ such that $r=f / g$. The order of $r$ along $V$ is defined by

$$
\operatorname{ord}_{V}(r)=l\left(\mathcal{O}_{V, W} /(f)\right)-l\left(\mathcal{O}_{V, W} /(g)\right) .
$$

Define a $k$-cycle $[\operatorname{div}(r)]$ on $X$ by $[\operatorname{div}(r)]=\sum \operatorname{ord}_{V}(r)[V]$, the sum runs over all codimension one subvarieties $V$ of $W$. Moreover, a $k$-cycle $\alpha$ is rationally equivalent to zero, denoted by $\alpha \sim 0$, if there are a finite number of $(k+1)$-dimensional subvarieties $W_{i}$ of $X$ and $r_{i} \in R(W)^{*}$ such that $\alpha=\sum\left[\operatorname{div}\left(r_{i}\right)\right]$. One denotes by $\operatorname{Rat}_{k}(X)$ the group of The cycles rationally equivalent to zero form a subgroup $\operatorname{Rat}_{k}(X)$ of $Z_{k}(X)$. The group of $k$-cycles modulo rational equivalence on $X$ is given by

$$
A_{k} X=Z_{k} X / \operatorname{Rat}_{k} X
$$

Let $f: X \longrightarrow Y$ be a proper morfism. Given any subvariety $V$ of $X$, the image $W=f(V)$ is a (closed) subvariety of $X$. It is known that if $W$ has the same dimension as $V$ then the induced imbedding of $R(W)$ in $R(V)$ is a finite field extension. Set

$$
\operatorname{deg}(V / W)= \begin{cases}{[R(V): R(W)]} & \text { if } \quad \operatorname{dim} W=V \\ 0 & \text { if } \quad \operatorname{dim} W<\operatorname{dim} V\end{cases}
$$

where $[R(V): R(W)]$ denotes the degree of the field extension. We define $f_{*}([V])=$ $\operatorname{deg}(V / W)[W]$. We can extend linearly to a homomorphism $f_{*}: Z_{k} X \longrightarrow Z_{k} Y$.

Theorem 1.5.1 Let $f: X \longrightarrow Y$ be a proper morphism and let $\alpha$ be a $k$-cycle on $X$, such that $\alpha$ is rationally equivalent to zero. Then $f^{*} \alpha$ is rationally equivalent to zero on $Y$. Therefore, there exists an induced homomorphism

$$
f_{*}: A_{k} X \longrightarrow A_{k} Y .
$$

So the $A_{*}$ is a covariant functor for proper morphisms.
Proof. See [Ful, Theorem 1.4].
Suppose that $X$ is complete, that is, $X$ is proper over $S=\operatorname{Spec}(K)$, with $K$ being the ground field. Consider $\alpha=\sum_{P} n_{P}[P]$ a zero-cycle on $X$.

Definition 1.5.2 The degree of $\alpha$, denoted $\operatorname{deg}(\alpha)$, or $\int_{X} \alpha$, is defined by

$$
\operatorname{deg}(\alpha)=\int_{X} \alpha=\sum_{P} n_{P}[R(P): K] .
$$

Equivalently, $\operatorname{deg}(\alpha)=p_{*}(\alpha)$, where $p$ is the structure morphism from $X$ to $S$, and $A_{0} S=\mathbb{Z}[S]$ is identified with $\mathbb{Z}$. One can extend the degree homomorphism to all of $A_{*} X=\bigoplus_{k=0}^{\operatorname{dim} X} A_{k} X$,

$$
\int_{X}: A_{*} X \longrightarrow \mathbb{Z}
$$

with $\int_{X} \alpha=0$ if $\alpha \in A_{*} X, k>0$. Now, for any morphism $f: X \longrightarrow Y$ of complete schemes, and any $\alpha \in A_{*} X$,

$$
\int_{X} \alpha=\int_{Y} f_{*}(\alpha) .
$$

Now, let $X$ be any scheme and let $X_{1}, \cdots X_{r}$ be the irreducible components of $X$. Note that, the local rings $\mathcal{O}_{X_{i}, X}$ are all zero-dimensional. We define the geometric multiplicity $m_{i}$ of $X_{i}$ in $X$ as being the length of $\mathcal{O}_{X_{i}, X}$, that is, $m_{i}=l_{\mathcal{O}_{X_{i}, X}}\left(\mathcal{O}_{X_{i}, X}\right)$. The fundamental cycle $[X]$ of $X$ is the cycle

$$
[X]=\sum_{i=1}^{r} m_{i}\left[X_{i}\right] .
$$

Let $f: X \longrightarrow Y$ be a flat morphism of relative dimension $n$. For any subvariety $V$ of $Y$, set $f^{*}[V]=\left[f^{-1}(V)\right]$, where $f^{-1}(V)$ is the inverse image scheme, which is a subscheme of $X$ of pure dimension $\operatorname{dim}(V)+n$. By linearity, we can extend to pull-back homomorphisms

$$
f^{*}: Z_{k} Y \longrightarrow Z_{k+n} X
$$

Theorem 1.5.3 Let $f: X \longrightarrow Y$ be a flat morphism of relative dimension $n$ and let $\alpha$ be a $k$-cycle on $Y$ wich is rationally equivalent to zero. Then $f^{*} \alpha$ is rationally equivalent to zero in $Z_{k+n} X$.

Proof. See [Ful, Theorem 1.7].
Consider $X$ and $Y$ two algebraic schemes over a field. Denote by $X \times Y$ the Cartesian (fiber) product of $X$ and $Y$ over the ground field. The exterior product

$$
Z_{k} X \otimes Z_{l} Y \xrightarrow{\times} Z_{k+l}(X \times Y)
$$

is defined by the formula $[V] \times[W]=[V \times W]$, for $V, W$ subvarieties of $X, Y$, respectively; and extending bilinearly to general cycles.

Proposition 1.5.4 (a) If $\alpha \sim 0$ or $\beta \sim 0$, then $\alpha \times \beta \sim 0$.
(b) Let $f: X^{\prime} \longrightarrow X, g: Y^{\prime} \longrightarrow Y$ be morphisms between algebraic schemes over a field. Denote by $f \times g$ the induced morphism from $X^{\prime} \times Y^{\prime}$ to $X \times Y$.
(i) If $f$ and $g$ are proper, then $f \times g$ is proper, and

$$
(f \times g)_{*}(\alpha \times \beta)=f_{*} \alpha \times g_{*} \beta
$$

for all cycles $\alpha \in X^{\prime}$ and $\beta \in Y^{\prime}$.
(ii) If $f$ and $g$ are flat of relative dimensions $m$ and $n$, respectively, then $f \times g$ is flat of relative dimension $m+n$, and

$$
(f \times g)^{*}(\alpha \times \beta)=f^{*} \alpha \times g^{*} \beta
$$

for all cycles $\alpha \in X^{\prime}$ and $\beta \in Y^{\prime}$.

Proof. See [Ful, Proposition 1.10].
Consequently, there are exterior products

$$
A_{k} X \otimes A_{l} Y \xrightarrow{\times} A_{k+l}(X \times Y) .
$$

### 1.5.3 Gysin map for divisors

Let $D$ be an effective Cartier divisor on a scheme $X$ and $i: D \longrightarrow X$ be the inclusion. There are Gysin homomorphisms

$$
i^{*}: Z_{k} X \longrightarrow A_{k-1} D
$$

given by $i^{*}(\alpha)=D \cdot \alpha$, where $D \cdot \alpha$ is the intersection class in $A_{k-1} D$.
Proposition 1.5.5 (a) If $\alpha$ is rationally equivalent to zero on $X$, then $i^{*} \alpha=0$.
Thus, there are induced homomorphisms

$$
i^{*}: A_{k} X \longrightarrow A_{k-1} D
$$

(b) If $\alpha$ is a $k$-cycle on $X$, then

$$
i_{*} i^{*}(\alpha)=c_{1}\left(\mathcal{O}_{X}(D)\right) \cap \alpha
$$

(c) If $\alpha$ is a $k$-cycle on $D$, then

$$
i_{*} i^{*}(\alpha)=c_{1}\left(i^{*} \mathcal{O}_{X}(D)\right) \cap \alpha
$$

(d) If $X$ is purely $n$-dimensional, then

$$
i^{*}[X]=[D] \in A_{n-1} D .
$$

(e) If $L$ is a line bundle on $X$ and $\alpha$ is a $k$-cycle on $X$, then

$$
i^{*}\left(c_{1}(L) \cap \alpha\right)=c_{1}\left(i^{*} L\right) \cap i^{*}(\alpha) \in A_{k-2} D
$$

Proof. See [Ful, Proposition 2.6].

### 1.5.4 Chern classes

Let $E$ be a vector bundle on $X$ of rank $r$. We define Segre class operators $s_{i}(E)$,

$$
s_{i}(E) \cap \cap_{-}: A_{k}(X) \longrightarrow A_{k-i}(X)
$$

as follow. Consider $p: \mathrm{P}(E) \longrightarrow X$ the projective bundle of $E, \mathcal{O}_{E}(1)$ the canonical line bundle on $\mathrm{P}(E)$, and $\alpha$ in $A_{k} X$. Set

$$
s_{i}(E) \cap \alpha=p_{*}\left(c_{1}\left(\mathcal{O}_{E}(1)\right)^{r-1+i} \cap p^{*} \alpha\right)
$$

where $p_{*}$ is flat
Note that $s_{i}(E)=0$ for $i<0$ and $s_{0}(E)=1$. We define Chern class operators

$$
c_{i}(E) \cap_{-}: A_{k}(X) \longrightarrow A_{k-i}(X)
$$

formally by $1+c_{1}(E)+c_{2}(E)+\cdots=\left(1+s_{1}(E)+s_{2}(E)+\cdots\right)^{-1}$. Explicity,

$$
\begin{aligned}
& c_{0}(E)=1, \quad c_{1}(E)=-s_{1}(E) \\
& c_{2}(E)=s_{2}(E)^{2}-s_{1}(E), \cdots \\
& c_{n}(E)=-s_{1}(E) c_{n-1}(E)-s_{2}(E) c_{n-2}(E)-\cdots-s_{n}(E) .
\end{aligned}
$$

The total Chern class of $E$ is the sum

$$
c(E):=1+c_{1}(E)+\cdots+c_{r}(E) .
$$

Theorem 1.5.6 We have that:
(a) (Vanishing) For all vector bundles $E$ on $X$, all $i>\operatorname{rank}(E)$,

$$
c_{i}(E)=0
$$

(b) (Commutativity) For all vector bundles $E, F$ on $X$, integers $i, j$, and cycles $\alpha$ on $X$,

$$
c_{i}(E) \cap\left(c_{j}(F) \cap \alpha\right)=c_{j}(F) \cap\left(c_{i}(E) \cap \alpha\right) .
$$

(c) (Projection formula) Let $E$ be a vector bundle on $X$ and let $f: X^{\prime} \longrightarrow X$ be a proper morphism. For all cycles $\alpha$ on $X^{\prime}$, and integers $i$, we have

$$
f_{*}\left(c_{i}\left(f^{*} E\right) \cap \alpha\right)=c_{i}(E) \cap f_{*}(\alpha) .
$$

(d) (Pull-back) Let $E$ be a vector bundle on $X$ and let $f: X^{\prime} \longrightarrow X$ be a flat morphism. For all cycles $\alpha$ on $X^{\prime}$, and integers $i$, we have

$$
c_{i}\left(f^{*} E\right) \cap f^{*} \alpha=f^{*}\left(c_{i}(E) \cap \alpha\right)
$$

(e) (Whitney sum) For any exact sequence

$$
0 \longrightarrow E^{\prime} \longrightarrow E \longrightarrow E^{\prime \prime} \longrightarrow 0
$$

of vector bundles on $X$, we have

$$
c(E)=c\left(E^{\prime}\right) \cdot c\left(E^{\prime \prime}\right)
$$

that is

$$
c_{k}(E)=\sum_{i+j=k} c_{i}\left(E^{\prime}\right) c_{j}\left(E^{\prime \prime}\right)
$$

(f) (Normalization) Let $L$ be a line bundle on $X$ and let $D$ be a Cartier divisor on $X$ with $\mathcal{O}(D) \cong L$. Then

$$
c_{1}(L) \cap[X]=[D] .
$$

Proof. See [Ful, Theorem 3.2].

### 1.5.5 Refined Gysin homomorphisms

Let $i: X \longrightarrow Y$ be a regular imbedding of codimension $d$, and let $f: Y^{\prime} \longrightarrow Y$ be a morphism. Form the fibre square


Now, we define homomorphisms $i^{!}: Z_{k} Y^{\prime} \longrightarrow A_{k-d} X^{\prime}$ given by $i^{!}\left(\sum n_{i}\left[V_{i}\right]\right)=\sum n_{i} X$. $V_{i}$, where $X \cdot V_{i}$ is the intersection product. The induced homomorphisms

$$
i^{!}: A_{k} Y^{\prime} \longrightarrow A_{k-d} X^{\prime}
$$

is called refined Gysin homomorphism.

Theorem 1.5.7 Consider a fibre diagram

where $i$ is a regular imbedding of codimension $d$.
(a) (Push-forward) If $p$ is proper, $\alpha \in A_{k} Y^{\prime \prime}$, then

$$
i^{!} p_{*}(\alpha)=q_{*}(i!\alpha)
$$

in $A_{k-d} X^{\prime}$.
(b) (Pull-back) If $p$ is flat of relative dimension $n$, and $\alpha \in A_{k} Y^{\prime}$, then

$$
i^{!} p^{*}(\alpha)=q^{*}\left(i^{!} \alpha\right)
$$

in $A_{k+n-d} X^{\prime \prime}$.
(c) (compatibility) Let $\alpha$ be another regular imbedding of codimension $d$. If $\alpha \in$ $A_{k} Y^{\prime \prime}$, then

$$
i^{!} \alpha=i^{\prime!} \alpha
$$

in $A_{k-d} X^{\prime \prime}$.

Proof. See [Ful, Theorem 6.2].
Proposition 1.5.8 Let $i: X \longrightarrow Y$ be a regular imbedding of codimension d,

a fibre square, and let $F$ be a vector bundle on $Y^{\prime}$. Then, for all $\alpha \in A_{k}\left(Y^{\prime}\right)$, and all $m \geqq 0$,

$$
i^{!}\left(c_{m}(F) \cap \alpha\right)=c_{m}\left(i^{\prime *} F\right) \cap i^{!}(\alpha)
$$

in $A_{k-d-m}\left(X^{\prime}\right)$.

Proof. See [Ful, Proposition 6.3].

Example 1.5.9 [Ful, Example 6.3.4] Consider E a vector bundle of rank $r$ on a scheme $Y$ and consider $s$ a regular section of $E$. Then, the inclusion $\iota$ of the zero-scheme $X=Z(s)$ in $Y$ is a regular imbedding of codimension $r$ and $N_{X} Y$ is the restriction of $E$ to $X$. If $f: Y^{\prime} \longrightarrow Y$ is a morphism, form the fibre square


Then

$$
j_{*} i^{!}(\alpha)=c_{r}\left(f^{*} E\right) \cap \alpha
$$

for all $\alpha \in A_{*} Y^{\prime}$.
Example 1.5.10 Let $i_{j}: X_{j} \longrightarrow Y_{j}$ be regular imbedding of codimensions $d_{j}, j=$ $1, \ldots, r$. Let $f_{j}: Y_{j}^{\prime} \longrightarrow Y_{j}$ be morphisms, $\alpha_{j} \in A_{k_{i}}\left(Y_{j}^{\prime}\right)$. Then $i_{1} \times \ldots \times i_{r}$ is a regular imbedding of $X_{1} \times \ldots \times X_{r}$ in $Y_{1} \times \ldots \times Y_{r}$, of codimension $\sum d_{i}$, and

$$
\left(i_{1} \times \cdots \times i_{r}\right)^{!}\left(\alpha_{1} \times \cdots \times \alpha_{r}\right)=i_{1}^{!}\left(\alpha_{1}\right) \times \cdots \times i_{r}^{!}\left(\alpha_{r}\right)
$$

in $A_{\sum\left(k_{j}-d_{j}\right)}\left(X_{1}^{\prime} \times \ldots \times X_{r}^{\prime}\right)$, with $X_{i}^{\prime}=X_{i} \times_{Y_{i}} Y_{i}^{\prime}$.
Example 1.5.11 Consider $X, Y$ schemes, $p$ and $q$ the projections from $X \times Y$ to $X$ and $Y$ and $E$ and $F$ vector bundles on $X$ and $Y$. Given $\alpha \in A_{*} X$ and $\beta \in A_{*} Y$, one has

$$
\left(c_{i}(E) \cap \alpha\right) \times \beta=c_{i}\left(p^{*} E\right) \cap(\alpha \times \beta)
$$

and

$$
(c(E) \cap \alpha) \times(c(F) \cap \beta)=c\left(p^{*} E \oplus q^{*} F\right) \cap(\alpha \times \beta)
$$

### 1.5.6 Segre Classes

Let $C$ be a cone over a scheme $X$, that is, $C=\operatorname{Spec}\left(S^{\bullet}\right)$, where $S^{\bullet}$ is a sheaf of graded $\mathcal{O}_{X}$-algebras. Let us assume $\mathcal{O}_{X} \longrightarrow S^{0}$ is surjective, $S^{1}$ is coherent and $S^{\bullet}$ is generated by $S^{1}$. For a variable $z$, denote by $S^{\bullet}[z]$ the graded algebra whose $n^{\text {th }}$ graded peace is $S^{n} \oplus S^{n-1} z \oplus \cdots \oplus S^{1} z^{n-1} \oplus S^{0} z^{n}$. Let $P(C \oplus 1)=\operatorname{Proj}(S \bullet[z])$ be the projective completion of $C$, with projection $q: P(C \oplus 1) \longrightarrow X$, and let $\mathcal{O}(1)$ be the canonical line bundle on $P(C \oplus 1)$. The Segre class of $C$, denoted by $s(C)$, is the class in $A_{*} X$ defined by the formula

$$
s(C)=q_{*}\left(\sum_{i \geq 0} c_{1}(\mathcal{O}(1))^{i} \cap[P(C \oplus 1)]\right) .
$$

If $E$ is a vector bundle on $X$, one has $s(E)=c(E)^{-1} \cap[X]$ see [Ful, Proposition 4.1(a)].
Now suppose $X$ can be a closed subscheme of a scheme $Y$. Let $C_{X} Y=$ $\operatorname{Spec}\left(\sum_{n=0}^{\infty} \mathcal{I}^{n} / \mathcal{I}^{n+1}\right)$ be the normal cone to $X$ in $Y$. The Segre class of $X$ in $Y$, denoted by $s(X, Y)$, is defined to be the Segre class of the normal cone $C_{X} Y$, that is,

$$
s(X, Y)=s\left(C_{X} Y\right) \in A_{*} X
$$

If $X$ is regularly imbedded in $Y$, it follow that the normal cone is a vector bundle on $X$, and consequently, $s(X, Y)$ is the cap product of the total inverse Chern class of the normal bundle with $[X]$.

Proposition 1.5.12 Let $X$ be a proper closed subscheme of a variety $Y$. Consider $\widetilde{Y}$ the blow-up of $Y$ along $X, \widetilde{X}=P(C)$ the exceptional divisor, $\eta: \widetilde{X} \longrightarrow X$ the projection. Then

$$
s(X, Y)=\sum_{i \geq 0} \eta_{*}\left(c_{1}(\mathcal{O}(1))^{i} \cap[P(C)]\right) .
$$

Consider $\mathcal{F}$ a coherent sheaf on a sheme $X$ and consider $P(\mathcal{F})=\operatorname{Proj}(\operatorname{Sym}(\mathcal{F}))$, with projection $p: P(\mathcal{F}) \longrightarrow X$. Denote by $\mathcal{O}_{\mathcal{F}}(1)$ the canonical invertible sheaf which is the universal quotient of $p^{*}(\mathcal{F})$. If the support of $\mathcal{F}$ is $X$, one defines its Segre class $s(\mathcal{F})$ in $A_{*}(X)$ by the formula

$$
\begin{aligned}
s(\mathcal{F}) & =p_{*}\left(\sum_{i \geq 0} c_{1}\left(\mathcal{O}_{\mathcal{F}}(1)\right)^{r} \cap[P(\mathcal{F})]\right) \\
& =p_{*}\left(c\left(\mathcal{O}_{\mathcal{F}}(1)\right)^{-1} \cap[P(\mathcal{F})]\right) .
\end{aligned}
$$

### 1.6 Milnor number

Consider $f:\left(\mathbb{C}^{n+1}, 0\right) \longrightarrow(\mathbb{C}, 0)$ a germ of holomorphic function. Set $Z$ to the complex hypersurface given by the zero set of $f$. One says that $Z$ is singular at 0 if the differential of $f$ vanishes at 0 , that is, the vector field ( $\partial f / \partial z_{0}, \cdots, \partial f / \partial z_{n}$ ) vanishes at 0 . One says that 0 is an isolated singularity of $Z$ if there is an open neibourhood $U$ of 0 such that $U \backslash\{0\}$ is non-singular, that is, 0 is the only point in $U$ that the vector field ( $\partial f / \partial z_{0}, \cdots, \partial f / \partial z_{n}$ ) vanishes. It is known that, 0 is an isolated singularity of $Z$ if and only if the quociente algebra $\mathcal{O}_{n+1} /\left(\partial f / \partial z_{0}, \ldots, \partial f / \partial z_{n}\right)$ is a $\mathbb{C}$-vector space of finite complex dimesion, see [L, Proposition 1.2]. Moreover, if 0 is an isolated singularity of $Z$, the index of the vector field $\left(\partial f / \partial z_{0}, \ldots, \partial f / \partial z_{n}\right)$ coincides with the
number $\operatorname{dim}_{\mathbb{C}} \mathcal{O}_{n+1} /\left(\partial f / \partial z_{0}, \ldots, \partial f / \partial z_{n}\right)$. The standard example is Pham-Brieskorn polynomial $f\left(z_{0}, \ldots, z_{n}\right)=z_{0}^{a_{0}}+\cdots+z_{n}^{a_{n}}$, with $a_{n}>1$. Note that $Z=f^{-1}(0)$ is a complex hypersurface with an isolated singularity at 0 .

Suppose that 0 is an isolated singularity of $Z$. For a $\varepsilon$ small enough, the sphere $S_{\varepsilon}=\left\{z \in \mathbb{C}^{n+1}| | z \mid=\varepsilon\right\}$ intersects transversally $Z$, see [Milnor, Corollary 2.9]. The smooth manifold $K:=S_{\varepsilon} \cap Z$ is called the link of the singularity of $Z$ at 0 , and its diffeomorphism type does not depend on $\varepsilon$.

Let us see below the classic Milnor's fibration theorem.
Theorem 1.6.1 [Milnor, Theorem 4.8] There is $\varepsilon_{1}>0$ such that, for all $\varepsilon$ with $\varepsilon_{1}>$ $\varepsilon>0$, the map

$$
\varphi_{\varepsilon}=\frac{f}{|f|}: S_{\varepsilon} \backslash K \longrightarrow \mathbb{S}^{1}
$$

is a locally trivial smooth fibration.
For $\varepsilon_{1}>\varepsilon>0$, all the fibrations $\varphi_{\varepsilon}$ are diffeomorphic. Now let $B_{\varepsilon}(0)$ be the open ball of $\mathbb{C}^{n+1}$ centered at 0 with radius $\varepsilon$ and let $\partial D_{\eta}$ be the boundary of the closed disc $D_{\eta}$ of $\mathbb{C}$ centered at 0 with radius $\eta$. Using the Ehresmann's fibration lemma on manifolds with boundary or the Thom's first isotopy lemma, one has the following alternative shape:

Theorem 1.6.2 (Milnor-Lê fibration theorem) There is $\varepsilon_{0}>0$ such that, for all $\varepsilon$ with $0<\varepsilon \leq \varepsilon_{0}$, there is $\eta>0$ such that, for all $\eta$ with $0<\eta \leq \eta_{\varepsilon}$. The map $f$ induces a locally trivial smooth fibration $\psi_{\epsilon, \eta}: B_{\varepsilon}(0) \cap f^{-1}\left(\partial D_{\eta}\right) \longrightarrow \partial D_{\eta}$.

For small enough $\epsilon>0$ and $\eta>0, \varphi_{\epsilon}$ and $\psi_{\epsilon, \eta}$ are diffeomorphic. Thus, $\varphi_{\epsilon}$ is called Milnor fibration of $f$ at 0 . If $f$ has an isolated singularity at 0 , then each fiber of the Milnor fibration has the homotopy type of a bouquet $\mathbb{S}^{n} \vee \ldots \vee \mathbb{S}^{n}$ of $n$-spheres, see [Milnor, Theorem 6.5]. The number of spheres in this bouquet is called the Milnor number of $f$ and is denoted by $\mu(f)$. Moreover, the Milnor number $\mu(f)$ coincides with the number $\operatorname{dim}_{\mathbb{C}} \mathcal{O}_{n+1} /\left(\partial f / \partial z_{0}, \ldots, \partial f / \partial z_{n}\right)$. In this way, assuming that $f$ has an isolated singularity at 0 , the Milnor number may be algebraically calculated.

There are some relevant generalizations to Milnor number. In 1971, H. Hamm extended the Milnor's fibration theorem for ICIS. The Lê number was introduced by David Massey, such numbers extend the notion of Milnor number to a setup of singularities not necessarily isolated. In 1988, Adam Parusiński presented a global definition for Milnor number, we will see in more detail in the next chapter.

### 1.7 Borel-Moore homology

The following is a brief explanation of the homology with locally finite supports, also known as Borel-Moore homology. We use as references [Ful] and [Ful2].

Let $X$ be a topological space imbedded as a closed subspace of $\mathbb{R}^{n}$. One may define the Borel-Moore homology groups $H_{i} X$ by

$$
H_{i} X \cong H^{n-i}\left(\mathbb{R}^{n}, \mathbb{R}^{n} \backslash X\right)
$$

where the group on the right is the relative singular cohomology with integer coefficients. More generally, if $X$ is a closed subspace of a topological space $Y$, there are cap products $H^{j}\left(Y^{n}, Y^{n} \backslash X\right) \otimes H_{k} Y \xrightarrow{\cap} H_{k-j} X$. Now assuming that $Y$ is an oriented connected real $n$-manifold, one has that $H_{n} Y$ is freely generated by a fundamental class $\mu_{Y}$, and capping $\mu_{Y}$ with determines an isomorphism $H^{n-i}\left(Y^{n}, Y^{n} \backslash X\right) \xrightarrow{n \mu_{Y}} H_{i} X$. In particular, when $X=Y$ one has the Poincaré duality, that is, the isomorphism $H^{n-i} Y \cong H_{i} Y$.

For any $n$-dimensional complex scheme $X$, we have $H_{i} X=0$ for all $i>2 n$, and $H_{2 n} X$ is a free abelian group with one generator for each irreducible component of $X$. Thus, the generator of $H_{2 n} X$ corresponding to an $n$-dimensional irreducible component $X_{i}$, which we denote by $\operatorname{cl}\left(X_{i}\right)$. More generally, if $V$ is a $k$-dimensional closed subvariety of $X$, define the cycle class by $c l_{X}(V)=i_{*} c l(V) \in H_{2 k} X$, where $i$ is the inclusion of $V$ in $X$. We may consider the homomorphism $c l: Z_{k} X \longrightarrow H_{2 k} X$ from the algebraic $k$ cycles to the Borel-Moore homology, which takes $\sum n_{i}\left[V_{i}\right]$ to $\sum n_{i} c l_{X}\left(V_{i}\right)$. This induces a homomorphism cl : $A_{*} X \longrightarrow H_{*} X$, the so-called cycle map. The cycle map has interesting properties such as being covariant for proper morphisms, and compatible with Chern classes of vector bundles.

### 1.8 Schwartz-MacPherson class

Let $X$ be a $n$-dimensional irreducible analytic (or algebraic) variety embedded in a smooth manifold $M$. Over the smooth part $X_{\text {reg }}$ of $X$, the tangent bundle of $X$ defines a section of the Grassmannian bundle $G_{n}(T M)$. We define the Nash blowup of $X$ as been $\nu: \widetilde{X} \longrightarrow X$, where $\widetilde{X}$ is the closure of the image of this section and $\nu$ is the restriction of the projection on $X$. The restriction of the tautological bundle over
$G_{n}(T M)$ is denoted by $\widetilde{T}$ or $\widetilde{T}_{X}$. One should notice that, $\widetilde{X}, \widetilde{T}$ and $\nu$ are analytically independents of the embedding chosen since, near each point, $X$ has a unique minimal local analytic embedding. We have that $\left.\widetilde{T}\right|_{\nu^{-1}\left(X_{\text {reg }}\right)}$ is isomorphic to $\nu^{*} T\left(X_{\text {reg }}\right)$. We define the Chern-Mather class of $X$ as been the element of the Borel-Moore homology $H_{*}(X)$ given as

$$
c_{M}=\nu_{*}(c(\widetilde{T}) \cap[\widetilde{X}]),
$$

where $[\widetilde{X}]$ is the fundamental class of $\widetilde{X}$. An analytic cycle on a variety $X$ is an element of free abelian group whose basis consists of all irreducible subvarieties of $X$. Given an analytic cycle $\sum n_{i} X_{i}$ on a variety $X$, with $n_{i}$ are integers and $X_{i}$ are irreducible subvarieties of $X$, one defines

$$
c_{M}\left(\sum n_{i} X_{i}\right):=\sum n_{i}\left(i n c l_{i}\right)_{*} c_{M a}\left(X_{i}\right)
$$

where incl $_{i}$ is the inclusion of $X_{i}$ in $X$. Note that, if $X$ is smooth, then $c_{M a}=c\left(T_{X}\right) \cap[X]$ is the total Chern class of $X$, where $T_{X}$ is the tangent bundle of $X$.

Deligne and Grothendieck conjectured on the existence of a natural transformation between the functor $C F$, which assigns to each variety its group of constructible functions and the functor $H_{*}$ of some homology theory such as Borel-Moore homology or rational equivalence theory, such that assigns the characteristic function of a nonsingular variety to the Poincaré dual of the its total Chern class. In [Mac], Robert D. Macpherson responded affirmatively to the Deligne and Grothendieck's conjecture. Macpherson defined, transcendentally, the local Euler obstruction $E u_{X}(x)$ of $X$ at $x \in X$. However, there exists an equivalent algebraic definition to the local Euler obstruction of $X$ at $x$ given by

$$
E u_{X}(x)=\int_{X} c(T X) \cap s\left(\nu^{-1}(x), \widetilde{X}\right),
$$

where $\nu: \widetilde{X} \longrightarrow X$ is the Nash blowup of $X, T X$ is the Nash tangent bundle of $X$ and $s$ denotes the Segre class, see [Ful, Chapter 4]. For this definition was used as motivation the formula of Gonzalez-Sprinberg and Verdier, see [G-S]. Now for each $V$ irreducible subvariety of $X$, the functions $E u_{V}(-)$ on $X$, defined as $E u_{V}(x)$ for all $x \in V$ and zero otherwise, are constructible, see [Mac] and [Ken]. MacPherson using the local Euler obstruction defined an isomorphism $T$ from free abelian group of analytic cycles on $X$
to the additive group of constructible functions on $X$ by

$$
T\left(\sum m_{i} V_{i}\right)=\sum n_{i} E u_{V_{i}}(-) .
$$

We define the MacPherson class of $X$ as been the element in $H_{*}(X)$ given by

$$
c_{*}(X):=c_{M}\left(T^{-1}\left(\mathbf{1}_{X}\right)\right) .
$$

Thus, there exists a natural tranformation $c_{*}: C F \longrightarrow H_{*}$, such that if $X$ is smooth then $c_{*}(X)=c\left(T_{X}\right) \cap[X] \in H_{*}(X)$, where $T_{X}$ is the tangent bundle of $X$. In this way, if $f: X \longrightarrow Y$ is a proper morphism of analytic varieties then the following diagram commutes

this is, $f_{*}\left(c_{*}(\alpha)\right)=c_{*}\left(f_{*}(\alpha)\right)$, for all $\alpha \in C F(X)$. Furthermore, for a compact analytic variety $X$, we have the a generalization of Gauss-Bonnet theorem to the singular setting

$$
\begin{equation*}
\chi(X)=\int_{X} c_{*}(X) . \tag{1.2}
\end{equation*}
$$

More generally, given a constructible function $\alpha$ on $X$, consider the constant function $\rho: X \longrightarrow\left\{p_{0}\right\}$. Then, $\rho_{*} \alpha\left(p_{0}\right)=\chi(X ; \alpha)$, and thus,

$$
\int_{X} c_{*}(\alpha)=\int_{\left\{p_{0}\right\}} \rho_{*} c_{*}(\alpha)=\int_{\left\{p_{0}\right\}} c_{*}\left(\rho_{*} \alpha\right)=\int_{\left\{p_{0}\right\}} c_{*}\left(\chi(X ; \alpha) \mathbf{1}_{p_{0}}\right)=\chi(X ; \alpha) .
$$

In particular, if $\alpha=\mathbf{1}_{X}$ we have the equation (1.2).
In the 1960s, Marie-Hélène Schwartz defined a generalization of Chern classes to singular varieties, a work independently using obstruction theory and radial vector fields. In [B-S], J. P. Brasselet and M.-H. Schwartz showed that if $X$ is a compact complex variety then, by Alexander isomorphism, the MacPherson class of $X$ coincides with the Schwartz class of $X$. Therefore, the class $c_{*}(X)$ is known as the SchwartzMacPherson class of $X$, also denotes by $c^{S M}(X)$.

Let $\mathcal{S}$ be a Whitney stratification of $X$ and let $E$ be a holomorphic vector bundle on $X$. Consider a holomorphic section $s$ of $E$ such that $s$ intersects, on each stratum of $\mathcal{S}$, the zero section of $E$ transversely. Consider $Z=s^{-1}(0)$ and $\iota: Z \longrightarrow X$ the inclusion.

## Capítulo 2

## Logarithmic Milnor number and some generalizations

In this chapter we will use the concept of logarithmic connection to generalize the Milnor number to a logarithmic setup. For this, let us review the approach on connections in the classical sense.

### 2.1 The logarithmic Milnor number

The main references in this section are [G-H] and [P].
Let $E \rightarrow M$ be a holomorphic $k$-vector bundle on a $n$-dimensional complex manifold $M$. A connection $\pi$ on $E$ is a $\mathbb{C}$-linear map

$$
\pi: \Gamma(E) \rightarrow \Omega^{1}(M) \otimes \Gamma(E)
$$

satisfying Leibnitz' rule

$$
\pi(g s)=d(g) \otimes s+g \pi(s)
$$

where $g: M \rightarrow \mathbb{C}$ is a holomorphic function and $s \in \Gamma(E)$. Let $\underline{s}=s_{1}, \ldots, s_{n}: U \rightarrow E$ be a frame, where $U$ is open subset in $M$. Given a connection $\pi$ on $E$, we can decompose $\pi\left(s_{i}\right)$ into its components, writing

$$
\pi\left(s_{i}\right)=\sum_{j=1}^{n} \theta_{i j} s_{j} .
$$

The matrix $\theta=\left(\theta_{i j}\right)$ of 1-forms is called the connection matrix of $\pi$ with repect to $\underline{s}$.

Now, we assume that $M$ is complex and $E$ is hermitian. Since $\Omega^{1}(M)=\Omega^{1,0}(M) \oplus$ $\Omega^{0,1}(M)$, we can write $\pi=\pi^{\prime}+\pi^{\prime \prime}$, where $\pi^{\prime}: \Gamma(E) \rightarrow \Omega^{1,0}(M) \otimes \Gamma(E)$ and $\pi^{\prime \prime}: \Gamma(E) \rightarrow$ $\Omega^{0,1}(M) \otimes \Gamma(E)$. We say that a connection $\pi$ on $E$ is compatible with the complex structure if $\pi^{\prime \prime}=\bar{\partial}$. Moreover, if $E$ is hermitian, the connection $\pi$ is said to be compatible with the metric if

$$
d(\xi, \eta)=(\pi(\xi), \eta)+(\xi, \pi(\eta))
$$

for all $\xi, \eta \in \Gamma(E)$. The existence and uniqueness of a connection that satisfies the above conditions is something that is answered in the result below.

Proposition 2.1.1 Given a hermitian vector bundle $E$ on $M$, there is a unique connection on $E$ compatible with both the metric and the complex structure.

Proof. See [G-H, Page 73] or [Huy, Proposition 4.2.14].
The unique connection compatible with the metric and the complex structures on $E$ is called metric connection or Chern connection. Note that the metric connection depends on the hermitian structure adopted for $E$.

For our goals, we will need a more general type of connection, the so-called logarithmic connection with poles along some divisor. Such a concept was first introduced by Pierre Deligne, in [Deligne].

Definition 2.1.2 Let $E \rightarrow M$ be a holomorphic vector bundle of rank $k$ on a $n$ dimensional complex manifold $M$, and let $D$ be a divisor (hypersurface) in $M$. A connection with logarithmic poles along $D$ or, simply, logarithmic connection on $E$ is $a \mathbb{C}$-linear map

$$
\nabla: \Gamma(E) \rightarrow \Omega_{M}^{1}(\log D) \otimes \Gamma(E)
$$

satisfying

$$
\nabla(g s)=d(g) \otimes s+g \nabla(s)
$$

where $g: M \rightarrow \mathbb{C}$ is a holomorphic function and $s \in \Gamma(E)$.

Consider $\underline{s}=s_{1}, \ldots, s_{k}: U \rightarrow E$ a frame, where $U$ is an open subset of $M$. Given $\omega \in\left(\Omega_{M}^{1}(\log D) \otimes \Gamma(E)\right)(U)$, can be written uniquely as $\sum \omega_{i} \otimes s_{i}$, with $\omega_{i} \in$ $\left.\Omega_{M}^{1}(\log D)\right|_{U}$. Then, given $\nabla$ a logarithmic connection on $E$, we have

$$
\nabla\left(s_{i}\right)=\sum_{j=1}^{k} \omega_{i j} \otimes s_{j}
$$

where $\left(\omega_{i j}\right)$ is a $k \times k$-matrix of elemets in $\left.\Omega_{M}^{1}(\log D)\right|_{U}$, which is called the logarithmic connection matrix with respect to $\underline{s}$. Now consider $\left(\omega_{i j}\right)$ the logarithmic connection matrix with respect to $\underline{s}$. For any $s \in \Gamma(E)$, we have $s=\sum_{i=1}^{k} f_{i} s_{i}$ for some $f_{1}, \ldots, f_{k} \in$ $\Omega^{0}(M)$, and then

$$
\begin{equation*}
\nabla(s)=\sum_{j=1}^{k} \nabla\left(f_{j} s_{j}\right)=\sum_{j=1}^{k}\left(d f_{j}+\sum_{i=1}^{k} f_{i} \omega_{i j}\right) \otimes s_{j} . \tag{2.1}
\end{equation*}
$$

In this way, given $\left(\omega_{i j}\right)$ a $k \times k$-matrix of elemets in $\Omega_{M}^{1}(\log D)$, we can define a logarithmic connection given by the expression of equation (2.1).

For each $r=1, \cdots, n$, we can construct a morphism $\mathbb{C}$-linear

$$
\nabla^{r}: \Omega_{M}^{r}(\log D) \otimes \Gamma(E) \longrightarrow \Omega_{M}^{r+1}(\log D) \otimes \Gamma(E)
$$

as follows. Given $\omega \in \Omega_{M}^{r}(\log D)$ and $s \in \Gamma(E)$, define

$$
\nabla^{r}(\omega \otimes s)=d \omega \otimes s+(-1)^{r} \omega \wedge \nabla(s) .
$$

When $\nabla^{r} \circ \nabla^{r+1}=0$ for all $r$, we say that the connection $\nabla$ is integrable. In this way, se $\nabla$ is integrable, one has the complex of $\mathcal{O}_{M}$-modules

$$
\begin{aligned}
0 \longrightarrow & \Gamma(E) \xrightarrow{\nabla} \Omega_{M}^{1}(\log D) \otimes \Gamma(E) \xrightarrow{\nabla^{1}} \cdots \xrightarrow{\nabla^{r-1}} \Omega_{M}^{r}(\log D) \otimes \Gamma(E) \xrightarrow{\nabla^{r}} \\
& \Omega_{M}^{r+1}(\log D) \otimes \Gamma(E) \xrightarrow{\nabla^{r+1}} \cdots \xrightarrow{\nabla^{n+1}} \Omega_{M}^{n}(\log D) \otimes \Gamma(E) \longrightarrow 0 .
\end{aligned}
$$

Assuming that $(E, h)$ is a hermitian vector bundle, we can to define the following operation, locally given by

$$
\begin{aligned}
h: \Omega^{1}(M) \otimes \mathcal{O}(D) \otimes \Gamma(E) \times \Omega^{1}(M) \otimes \Gamma(E) & \longrightarrow \Omega^{1}(M) \otimes \mathcal{O}(D) \\
\left(\sum_{i} \frac{w_{i}}{f} \otimes s_{i}, \sum_{j} \eta_{j} \otimes t_{j}\right) & \longmapsto \sum_{i, j} \frac{w_{i} \wedge \overline{\eta_{j}}}{f} h\left(s_{i}, t_{j}\right)
\end{aligned}
$$

Similarly, we can to define $h: \Omega^{1}(M) \otimes \Gamma(E) \times \Omega^{1}(M) \otimes \mathcal{O}(D) \otimes \Gamma(E) \rightarrow \Omega^{1}(M) \otimes$ $\mathcal{O}(D)$. We say that a logarithmic connection $\nabla$ on a hermitian vector bundle $(E, h)$ is compatible with the metric $h$ with respect to divisor $D$ if, locally, we have

$$
\frac{d h\left(s_{i}, s_{j}\right)}{f}=h\left(\nabla\left(s_{i}\right), s_{j}\right)+h\left(s_{i}, \nabla\left(s_{j}\right)\right),
$$

where $f$ is a local defining function of $D$ and $s_{1}, \ldots, s_{n}$ is a local frame for $E$. Motivated by the Proposition 2.1.1, it is natural to inquire the existence and uniqueness of a
logarithmic connection that satisfies such definition. Suppose there is such connection, we say $\nabla$. Consider $\underline{s}=s_{1}, \ldots, s_{n}: U \rightarrow E$ a local frame, with $U$ an open subset of $M$. Let $h_{i j}:=h\left(s_{i}, s_{j}\right)$ and let $H=\left(h_{i j}\right)$. Thus,

$$
\begin{aligned}
\frac{d h_{i j}}{f} & =h\left(\nabla\left(s_{i}\right), s_{j}\right)+h\left(s_{i}, \nabla\left(s_{j}\right)\right) \\
& =\sum_{k} \omega_{i k} h_{k j}+\sum_{k} \overline{\omega_{k j}} h_{i k}
\end{aligned}
$$

and then

$$
\left\{\begin{array}{c}
\frac{\partial H}{f}=\omega H \\
\frac{\bar{\partial} H}{f}=H \bar{\omega}^{t}
\end{array}\right.
$$

where $\omega=\left(\omega_{i j}\right)$. Therefore, the unique solution of the system above is $w=\frac{\partial H H^{-1}}{f}$. Note that, the form $\omega$ not necessarily has logarithmic poles along of $D$, this is, the existence of a logarithmic connection compatible with the metric is not always guaranteed. In some cases, we can do it, for exemplo, if the manifold $M$ is a Riemann surface, see [Saito, pg 267]. However, if there is a logarithmic connection on a hermitian vector bundle ( $E, h$ ) compatible with the metric $h$ with respect to divisor $D$, then such connection is unique.

We will consider $M$ a $n$-dimensional connected complex manifold and $(L, h)$ a hermitian line bundle on $M$. Let $X:=v^{-1}(0)$ be a (nowhere dense) divisor in $M$, where $v$ is a holomorphic section of $L$. Let $Y$ be a compact connected component of Sing $X$ and $U$ a small neighbourhood of $Y$. In [P], A. Parusiński defined the $\mu$-number of $X$ at $Y$ as been $\mu(X, Y):=\operatorname{ind}_{U} \pi^{\prime} v$, where $\pi=\pi^{\prime}+\pi^{\prime \prime}$ is the decomposition of metric connection of $L$. The original setup of Milnor number is with an isolated points. This covers several cases and there are many works in this setup. The definition of a $\mu$-number due Parusiński is a remarkable generalization of the Milnor number in the following sense: if $x_{0}$ is an isolated singularity of $X$, then $\mu\left(X,\left\{x_{0}\right\}\right)$ is equals the Milnor number of $X$ at $x_{0}$, see [ P , Proposition 1.4].

Theorem 2.1.3 [P, Proposition 1.6] Suppose $M$ be compact. Then,

$$
\mu(X)=(-1)^{n} \chi(X)+\int c_{n}\left(T^{*^{\prime}} M \otimes L\right) \cap[M]-(-1)^{n} \chi(M)
$$

With a computation involving two divisors, one has the following consequence:

Corollary 2.1.4 [P, Corollary 1.7] With the notation above, we have

$$
\mu(X)-\mu(Z)=(-1)^{n}(\chi(X)-\chi(Z))
$$

Now let $D$ be another divisor in $M$. Assume that $X$ and $D$ are disjoint. Suppose there exists a logarithmic connection $\nabla: \Gamma(L) \rightarrow \Omega_{M}^{1}(\log D) \otimes \Gamma(L)$ compatible with the metric $h$ with respect to divisor $D$. Note that, the logarithmic connection forms of $\nabla$ with respect to any holomorphic frames are holomorphic outside of $D$ and then, by analog arguments to $[\mathrm{P}]$, we have $\operatorname{Sing} X=\{x \in X: \nabla v(x)=0\}$ and $\operatorname{Sing} X$ is closed and open in $\{x \in M: \nabla v(x)=0\}$.

Definition 2.1.5 We define the $\mu$-number of $X$ at $Y$ with respect to $D$ as been the intersection index $\operatorname{ind}_{U} \nabla v$. We denote by $\mu_{D}(X, Y)$. If $X$ is compact, then intersection index of the zero section of $\Omega_{M}^{1}(\log D) \otimes L$ and $\nabla v$ will be called $\mu$-number of $X$ with respect to $D$, and we will denote by $\mu_{D}(X)$.

Note that, the intersection index over $U$ of $\nabla v$ is the number of the zero, in $U$, of a small perturbation of $\nabla v$ transversal to the zero section in $U$, counted with signs. Moreover, we have

$$
\mu_{D}(X)=\sum_{i=1}^{r} \mu_{D}\left(X, Y_{i}\right)
$$

where $Y_{1}, \ldots, Y_{r}$ are the connected components of Sing $X$.
We use as motivation the Theorem 2.1.3 due to Parusiński to obtain the following result:

Proposition 2.1.6 Assume $M$ compact. Then, $\mu_{D}(X)=\int_{U} c_{n}\left(\left.\Omega_{M}^{1}(\log D)\right|_{U}\right) \cap[U]+(-1)^{n-1} \int_{M} c(L)^{-1} c_{1}(L) c\left(\operatorname{Der}_{M}(-\log D)\right) \cap[M]$, where $U=\{x \in M:|v(x)| \leq \epsilon\}$, for a fixed $0<\epsilon \ll 1$.

Proof. Consider $U=\{x \in M:|v(x)| \leq \epsilon\}$ for a $\epsilon$ sufficiently small such that $U \cap D=\emptyset$. Then,

$$
\begin{aligned}
\mu_{D}(X)= & \operatorname{ind}_{M} \nabla v-\operatorname{ind}_{M \backslash U} \nabla v \\
= & \operatorname{ind}_{M} \nabla v-i n d_{M \backslash U} h(\nabla v, v) \\
= & \operatorname{ind}_{M} \nabla^{\prime} v-(-1)^{n} i n d_{M \backslash U} d|v|^{2} \\
= & \int_{M} c_{n}\left(\Omega_{M}^{1}(\log D) \otimes L\right) \cap[M]- \\
& -\left(\int_{M} c_{n}\left(\Omega_{M}^{1}(\log D)\right) \cap[M]-\int_{U} c_{n}\left(\left.\Omega_{M}^{1}(\log D)\right|_{U}\right) \cap[U]\right) .
\end{aligned}
$$

The last equality follows by Theorem 1.1.3. Lastly, note that

$$
\begin{aligned}
& \int_{M} c(L)^{-1} c_{1}(L) c\left(\operatorname{Der}_{M}(-\log D)\right) \cap[M] \\
= & \int_{M}\left(\sum_{k \geq 0}(-1)^{k} c_{1}(L)^{k+1}\right)\left(\sum_{j \geq 0} c_{j}\left(\operatorname{Der}_{M}(-\log D)\right)\right) \cap[M] \\
= & \int_{M} \sum_{k, j \geq 0}(-1)^{k} c_{1}(L)^{k+1} c_{j}\left(\operatorname{Der}_{M}(-\log D)\right) \cap[M] \\
= & \int_{M} \sum_{k=0}^{n-1}(-1)^{k} c_{1}(L)^{k+1} c_{n-1-k}\left(\operatorname{Der}_{M}(-\log D)\right) \cap[M] \\
= & (-1)^{n-1} \int_{M} \sum_{k=0}^{n-1} c_{1}(L)^{k+1} c_{n-1-k}\left(\Omega_{M}^{1}(\log D)\right) \cap[M]
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_{M} c_{n}\left(\Omega_{M}^{1}(\log D) \otimes L\right) \cap[M] \\
= & \int_{M} \sum_{i=0}^{n} c_{1}(L)^{i} c_{n-i}\left(\Omega_{M}^{1}(\log D)\right) \cap[M] \\
= & \int_{M} c_{n}\left(\Omega_{M}^{1}(\log D)\right) \cap[M]+\int_{M} \sum_{i=1}^{n} c_{1}(L)^{i} c_{n-i}\left(\Omega_{M}^{1}(\log D)\right) \cap[M] \\
= & \int_{M} c_{n}\left(\Omega_{M}^{1}(\log D)\right) \cap[M]+\int_{M} \sum_{k=0}^{n-1} c_{1}(L)^{k+1} c_{n-k-1}\left(\Omega_{M}^{1}(\log D)\right) \cap[M] .
\end{aligned}
$$

The case in which $X$ and $D$ are disjoint is not very interesting. Then, we would like to define a number involving two divisors in $M$ not necessarily disjoints. Therefore, consider $M$ a $n$-dimensional connected complex variety. Let $X:=v^{-1}(0)$ be a divisor in $M$, where $v$ is a holomorphic section of a hermitian line bundle $L$ on $M$, and let $D$ be another divisor in $M$. The number

$$
\begin{aligned}
& \mu_{D}(X)=(-1)^{n}\left(\int_{U} c_{n}\left(\left.\operatorname{Der}_{M}(-\log D)\right|_{U}\right) \cap[U]-\right. \\
&\left.-\int_{M} c(L)^{-1} c_{1}(L) c\left(\operatorname{Der}_{M}(-\log D)\right) \cap[M]\right)
\end{aligned}
$$

is called logarithmic Milnor number of $X$ with poles along $D$ or, simply, logarithmic Milnor number of $X$, with respect to a $0<\epsilon \ll 1$.

In [Alu2], Paolo Aluffi established the follow conjeture: for $D$ a locally quasihomogeneous free divisor in a $n$-dimensional nonsingular variety $S$, is it true that
$c_{*}\left(\mathbf{1}_{S \backslash D}\right)=c_{n}\left(\operatorname{Der}_{S}(-\log D)\right) \cap[S]$ ? This question was answered in the following settings:
$\left(\mathrm{D}_{1}\right) V$ is a nonsingular complex surface and the Tjurina number equals the Milnor number for all singularities of $D$, or then, $D$ is a locally quasi-homogeneous divisors, see [Liao1].
$\left(\mathrm{D}_{2}\right) V$ is a nonsingular projective complex variety and $D$ is a locally quasi-homogeneous free divisor, see [Liao2].
$\left(\mathrm{D}_{3}\right) V$ is a nonsingular compact complex variety and $D$ is a certain class of divisors that Aluffi called "free hypersurface arrangement", see [Alu1].
$\left(\mathrm{D}_{4}\right) V$ is a nonsingular (complex) variety defined over an algebraically closed field $k$ of characteristic 0 and $D$ is a free divisor with Jacobian ideal of linear type, see [Liao3].

This motivates the following definition:
Definition 2.1.7 Let $M$ be a $n$-dimensional connected complex manifold and let $D$ be a divisor in $M$. We say that $D$ is a chern logarithmic divisor in $M$ if

$$
c_{*}\left(\mathbf{1}_{M \backslash D}\right)=c\left(\operatorname{Der}_{M}(-\log D)\right) \cap[M] .
$$

Proposition 2.1.8 Consider $M$ and $U$ as in the Proposition 2.1.6. Suppose that $D$ is a chern logarithmic divisor in $M$, such that $c_{*}\left(\mathbf{1}_{U \backslash U \cap D}\right)=c\left(\left.\operatorname{Der}_{M}(-\log D)\right|_{U}\right) \cap[U]$. Then

$$
\mu_{D}(X)=(-1)^{n}\left(\chi\left(X ; \mathbf{1}_{X \backslash D}\right)-\int_{M} c(L)^{-1} c_{1}(L) c\left(\operatorname{Der}_{M}(-\log D)\right) \cap[M]\right) .
$$

Proof. Thus,

$$
\begin{aligned}
\mu_{D}(X) & =(-1)^{n}\left(\int_{U} c_{n}\left(\left.\operatorname{Der}_{M}(-\log D)\right|_{U}\right) \cap[U]-\int_{M} c(L)^{-1} c_{1}(L) c\left(\operatorname{Der}_{M}(-\log D)\right) \cap[M]\right) \\
& =(-1)^{n}\left(\int_{U} c_{*}\left(\mathbf{1}_{U \backslash U \cap D}\right)-\int_{M} c(L)^{-1} c_{1}(L) c\left(\operatorname{Der}_{M}(-\log D)\right) \cap[M]\right) \\
& =(-1)^{n}\left(\int_{U} c_{*}(U)-\int_{U} c_{*}\left(\mathbf{1}_{U \cap D}\right)-\int_{M} c(L)^{-1} c_{1}(L) c\left(\operatorname{Der}_{M}(-\log D)\right) \cap[M]\right) \\
& =(-1)^{n}\left(\chi(U)-\chi\left(U ; \mathbf{1}_{U \cap D}\right)-\int_{M} c(L)^{-1} c_{1}(L) c\left(\operatorname{Der}_{M}(-\log D)\right) \cap[M]\right) \\
& =(-1)^{n}\left(\chi\left(X ; \mathbf{1}_{X \backslash D}\right)-\int_{M} c(L)^{-1} c_{1}(L) c\left(\operatorname{Der}_{M}(-\log D)\right) \cap[M]\right) .
\end{aligned}
$$

Definition 2.1.9 We define the logarithmic Milnor number of $X$ with poles along $D$ or, simply, logarithmic Milnor number of $X$ as been

$$
\mu_{D}(X)=(-1)^{n}\left(\chi\left(X ; \boldsymbol{1}_{X \backslash D}\right)-\int_{M} c(L)^{-1} c_{1}(L) c\left(\operatorname{Der}_{M}(-\log D)\right) \cap[M]\right) .
$$

Example 2.1.10 Consider the crossing divisor $D=\{x y z=0\}=\{x=0\} \cup\{y=$ $0\} \cup\{z=0\}$ and the hypersurface $X=\left\{y^{2} z-x^{3}=0\right\}$ in $\mathbb{P}^{2}$. We have that $X \cap D=$ $\{p=(0: 0: 1), q=(0: 1: 0)\}$ and $\operatorname{Sing} X=\{p\}$. Then, $\chi\left(X ; \boldsymbol{1}_{X \backslash D}\right)=\chi(X \backslash\{p, q\})=$ $\chi(X)-\chi(\{p\})-\chi(\{q\})=0$, because $\chi(X)=2$. It is known that

$$
c\left(\Omega_{\mathbb{P}^{2}}^{1}(\log D)\right)=c\left(\Omega_{\mathbb{P}^{2}}^{1}\right) \prod_{i=1}^{3} c\left(\mathcal{O}_{D_{i}}\right)
$$

where $D_{1}=\{x=0\}, D_{2}=\{y=0\}$ and $D_{3}=\{z=0\}$, see [Alu5, pg 12], [Sil, 3.1], [D-A, Proposition 2.3]. Set $h=c_{1}\left(\mathcal{O}_{\mathbb{P}^{2}}(1)\right)$ the class of a hyperplane in $\mathbb{P}^{2}$. The line bundle of $X$ is $L=\mathcal{O}(3 h)$. Hence,

$$
c\left(\Omega_{\mathbb{P}^{2}}^{1}(\log D)\right)=c\left(\Omega_{\mathbb{P}^{2}}^{1}\right) \prod_{i=1}^{3} c\left(\mathcal{O}_{D_{i}}\right)=\left(1-3 h+3 h^{2}\right)(1+h)^{3}=1-3 h^{2}
$$

and then $c\left(\operatorname{Der}_{M}(-\log D)\right)=1-3 h^{2}$. Note that $c(L)=1+3 h$, and then, $c(L)^{-1}=$ $1-3 h+9 h^{2}$. Therefore,

$$
\begin{aligned}
\mu_{D}(X) & =\chi\left(X ; \boldsymbol{1}_{X \backslash D}\right)-\int_{\mathbb{P}^{2}} c(L)^{-1} c_{1}(L) c\left(\operatorname{Der}_{\mathbb{P}^{2}}(-\log D)\right) \cap\left[\mathbb{P}^{2}\right] \\
& =-\int_{\mathbb{P}^{2}}\left(\left(1-3 h+9 h^{2}\right) 3 h\left(1-3 h^{2}\right)\right) \cap\left[\mathbb{P}^{2}\right]=9 .
\end{aligned}
$$

In [P], Parusiński presented a remarkable definition that generalizes the initial notion of Milnor number to a setup not necessarily involving isolated singularities as follows. Let $M$ be a compact $n$-dimensional complex manifold and let $L$ be a holomorphic line bundle over $M$. Consider $X:=v^{-1}(0)$ a divisor in $M$, where $v$ is a regular holomorphic section of $L$. Parusiński defined a generalization of the Milnor number given by

$$
\mu(X)=(-1)^{n-1}(\chi(M \mid L)-\chi(X)),
$$

where for a vector bundle $E$ over $M$,

$$
\chi(M \mid E):=\int_{M} c(E)^{-1} c_{\text {top }}(E) c(M) \cap[M] .
$$

Define $\chi: X \longrightarrow \mathbb{Z}$ given by $\chi(x):=\chi\left(F_{x}\right)$ for all $x \in X$, where $F_{x}$ denotes de Milnor fibre at $x$ and $\chi\left(F_{x}\right)$ denotes the Euler characteristic of $F_{x}$. Now define the function $\mu: X \longrightarrow \mathbb{Z}$ by $\mu=(-1)^{n-1}\left(\chi-\mathbf{1}_{X}\right)$.

Let us fix a stratification $\mathcal{S}=\{S\}$ of $X$ such that $\mu$ is constant on the strata of $\mathcal{S}$. It is known that the topological type of the Milnor fibres is constant along the strata of a Whitney stratification of $X$. Therefore, a Whitney stratification of $X$ satisfies the condition desired above, see [P3], [B-M-M] and [PP]. Thus, for each $S \in \mathcal{S}$ we denote by $\mu_{S}$ the value of $x \longmapsto \mu(X, x)$ on $S$. Set

$$
\gamma(S)=\mu_{S}-\sum_{S^{\prime} \neq S, \overline{S^{\prime}} \supset S} \gamma\left(S^{\prime}\right)
$$

the numbers defined inductively on descending dimension of $S$.
Theorem 2.1.11 Let $M$ be a nonsingular suvariety of $\mathbb{P}^{N}$ and let $L$ be a holomorphic line bundle on $X$. Consider $X$ as being the zero set of a holomorphic section of $L$. Given $\mathcal{S}$ a Whitney stratification of $X$, we have

$$
\mu(X)=\sum_{s \in \mathcal{Z}} \gamma(S) \int_{\bar{S}}\left(c\left(\left.L\right|_{\bar{S}}\right)^{-1} \cap c_{*}(\bar{S})\right)
$$

where $\gamma(S)=\mu_{S}-\sum_{S^{\prime} \neq S, \overline{S^{\prime}} \supset S} \gamma\left(S^{\prime}\right)$.
Proof. See [PP, Theorem 4].
Suppose that $\operatorname{Sing}(X)=\left\{x_{0}\right\}$. Then, we can take the Whitney stratification $\left\{S_{0}:=X \backslash\left\{x_{0}\right\}, S_{1}:=\left\{x_{0}\right\}\right\}$ of $X$. Note that, $\gamma\left(S_{0}\right)=\mu_{S_{0}}=\mu(X, x)=0$, for all $x \in X \backslash\left\{x_{0}\right\}$ and $\gamma\left(S_{1}\right)=\mu_{S_{1}}-\gamma\left(S_{0}\right)=\mu_{S_{1}}=\mu\left(X, x_{0}\right)$. Then, using the Theorem 2.1.11, it follow that

$$
\begin{aligned}
\mu(X) & =\gamma\left(S_{1}\right) \int_{\overline{S_{1}}} c\left(\left.L\right|_{\overline{S_{1}}}\right)^{-1} \cap c_{*}\left(\overline{S_{1}}\right) \\
& =\mu\left(X, x_{0}\right) \int_{\left\{x_{0}\right\}}\left[x_{0}\right]=\mu\left(X, x_{0}\right)
\end{aligned}
$$

because $c\left(\left.L\right|_{\overline{S_{1}}}\right)=1$ and $c_{*}\left(\overline{S_{1}}\right)=c_{*}\left(x_{0}\right)=\left[x_{0}\right]$. Moreover, one can generalize this fact as follows.

Example 2.1.12 [PP] Assume that the $\operatorname{Sing}(X)=\left\{x_{1}, \cdots, x_{r}\right\}$. We have that

$$
\mu(X)=(-1)^{n} \sum_{i=1}^{r} \mu\left(X, x_{i}\right)
$$

In [SS, Theorem 2.4], Seade and Suwa presented a generalization of this fact to "strong"local complete intersections.

In this way, we can show the following relation between the Milnor number due to Parusiński and the logarithmic Milnor number defined above:

Proposition 2.1.13 Let $D$ a divisor in $M$. If $D$ is a chern logarithmic divisor in $M$, then

$$
\mu_{D}(X)=\mu(X)-(-1)^{n} \chi\left(X ; \boldsymbol{1}_{X \cap D}\right)+(-1)^{n} \int_{M} c(L)^{-1} c_{1}(L) c_{*}(D)
$$

Proof. Indeed, we have

$$
\begin{aligned}
\mu_{D}(X) & =(-1)^{n}\left(\chi\left(X ; \mathbf{1}_{X \backslash X \cap D}\right)-\int_{M} c(L)^{-1} c_{1}(L) c\left(\operatorname{Der}_{M}(-\log D)\right) \cap[M]\right) \\
& =(-1)^{n}\left(\int_{X} c_{*}\left(\mathbf{1}_{X \backslash X \cap D}\right)-\int_{M} c(L)^{-1} c_{1}(L) c_{*}\left(\mathbf{1}_{M \backslash D}\right)\right) \\
& =(-1)^{n}\left(\int_{X} c_{*}\left(\mathbf{1}_{X}\right)-\int_{X} c_{*}\left(\mathbf{1}_{X \cap D}\right)-\int_{M} c(L)^{-1} c_{1}(L) c_{*}\left(\mathbf{1}_{M \backslash D}\right)\right) \\
& =(-1)^{n}\left(\chi(X)-\int_{M} c(L)^{-1} c_{1}(L) c_{*}(M)-\chi\left(X ; \mathbf{1}_{X \cap D}\right)+\int_{M} c(L)^{-1} c_{1}(L) c_{*}(D)\right) \\
& =\mu(X)-(-1)^{n} \chi\left(X ; \mathbf{1}_{X \cap D}\right)+(-1)^{n} \int_{M} c(L)^{-1} c_{1}(L) c_{*}(D) .
\end{aligned}
$$

Now let $E$ be a holomorphic vector bundle over $M$ of $\operatorname{rank} d$. Set $X:=\{p \in M$ : $s(p)=0\}$, where $s$ is a regular holomorphic section of $E$, that is, at any point $p \in X$, the germs of the components of $s$ with repect to a holomorphic frame near $p$ form a regular sequence in the $\mathcal{O}_{M, p}$ of germs holomorphic functions at $p$. Then, $X$ is a closed subvariety of $M$ of pure dimension $n-d$, see [Ful, B.3].

Definition 2.1.14 Given $\alpha$ a constructible function on $M$, we define the Milnor number relative to $\alpha$ as being

$$
\mu(X ; \alpha)=(-1)^{\operatorname{dim} X}\left(\int_{M} c(E)^{-1} c_{\text {top }}(E) c_{*}(\alpha)-\chi\left(X ;\left.\alpha\right|_{X}\right)\right) .
$$

First fact that we highlight below is how we recover our definition of the logarithmic Milnor number.

Example 2.1.15 Let $D$ be a chern logarithmic divisor in $M$. When the rank of the vector bundle $E$ is 1 , this is $\operatorname{dim}(X)=n-1$, and $\alpha=\boldsymbol{1}_{M \backslash D}$, we have that

$$
\begin{aligned}
\mu\left(X ; \mathbf{1}_{M \backslash D}\right) & =(-1)^{\operatorname{dim} X}\left(\int_{M} c(E)^{-1} c_{t o p}(E) c_{*}\left(\mathbf{1}_{M \backslash D}\right)-\chi\left(X ;\left.\left(\mathbf{1}_{M \backslash D}\right)\right|_{X}\right)\right) \\
& =(-1)^{n-1}\left(\int_{M} c(E)^{-1} c_{\text {top }}(E) c\left(\operatorname{Der}_{M}(-\log D)\right) \cap[M]-\chi\left(X ; \mathbf{1}_{X \backslash D}\right)\right) \\
& =\mu_{D}(X) .
\end{aligned}
$$

### 2.2 On the generalization of Schürmann for Milnor classes

Let $X$ be a scheme which can be imbedded as a closed subscheme of a nonsingular variety $M$. One define the Fulton class of $X$ as being the class

$$
c^{F}(X)=c\left(\left.T M\right|_{X}\right) \cap s(X, M)
$$

in $A_{*}(X)$. Fulton showed that $c^{F J}(X)$ does not depend on the choice of imbedding, see [Ful, Example 4.2.6(a)]. Moreover, the Fulton-Johnson class of $X$ is defined by

$$
c^{F J}(X)=c\left(\left.T M\right|_{X}\right) \cap s(\mathcal{N})
$$

where $s(\mathcal{N})$ is the total Chern class of the conormal sheaf of the embedding of $X$ in $M$, see [FJ]. Assuming that $X$ is a local complete intersection, the Fulton and Fulton-Johnson classes coincide and are equal to

$$
c\left(\left.T M\right|_{X}\right) c\left(N_{X} M\right)^{-1} \cap[X]=c\left(T_{X}\right) \cap[X]
$$

where $T_{X}=\left.T M\right|_{X}-N_{X} M$ denotes the virtual tangent bundle on $X$, which is a welldefined element of the Grothendieck group of vector bundles on $X$, see [Ful, Example 3.2.7].

Let $M$ be a nonsingular compact complex analytic variety of pure dimension $n$ and let $L$ be a holomorphic line bundle over $M$. Consider $X:=v^{-1}(0)$ a divisor in $M$, where $v$ is a regular holomorphic section of $L$. In this way, we have

$$
c^{F}(X)=c\left(\left.T M\right|_{X}-\left.L\right|_{X}\right) \cap[X] .
$$

Definition 2.2.1 [PP3], [BLSS], [Y], [A4] The Milnor class of $X$ is defined as being

$$
\mathcal{M}(X)=(-1)^{n-1}\left(c^{F J}(X)-c_{*}(X)\right)
$$

Due to Parusiński and Pragacz, there is the following formula to Milnor number $\mathcal{M}(X)$ in terms of a Whitney stratification of $X$.

Theorem 2.2.2 Take $\mathcal{S}=\{S\}$ a Whitney stratification of $X$. Then

$$
\mathcal{M}(X)=\sum_{s \in \mathcal{S}} \gamma(S) c\left(\left.L\right|_{X}\right)^{-1} \cap\left(\iota_{\bar{S}, X}\right)_{*} c_{*}(\bar{S})
$$

where $\gamma(S)=\mu_{S}-\sum_{S^{\prime} \neq S, \overline{S^{\prime}} \supset S} \gamma\left(S^{\prime}\right)$ and the aplication $\iota_{\bar{S}, X}$ denotes the inclusion from $\bar{S}$ to $X$.

Proof. See [PP3, Theorem 0.2].
In particular, one has

$$
\int_{X} \mathcal{M}(X)=\sum_{s \in \mathcal{S}} \gamma(S) \int_{\bar{S}} c\left(\left.L\right|_{\bar{S}}\right)^{-1} \cap c_{*}(\bar{S})
$$

Note that Parusiński and Pragacz had already proven the above formula to $M$ projective and $X$ not necessarily compact, see Theorem 2.1.11.

Example 2.2.3 [PP3, Example 0.1] Assume that $\operatorname{Sing}(X)=\left\{x_{1}, \cdots, x_{r}\right\}$. Then, one has

$$
\mathcal{M}(X)=\sum_{i=1}^{r} \mu\left(X, x_{i}\right)\left[x_{i}\right] \in H_{0}(X) .
$$

There is a generalization of this result for "strong"local complete intersections, for more details see [Suwa].

Let $\iota: X \hookrightarrow Z$ be a regular embedding between algebraic (or complex) possibly singular varieties and let $N_{X} Z$ be a normal bundle of $X$ in $Z$. The diagram

does not commute in general, where $\iota^{*}: H_{*}(Z) \longrightarrow H_{*}(X)$ is the Gysin homomorphism. Indeed, suppose the particular case in which $Z=M$ a smooth variety. On the one hand, $c_{*}\left(\iota^{*}\left(\mathbf{1}_{M}\right)\right)=c_{*}\left(\mathbf{1}_{X}\right)$ which is equals to Schwartz-MacPherson class of $X$. On the other hand,

$$
\begin{aligned}
c\left(N_{X} M\right)^{-1} \cap \iota^{*}\left(c_{*}\left(\mathbf{1}_{M}\right)\right) & =c\left(N_{X} M\right)^{-1} \cap \iota^{*}(c(T M) \cap[M]) \\
& =c\left(N_{X} M\right)^{-1} \cdot c\left(\iota^{*} T M\right) \cap \iota^{*}[M] \\
& =c\left(N_{X} M\right)^{-1} \cdot c\left(\left.T M\right|_{X}\right) \cap[X]=c^{F J}(X)
\end{aligned}
$$

and it is known that, in general, the Schwartz-MacPherson class and the FultonJohnson class are different, see [Schür1], [PP3], [BLSS], [Y], [A4]. This motivated J. Schürmann the following definition:

Definition 2.2.4 [Schür1] Let $\iota: X \hookrightarrow Z$ be a regular embedding and let $\alpha$ be a constructible function on $Z$. One defines the Milnor class of the pair $X \subset Z$ relative to $\alpha$ as been

$$
\mathcal{M}(X \subset Z ; \alpha)=(-1)^{\operatorname{dim} X}\left(c\left(N_{X} Z\right)^{-1} \cap \iota^{*}\left(c_{*}(\alpha)\right)-c_{*}\left(\iota^{*}(\alpha)\right)\right) \in H_{*}(X),
$$

where $N_{X} Z$ is the normal cone of $X$ in $Z$.

When $Z$ is a smooth variety, we also denote $\mathcal{M}(X \subset Z ; \alpha)$ simply by $\mathcal{M}(X ; \alpha)$.
Theorem 2.2.5 Suppose that $Z$ is a smooth variety and suppose that $X$ is the zeroscheme of a regular section of a vector bundle $E$ on $Z$. Given a constructible function $\alpha$ on $Z$, it follows that

$$
\mu(X ; \alpha)=\int_{X} \mathcal{M}(X ; \alpha) .
$$

Proof. We will take a particular situation of the Example 1.5.9 given by the following fiber square


Therefore, $\iota_{*} \iota^{*}(\beta)=c_{\text {top }}\left(I d^{*} E\right) \cap \beta=c_{\text {top }}(E) \cap \beta$, for all $\beta \in A_{*} Z$. By definition, we have

$$
\mu(X ; \alpha)=(-1)^{\operatorname{dim} X}\left(\int_{X} c(E)^{-1} c_{\text {top }}(E) c_{*}(\alpha)-\chi\left(X ;\left.\alpha\right|_{X}\right)\right) .
$$

Thus, using the Example 1.5.9 and Proposition 1.5.8, we obtain

$$
\begin{aligned}
\mu(X ; \alpha) & =(-1)^{\operatorname{dim} X}\left(\int_{X} c(E)^{-1} \cap \iota_{*} \iota^{*} c_{*}(\alpha)-\int_{X} c_{*}\left(\iota^{*} \alpha\right)\right) \\
& =(-1)^{\operatorname{dim} X}\left(\int_{X} \iota_{*}\left(c\left(\iota^{*} E\right)^{-1} \cap \iota^{*} c_{*}(\alpha)\right)-\int_{X} c_{*}\left(\iota^{*} \alpha\right)\right) \\
& =(-1)^{\operatorname{dim} X}\left(\int_{X} c\left(N_{X} Z\right)^{-1} \cap \iota^{*} c_{*}(\alpha)-\int_{X} c_{*}\left(\iota^{*} \alpha\right)\right) \\
& =\int_{X} \mathcal{M}(X ; \alpha)
\end{aligned}
$$

because $\iota^{*} E=\left.E\right|_{X}=N_{X} Z$ and $\chi\left(X ;\left.\alpha\right|_{X}\right)=\int_{X} c_{*}\left(\iota^{*} \alpha\right)$.
In this way, the class $c^{F J}(X \subset Z ; \alpha):=c\left(N_{X} Z\right)^{-1} \cap \iota^{*}\left(c_{*}(\alpha)\right)$ is called FultonJohnson class of the pair $X \subset Z$ relative to $\alpha$; and the class $c^{S M}(X \subset Z ; \alpha)=c_{*}\left(\iota^{*} \alpha\right)$ is called Schwartz-MacPherson class of the pair $X \subset Z$ relative to $\alpha$.

Question. Let $L$ be a holomorphic line bundle over a smooth manifold $M$ and let $X:=v^{-1}(0)$ be a divisor in $M$, where $v$ is a regular holomorphic section of $L$. Consider $D$ another divisor in $M$. Assume that $\operatorname{Sing} X=\left\{x_{1}, \ldots, x_{s}\right\} \subset D$. One defines the number

$$
\mu^{B R}(X, D):=\sum_{i=1}^{s} \mu^{B R}\left(f_{i}, D\right)
$$

where $f_{1}, \ldots, f_{s}$ are local defining functions of $X$ at $x_{1}, \ldots, x_{s}$, respectively, with $f_{i}$ being finitely $\mathcal{R}(D)$-determined, for more details see [BR].

Is there any relationship between the numbers $\mu^{B R}(X, D)$ and $\mu_{D}(X)$ ?

## Capítulo 3

## Intersection product formulas relative to constructible functions

This chapter is about some product formulas and some intersection formulas of varieties, generalized for arbitrary constructible functions.

### 3.1 Product formulas for the Milnor class of constructible functions

Let $X$ and $Y$ be manifolds and let $\alpha$ and $\beta$ be constructible functions on $X$ and $Y$, respectively. One defines $(\alpha \otimes \beta)(x, y):=\alpha(x) \beta(y)$, for all $(x, y) \in X \times Y$. We have that $\alpha \otimes \beta$ is a constuctible function on $X \times Y$.

Theorem 3.1.1 [Kwie],[Kwei-Yoka] Consider $\alpha$ a constructible function on $X$ and $\beta$ a constructible functions on $Y$. Then,

$$
c_{*}(\alpha \otimes \beta)=c_{*}(\alpha) \times c_{*}(\beta) .
$$

In particular, $c_{*}(X \times Y)=c_{*}(X) \times c_{*}(Y)$ and $c_{M a}(X \times Y)=c_{M a}(X) \times c_{M a}(Y)$, because $1_{X} \otimes 1_{Y}=1_{X \times Y}$ and $E u_{X \times Y}=E u_{X} \otimes E u_{Y}$.

For each $i=1,2$, let $M_{i}$ be an $\left(n_{i}+k_{i}\right)$-dimensional compact complex analytic manifold and let $E_{i}$ be a holomorphic vector bundle over $M_{i}$ of rank $k_{i}$. Consider $s_{i}$ a regular holomorphic section of $E_{i}$, and consider the $n_{i}$-dimensional local complete intersection. For each $i=1,2$, consider the projection $p_{i}: M_{1} \times M_{2} \longrightarrow M_{i}$. Then,
consider the holomorphic section $s_{1} \oplus s_{2}: M_{1} \times M_{2} \longrightarrow p_{1}^{*} E_{1} \oplus p_{2}^{*} E_{2}$ given by $\left(s_{1} \oplus\right.$ $\left.s_{2}\right)(x, y)=\left(s_{1}(x), s_{2}(y)\right)$, for all $(x, y) \in M_{1} \times M_{2}$. Note that $X_{1} \times X_{2}=\left(s_{1} \oplus s_{2}\right)^{-1}(0)$. Let $\alpha$ and $\beta$ be constructible functions on $M_{1}$ and $M_{2}$, respectively.

## Proposition 3.1.2 We have

$$
c_{*}\left(X_{1} \times X_{2} \subset M_{1} \times M_{2} ; \alpha \otimes \beta\right)=c_{*}\left(X_{1} \subset M_{1} ; \alpha\right) \times c_{*}\left(X_{2} \subset M_{2} ; \beta\right) .
$$

Proof. For all $\left(x_{1}, x_{2}\right) \in X_{1} \times X_{2}$, it follows that

$$
\begin{aligned}
\left(\iota_{1} \times \iota_{2}\right)^{*}(\alpha \otimes \beta)\left(x_{1}, x_{2}\right) & =(\alpha \otimes \beta)\left(\iota_{1}\left(x_{1}\right), \iota_{2}\left(x_{2}\right)\right) \\
& =\alpha\left(\iota_{1}\left(x_{1}\right)\right) \beta\left(\iota_{2}\left(x_{2}\right)\right) \\
& =\left(\left(\iota_{1}^{*} \alpha\right) \otimes\left(\iota_{2}^{*} \beta\right)\right)\left(x_{1}, x_{2}\right) .
\end{aligned}
$$

Thus, $\left(\iota_{1} \times \iota_{2}\right)^{*}(\alpha \otimes \beta)=\left(\iota_{1}{ }^{*} \alpha\right) \otimes\left(\iota_{2}{ }^{*} \beta\right)$. Then, using Theorem 3.1.1

$$
\begin{aligned}
c_{*}\left(X_{1} \times X_{2} \subset M_{1} \times M_{2} ; \alpha \otimes \beta\right) & =c_{*}\left(\left(\iota_{1} \times \iota_{2}\right)^{*}(\alpha \otimes \beta)\right) \\
& =c_{*}\left(\left(\iota_{1}^{*} \alpha\right) \otimes\left(\iota_{2}^{*} \beta\right)\right) \\
& =c_{*}\left(\iota_{1}^{*} \alpha\right) \times c_{*}\left(\iota_{2}^{*} \beta\right) \\
& =c_{*}\left(X_{1} \subset M_{1} ; \alpha\right) \times c_{*}\left(X_{2} \subset M_{2} ; \beta\right) .
\end{aligned}
$$

One should notice that the above proposition generalizes the Theorem 3.1.1. For this, take $\alpha=\mathbf{1}_{X_{1}}$ and $\beta=\mathbf{1}_{X_{2}}$. Now we look for, in similar fashion, a result for the Fulton-Johnson classes.

Proposition 3.1.3 In the same conditions, we have

$$
c^{F J}\left(X_{1} \times X_{2} \subset M_{1} \times M_{2} ; \alpha \otimes \beta\right)=c^{F J}\left(X_{1} \subset M_{1} ; \alpha\right) \times c^{F J}\left(X_{2} \subset M_{2} ; \beta\right) .
$$

Proof. By defintion,

$$
c^{F J}\left(X_{1} \times X_{2} \subset M_{1} \times M_{2} ; \alpha \otimes \beta\right)=c\left(N_{X_{1} \times X_{2}} M_{1} \times M_{2}\right)^{-1} \cap\left(\iota_{1} \times \iota_{2}\right)^{*}\left(c_{*}(\alpha \otimes \beta)\right) .
$$

By Theorem 3.1.1 and Proposition 1.5.4(ii), it follows that

$$
\begin{aligned}
\left(\iota_{1} \times \iota_{2}\right)^{*}\left(c_{*}(\alpha \otimes \beta)\right) & =\left(\iota_{1} \times \iota_{2}\right)^{*}\left(c_{*}(\alpha) \times c_{*}(\beta)\right) \\
& =\left(\iota_{1}\right)^{*}\left(c_{*}(\alpha)\right) \times\left(\iota_{2}\right)^{*}\left(c_{*}(\beta)\right) .
\end{aligned}
$$

Since $N_{X_{1} \times X_{2}} M_{1} \times M_{2}=p_{1}^{*}\left(N_{X_{1}} M_{1}\right) \oplus p_{2}^{*}\left(N_{X_{2}} M_{2}\right)$ and using the Example 1.5.11, for all $\gamma \in A_{*} X_{1}$ and $\eta \in A_{*} X_{2}$, we have

$$
\begin{aligned}
\gamma \times \eta & =\left(\left(c\left(N_{X_{1}} M_{1}\right) c\left(N_{X_{1}} M_{1}\right)^{-1}\right) \cap \gamma\right) \times\left(\left(c\left(N_{X_{2}} M_{2}\right) c\left(N_{X_{2}} M_{2}\right)^{-1}\right) \cap \eta\right) \\
& =c\left(p_{1}^{*}\left(N_{X_{1}} M_{1}\right)\right) c\left(p_{2}^{*}\left(N_{X_{2}} M_{2}\right)\right)\left(c\left(N_{X_{1}} M_{1}\right)^{-1} \cap \gamma\right) \times\left(c\left(N_{X_{2}} M_{2}\right)^{-1} \cap \eta\right) \\
& =c\left(p_{1}^{*}\left(N_{X_{1}} M_{1}\right) \oplus p_{2}^{*}\left(N_{X_{2}} M_{2}\right)\right)\left(c\left(N_{X_{1}} M_{1}\right)^{-1} \cap \gamma\right) \times\left(c\left(N_{X_{2}} M_{2}\right)^{-1} \cap \eta\right)
\end{aligned}
$$

where the last equality follows from the Whitney product formula. Consequently,

$$
c\left(p_{1}^{*}\left(N_{X_{1}} M_{1}\right) \oplus p_{2}^{*}\left(N_{X_{2}} M_{2}\right)\right)^{-1} \cap(\gamma \times \eta)=\left(c\left(N_{X_{1}} M_{1}\right)^{-1} \cap \gamma\right) \times\left(c\left(N_{X_{2}} M_{2}\right)^{-1} \cap \eta\right)
$$

Therefore,

$$
\begin{aligned}
c^{F J}\left(X_{1} \times X_{2} \subset M_{1} \times M_{2} ; \alpha \otimes \beta\right)= & c\left(p_{1}^{*}\left(N_{X_{1}} M_{1}\right) \oplus p_{2}^{*}\left(N_{X_{2}} M_{2}\right)\right)^{-1} \cap \\
& \cap\left(\iota_{1}^{*}\left(c_{*}(\alpha)\right) \times \iota_{2}^{*}\left(c_{*}(\beta)\right)\right) \\
= & \left(c\left(p_{1}^{*}\left(N_{X_{1}} M_{1}\right)\right)^{-1} \cap \iota_{1}^{*}\left(c_{*}(\alpha)\right)\right) \times \\
& \times\left(c\left(p_{2}^{*}\left(N_{X_{2}} M_{2}\right)\right)^{-1} \cap \iota_{2}^{*}\left(c_{*}(\beta)\right)\right) \\
= & c^{F J}\left(X_{1} \subset M_{1} ; \alpha\right) \times c^{F J}\left(X_{2} \subset M_{2} ; \beta\right) .
\end{aligned}
$$

By definition, the Milnor class is the difference, up to sign, between the SchwartzMacPherson class and the Fulton-Johnson class. Then, we can use the above results to show the following theorem.

Theorem 3.1.4 Let $M_{1}, \cdots, M_{r}$ be compact complex manifolds of dimension $n_{i}$, respectively. For each $i$, consider a holomorphic vector bundle $E_{i}$ of rank $d_{i}$ over $M_{i}$, a regular holomorphic section $s_{i}: M_{i} \longrightarrow E_{i}$ and $X_{i}:=s_{i}^{-1}(0)$ the local complete intersection of dimension $n_{i}-d_{i}$. For each $i$, let $\alpha_{i}$ be a constructible function on $M_{i}$. Then,

$$
\mathcal{M}\left(X_{1} \times \cdots \times X_{r} ; \alpha_{1} \otimes \cdots \otimes \alpha_{r}\right)=\sum(-1)^{\left(n_{1}-d_{1}\right) \varepsilon_{1}+\cdots+\left(n_{r}-d_{r}\right) \varepsilon_{r}} P_{1} \cdots P_{r}
$$

where the sum runs over all choices of $P_{i} \in\left\{c_{*}\left(X_{i} ; \alpha_{i}\right), \mathcal{M}\left(X_{i} ; \alpha_{i}\right)\right\}, i=1, \cdots, r$, except $\left(P_{1}, \cdots, P_{r}\right)=\left(c_{*}\left(X_{1} ; \alpha_{1}\right), \cdots, c_{*}\left(X_{r} ; \alpha_{r}\right)\right)$ and

$$
\varepsilon_{i}=\left\{\begin{array}{l}
1, \text { if } P_{i}=c_{*}\left(X_{i} ; \alpha_{i}\right) \\
0, \text { if } P_{i}=\mathcal{M}\left(X_{i} ; \alpha_{i}\right)
\end{array}\right.
$$

Proof. First of all, on the one hand, we have

$$
\begin{aligned}
\mathcal{M}\left(X_{1} \times X_{2} ; \alpha_{1} \otimes \alpha_{2}\right)= & (-1)^{n_{1}+n_{2}-d_{1}-d_{2}}\left(c^{F J}\left(X_{1} \times X_{2} ; \alpha_{1} \otimes \alpha_{2}\right)-\right. \\
& \left.-c_{*}\left(X_{1} \times X_{2} ; \alpha_{1} \otimes \alpha_{2}\right)\right) \\
= & (-1)^{n_{1}+n_{2}-d_{1}-d_{2}}\left(c^{F J}\left(X_{1} ; \alpha_{1}\right) \times c^{F J}\left(X_{2} ; \alpha_{2}\right)-\right. \\
& -c_{*}\left(X_{1} ; \alpha_{1}\right) \times c_{*}\left(X_{2} ; \alpha_{2}\right) .
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
& \mathcal{M}\left(X_{1} ; \alpha_{1}\right) \times \mathcal{M}\left(X_{2} ; \alpha_{2}\right)+(-1)^{n_{1}-d_{1}} c_{*}\left(X_{1} ; \alpha_{1}\right) \times \mathcal{M}\left(X_{2} ; \alpha_{2}\right)+ \\
& +(-1)^{n_{2}-d_{2}} \mathcal{M}\left(X_{1} ; \alpha_{1}\right) \times c_{*}\left(X_{2} ; \alpha_{2}\right) \\
= & (-1)^{n_{1}-d_{1}}\left(c^{F J}\left(X_{1} ; \alpha_{1}\right)-c_{*}\left(X_{1} ; \alpha_{1}\right)\right) \times(-1)^{n_{2}-d_{2}}\left(c^{F J}\left(X_{2} ; \alpha_{2}\right)-c_{*}\left(X_{2} ; \alpha_{2}\right)\right)+ \\
& +(-1)^{n_{1}-d_{1}} c_{*}\left(X_{1} ; \alpha_{1}\right) \times(-1)^{n_{2}-d_{2}}\left(c^{F J}\left(X_{2} ; \alpha_{2}\right)-c_{*}\left(X_{2} ; \alpha_{2}\right)\right)+ \\
& +(-1)^{n_{2}-d_{2}}(-1)^{n_{1}-d_{1}}\left(c^{F J}\left(X_{1} ; \alpha_{1}\right)-c_{*}\left(X_{1} ; \alpha_{1}\right)\right) \times c_{*}\left(X_{2} ; \alpha_{2}\right) \\
= & (-1)^{n_{1}+n_{2}-d_{1}-d_{2}}\left(c^{F J}\left(X_{1} ; \alpha_{1}\right) \times c^{F J}\left(X_{2} ; \alpha_{2}\right)-c^{F J}\left(X_{1} ; \alpha_{1}\right) \times c_{*}\left(X_{2} ; \alpha_{2}\right)-\right. \\
& \left.-c_{*}\left(X_{1} ; \alpha_{1}\right) \times c^{F J}\left(X_{2} ; \alpha_{2}\right)+c_{*}\left(X_{1} ; \alpha_{1}\right) \times c_{*}\left(X_{2} ; \alpha_{2}\right)\right)+ \\
& +(-1)^{n_{1}+n_{2}-d_{1}-d_{2}}\left(c_{*}\left(X_{1} ; \alpha_{1}\right) \times c^{F J}\left(X_{2} ; \alpha_{2}\right)-c_{*}\left(X_{1} ; \alpha_{1}\right) \times c_{*}\left(X_{2} ; \alpha_{2}\right)\right)+ \\
& +(-1)^{n_{1}+n_{2}-d_{1}-d_{2}}\left(c^{F J}\left(X_{1} ; \alpha_{1}\right) \times c_{*}\left(X_{2} ; \alpha_{2}\right)-c_{*}\left(X_{1} ; \alpha_{1}\right) \times c_{*}\left(X_{2} ; \alpha_{2}\right)\right) .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathcal{M}\left(X_{1} \times X_{2} ; \alpha_{1} \otimes \alpha_{2}\right)= & (-1)^{n_{1}-d_{1}} c_{*}\left(X_{1} ; \alpha_{1}\right) \times M\left(X_{2} ; \alpha_{2}\right)+ \\
& +(-1)^{n_{2}-d_{2}} \mathcal{M}\left(X_{1} ; \alpha_{1}\right) \times c_{*}\left(X_{2} ; \alpha_{2}\right)+\mathcal{M}\left(X_{1} ; \alpha_{1}\right) \times \mathcal{M}\left(X_{2} ; \alpha_{2}\right) .
\end{aligned}
$$

Now take $r>2$ and suppose that the result holds for $r-1$. Thus,

$$
\mathcal{M}\left(X_{1} \times \cdots \times X_{r-1} ; \alpha_{1} \otimes \cdots \otimes \alpha_{r-1}\right)=\sum(-1)^{n_{1} \varepsilon_{1}+\cdots+n_{r-1} \varepsilon_{r-1}} P_{1} \cdots P_{r-1},
$$

where the sum runs over all choices of $P_{i} \in\left\{c_{*}\left(X_{i} ; \alpha_{i}\right), \mathcal{M}\left(X_{i} ; \alpha_{i}\right)\right\}, i=1, \cdots, r-1$, except $\left(P_{1}, \cdots, P_{r-1}\right)=\left(c_{*}\left(X_{1} ; \alpha_{1}\right), \cdots, c_{*}\left(X_{r-1} ; \alpha_{r-1}\right)\right)$ and

$$
\varepsilon_{i}=\left\{\begin{array}{l}
1, \text { if } P_{i}=c_{*}\left(X_{i} ; \alpha_{i}\right) \\
0, \text { if } P_{i}=\mathcal{M}\left(X_{i} ; \alpha_{i}\right)
\end{array}\right.
$$

Then, it follows that

$$
\begin{aligned}
& \mathcal{M}\left(X_{1} \times \cdots \times X_{r} ; \alpha_{1} \otimes \cdots \otimes \alpha_{r}\right) \\
= & (-1)^{n_{1}+\cdots+n_{r-1}} c_{*}\left(X_{1} \times \cdots \times X_{r-1} ; \alpha_{1} \otimes \cdots \otimes \alpha_{r-1}\right) \times \mathcal{M}\left(X_{r} ; \alpha_{r}\right)+ \\
& +(-1)^{n_{r}} \mathcal{M}\left(X_{1} \times \cdots \times X_{r-1} ; \alpha_{1} \otimes \cdots \otimes \alpha_{r-1}\right) \times c_{*}\left(X_{r} ; \alpha_{r}\right)+ \\
& +\mathcal{M}\left(X_{1} \times \cdots \times X_{r-1} ; \alpha_{1} \otimes \cdots \otimes \alpha_{r-1}\right) \times \mathcal{M}\left(X_{r} ; \alpha_{r}\right) \\
= & (-1)^{n_{1}+\cdots+n_{r-1}} c_{*}\left(X_{1} ; \alpha_{1}\right) \times \cdots \times c_{*}\left(X_{r-1} ; \alpha_{r-1}\right) \times \mathcal{M}\left(X_{r} ; \alpha_{r}\right)+ \\
& +(-1)^{n_{r}} \sum(-1)^{n_{1} \varepsilon_{1}+\cdots+n_{r-1} \varepsilon_{r-1}} P_{1} \cdots \cdots P_{r-1} \times c_{*}\left(X_{r} ; \alpha_{r}\right)+ \\
& +\sum(-1)^{n_{1} \varepsilon_{1}+\cdots+n_{r-1} \varepsilon_{r-1}} P_{1} \cdots P_{r-1} \times \mathcal{M}\left(X_{r} ; \alpha_{r}\right) \\
= & \sum(-1)^{n_{1} \varepsilon_{1}+\cdots+n_{r} \varepsilon_{r}} P_{1} \cdots P_{r} .
\end{aligned}
$$

In particular, when $\alpha_{i}=\mathbf{1}_{X_{i}}$, for all $i$, we obtain the principal result of $[\mathrm{O}-\mathrm{Y}]$. Explicitly:

Corollary 3.1.5 In the same conditions, one has

$$
\mathcal{M}\left(X_{1} \times \cdots \times X_{r}\right)=\sum(-1)^{\left(n_{1}-d_{1}\right) \varepsilon_{1}+\cdots+\left(n_{r}-d_{r}\right) \varepsilon_{r}} P_{1} \cdots P_{r}
$$

where the sum runs over all choices of $P_{i} \in\left\{c_{*}\left(X_{i}\right), \mathcal{M}\left(X_{i}\right)\right\}, i=1, \cdots, r$, except $\left(P_{1}, \cdots, P_{r}\right)=\left(c_{*}\left(X_{1}\right), \cdots, c_{*}\left(X_{r}\right)\right)$ and

$$
\varepsilon_{i}=\left\{\begin{array}{l}
1, \text { if } P_{i}=c_{*}\left(X_{i}\right) \\
0, \text { if } P_{i}=\mathcal{M}\left(X_{i}\right)
\end{array}\right.
$$

Proof. Note that $\mathcal{M}\left(X_{1} \times \cdots \times X_{r} ; \mathbf{1}_{X_{1}} \otimes \cdots \otimes \mathbf{1}_{X_{r}}\right)=\mathcal{M}\left(X_{1} \times \cdots \times X_{r} ; \mathbf{1}_{X_{1} \times \cdots \times X_{r}}\right)=\mathcal{M}\left(X_{1} \times\right.$ $\left.\cdots \times X_{r}\right)$. Moreover, for each $i=1, \cdots, r$, we know that $c_{*}\left(X_{i} ; \mathbf{1}_{X_{i}}\right)=c_{*}\left(X_{i}\right)$ and $\mathcal{M}\left(X_{i} ; \mathbf{1}_{X_{i}}\right)=\mathcal{M}\left(X_{i}\right)$.

### 3.2 On the diagonal embedding

Consider $M$ a complex manifold and $X$ an analytic subvariety of $M$. Let $\mathcal{S}$ be a Whitney stratication of $M$ adapted to $X$. Given $x \in X \cap S$ for some $S \in \mathcal{S}$, consider $g:(M, x) \longrightarrow(\mathbb{C}, 0)$ a germ of holomorphic function such that $d_{x} g$ is a non-degenerate covector at $x$ with respect to $\mathcal{S}$, that is, $d_{x} g \in T_{S}^{*} M$ and $d_{x} g \notin T_{S^{\prime}}^{*} M$ for all stratum $S^{\prime} \neq S$. For $N$ a germ of a closed complex submanifold of $M$ which is transversal to $\mathcal{S}$
with $N \cap S=\{x\}$, one define the complex link $l_{S}$ of $S$ by

$$
l_{S}:=X \cap N \cap B_{\delta}(x) \cap\{g=\omega\}
$$

for $0<|\omega| \ll \delta \ll 1$, and one defines the normal Morse index

$$
\eta\left(S ; F^{\bullet}\right):=\chi\left(X \cap N \cap B_{\delta}(x), l_{S} ; F^{\bullet}\right)
$$

where the right-hand side is the Euler characteristic of the relative hypercohomology. the number $\eta\left(S ; F^{\bullet}\right)$ does not depend on the choices of $x \in S, g$ and $N$, see [G-M, Section 2.3]. Moreover, we have

$$
\eta\left(S ; F^{\bullet}\right):=\chi\left(X \cap N \cap B_{\delta}(x) ; F^{\bullet}\right)-\chi\left(l_{S} ; F^{\bullet}\right) .
$$

The conormal variety of a subvariety $X$ in a complex manifold $M$ is given by

$$
T_{X}^{*} M:=\operatorname{closure}\left\{(x, \theta) \in T^{*} M \mid x \in X_{\mathrm{reg}} \text { and }\left.\theta\right|_{T_{x} X_{r e g}} \equiv 0\right\} .
$$

Let $L(M)$ be the free abelian group generated by all the conormal spaces $T_{X}^{*} M$, where $X$ varies over all subvarieties of $M$. Given a constructible function $\alpha$ on $M$ with respect to a Whitney stratication $\mathcal{S}$ defines an element in $L(M)$ by

$$
\operatorname{Ch}(\alpha):=\sum_{S \in \mathcal{S}}(-1)^{\operatorname{dim} S} \eta(S ; \alpha) \cdot T_{\frac{*}{S}}^{*} M .
$$

Inducing an isomorphism $\mathrm{Ch}: C F(M) \longrightarrow L(M)$.
Let $M$ be an $n$-dimensional compact complex analytic manifold. Define $M^{(r)}:=$ $M \times \cdots \times M$. And let $E$ be a holomorphic vector bundle over $M^{(r)}$ of rank $d$. Consider $\Delta: M \longrightarrow M^{(r)}$ the diagonal morphism, which is a regular embedding of codimension $n r-n$. Let $t$ be a regular holomorphic section of $E$. This means that set $Z(t):=\{p \in$ $\left.M^{(r)}: t(p)=0\right\}$ is a closed subvariety of $M^{(r)}$ of dimension $n r-d$. In addition, the morphism $\Delta$ induces the refined Gysin homomorphism

$$
\Delta^{!}: H_{2 k}(Z(t)) \longrightarrow H_{2(k-n r+n)}\left(Z\left(\Delta^{*}(t)\right)\right) .
$$

The refined intersection product is defined by $\gamma_{1} \cdot \ldots \cdot \gamma_{r}:=\Delta^{!}\left(\gamma_{1} \times \cdots \times \gamma_{r}\right)$, see [Ful, Example 8.1.9].

Consider the projectivized cotangent bundles $\mathbb{P}\left(T^{*} M\right)$ and $\mathbb{P}\left(T^{*}\left(M^{(r)}\right)\right)$. We will denote the vector bundle $\mathbb{P}\left(T^{*} M \oplus \cdots \oplus T^{*} M\right)$ by $\mathbb{P}\left(T^{*}\left(M^{(r)}\right)\right)$. We have the following fibre square diagram:

where $\pi^{(r)}$ is the natural proper map. Let $i: \mathbb{P}\left(T^{*} M\right) \longrightarrow \mathbb{P}\left(\left(T^{*} M\right)^{\oplus r}\right)$ be the morphism induced by the diagonal embedding $T^{*} M \longrightarrow T^{*} M \oplus \cdots \oplus T^{*} M$.

Lemma 3.2.1 Let $\beta$ be a constructible function on $M^{(r)}$ with respect to a Whitney stratication $\mathcal{S}$, that is, $\beta$ is constant at each stratum of $\mathcal{S}$. We assume transversal to $\Delta(M)$ and such that the intersections $S \cap \Delta(M)$ are connected. Then

$$
\delta^{!}[\mathbb{P}(\operatorname{Ch}(\beta))]=(-1)^{n r-n} i_{*}\left[\mathbb{P}\left(\operatorname{Ch}\left(\Delta^{*} \beta\right)\right)\right]
$$

Proof. See [B-M-S, Proposition 1.4].
There is a classical description of the Schwartz-MacPherson class due to C. Sabbah, see [Sab]. In our context, such a generalization is as follows: given a constructible function $\alpha$ on $M^{(r)}$, we have

$$
\begin{equation*}
c_{*}(Z(t) ; \alpha)=(-1)^{n r-1} c\left(\left.T M^{(r)}\right|_{Z(t)}\right) \cap \pi_{*}^{(r)}\left(c\left(\mathcal{O}_{r}(1)^{-1}\right) \cap[\mathbb{P}(\mathrm{Ch}(\alpha))]\right) \tag{3.2}
\end{equation*}
$$

where $\mathcal{O}_{r}(1)$ is the tautological line bundle on the projectivisation $\mathbb{P}\left(T^{*} M^{(r)}\right) \longrightarrow M^{(r)}$, see [P4, pg 13], [P5, pg 352], [PP3, pg 4] and [Ken].

Proposition 3.2.2 With the same notation, we have

$$
\Delta^{!}\left(c_{*}(Z(t) ; \alpha)\right)=c\left(\left(\left.T M\right|_{Z\left(\Delta^{*} t\right)}\right)^{\oplus r-1}\right) \cap c_{*}\left(Z\left(\Delta^{*} t\right) ; \Delta^{*} \alpha\right)
$$

Proof. Using the description (3.2) and Proposition 1.5.8, one has

$$
\Delta^{!} c_{*}(Z(t) ; \alpha)=(-1)^{n r-1} c\left(\Delta^{*}\left(\left.T M^{(r)}\right|_{Z(t)}\right)\right) \cap \Delta^{!} \pi_{*}^{(r)}\left(c\left(\mathcal{O}_{r}(1)\right)^{-1} \cap[\mathbb{P}(\operatorname{Ch}(\alpha))]\right)
$$

By the Theorem 1.5.7 and the Proposition 1.5.8, it follows that

$$
\begin{aligned}
\Delta^{!} \pi_{*}^{(r)}\left(c\left(\mathcal{O}_{r}(1)\right)^{-1} \cap[\mathbb{P}(\operatorname{Ch}(\alpha))]\right) & =p_{*} \delta^{!}\left(c\left(\mathcal{O}_{r}(1)\right)^{-1} \cap[\mathbb{P}(\operatorname{Ch}(\alpha))]\right) \\
& =p_{*}\left(c\left(\delta^{*} \mathcal{O}_{r}(1)\right)^{-1}\right) \cap \delta^{!}[\mathbb{P}(\operatorname{Ch}(\alpha))] .
\end{aligned}
$$

It is known that $\delta^{*} \mathcal{O}_{r}(1)=c\left(\mathcal{O}_{\mathbb{P}\left(\left(T^{*} M\right)^{\oplus r}\right)}(1)\right)$ is the tautological line bundle on the projectivition $\mathbb{P}\left(\left(T^{*} M\right)^{\oplus r}\right) \longrightarrow M^{(r)}$ and $c\left(\Delta^{*}\left(\left.T M^{(r)}\right|_{Z(t)}\right)\right)=c\left(\left(\left.T M\right|_{Z\left(\Delta^{*} t\right)}\right)^{\oplus r}\right)$. Thus, by Lemma 3.2.1, we have that $\delta^{!}[\mathbb{P}(\operatorname{Ch}(\alpha))]=(-1)^{n r-n} i_{*}\left[\mathbb{P}\left(\operatorname{Ch}\left(\Delta^{*} \alpha\right)\right)\right]$, and then

$$
\begin{aligned}
\Delta^{!} c_{*}(Z(t) ; \alpha)= & (-1)^{n-1} c\left(\left.T M^{\oplus r}\right|_{Z\left(\Delta^{*} t\right)}\right) \cap \\
& \cap p_{*}\left(c\left(\mathcal{O}_{\mathbb{P}\left(\left(T^{*} M\right)^{\oplus r}\right)}(1)\right)^{-1}\right) \cap i_{*}\left[\mathbb{P}\left(\operatorname{Ch}\left(\Delta^{*} \alpha\right)\right)\right] \\
= & (-1)^{n-1} c\left(\left.T M^{\oplus r}\right|_{Z\left(\Delta^{*} t\right)}\right) \cap \\
& \cap(p \circ i)_{*}\left(c\left(i_{*} \mathcal{O}_{\mathbb{P}\left(\left(T^{*} M\right)^{\oplus r}\right)}(1)\right)^{-1} \cap\left[\mathbb{P}\left(\operatorname{Ch}\left(\Delta^{*} \alpha\right)\right)\right]\right),
\end{aligned}
$$

where the last equality follows from the projection formula. Since that $i_{*} \mathcal{O}_{\mathbb{P}\left(\left(T^{*} M\right)^{\oplus r}\right)}(1)=$ $\mathcal{O}_{\mathbb{P}\left(T^{*} M\right)}(1)$ and that $q:=(p \circ i): \mathbb{P}\left(T^{*} M\right) \longrightarrow M$ is the projectived cotangent morphism, one has

$$
\begin{aligned}
\Delta^{\prime} c_{*}(Z(t) ; \alpha) & =(-1)^{n-1} c\left(\left.T M^{\oplus r}\right|_{Z\left(\Delta^{*} t\right)}\right) \cap q_{*}\left(c\left(\mathcal{O}_{\mathbb{P}\left(T^{*} M\right)}(1)\right)^{-1} \cap\left[\mathbb{P}\left(\operatorname{Ch}\left(\Delta^{*} \alpha\right)\right)\right]\right) \\
& =c\left(\left(\left.T M\right|_{Z\left(\Delta^{*} t\right)}\right)^{\oplus r-1}\right) \cap c_{*}\left(Z\left(\Delta^{*} t\right) ; \Delta^{*} \alpha\right) .
\end{aligned}
$$

Through this text we will assume that $Z(t) \subset \operatorname{Supp}(\alpha)$.
Proposition 3.2.3 With the same notation, we have

$$
\Delta^{!}\left(c^{F J}(Z(t) ; \alpha)\right)=c\left(\left(\left.T M\right|_{Z\left(\Delta^{*} t\right)}\right)^{\oplus r-1}\right) \cap c^{F J}\left(Z\left(\Delta^{*} t\right) ; \Delta^{*} \alpha\right)
$$

Proof. By definition of Fulton-Johnson class of $Z(t)$ relative to $\alpha$ and by the commutativity of the diagram (3.1), one has

$$
\begin{aligned}
\Delta^{!} c^{F J}(Z(t) ; \alpha) & =\Delta^{!}\left(c\left(\left.E\right|_{Z(t)}\right)^{-1} \cap \iota^{*} c_{*}(\alpha)\right) \\
& =c\left(\Delta^{*}\left(\left.E\right|_{Z(t)}\right)\right)^{-1} \cap \Delta^{!} \iota^{*} c_{*}(\alpha) \\
& =c\left(\left.\Delta^{*} E\right|_{Z\left(\Delta^{*} t\right)}\right)^{-1} \cap \iota^{*} \Delta^{!} c_{*}(\alpha)
\end{aligned}
$$

In [PP3], the Schwartz-MacPherson class of the constructible function $\alpha$ has the following description:

$$
c_{*}(\alpha)=(-1)^{n r-1} c\left(\left.T M^{(r)}\right|_{\operatorname{Supp}(\alpha)}\right) \cap \pi_{*}^{(r)}\left(c\left(\mathcal{O}_{r}(1)^{-1}\right) \cap[\mathbb{P}(\operatorname{Ch}(\alpha))]\right),
$$

where $\mathcal{O}_{r}(1)$ is the tautological line bundle on the projectivisation $\mathbb{P}\left(T^{*} M^{(r)}\right) \longrightarrow M^{(r)}$. Then,

$$
\Delta^{!} c_{*}(\alpha)=(-1)^{n r-1} c\left(\Delta^{*}\left(\left.T M^{(r)}\right|_{\operatorname{Supp}(\alpha)}\right)\right) \cap \Delta^{!} \pi_{*}^{(r)}\left(c\left(\mathcal{O}_{r}(1)\right)^{-1} \cap[\mathbb{P}(\operatorname{Ch}(\alpha))]\right)
$$

Similarly the proof of the Proposition 3.2.2, we have that

$$
\Delta^{\prime} \pi_{*}^{(r)}\left(c\left(\mathcal{O}_{r}(1)\right)^{-1} \cap[\mathbb{P}(\operatorname{Ch}(\alpha))]\right)=q_{*}\left(c\left(\mathcal{O}_{\mathbb{P}\left(T^{*} M\right)}(1)\right)^{-1} \cap\left[\mathbb{P}\left(\operatorname{Ch}\left(\Delta^{*} \alpha\right)\right)\right]\right.
$$

where $q: \mathbb{P}\left(T^{*} M\right) \longrightarrow M$ is the projectived cotangent morphism. Since that $c\left(\Delta^{*}\left(\left.T M^{(r)}\right|_{\operatorname{Supp}(\alpha)}\right)\right)=c\left(\left(\left.T M\right|_{\operatorname{Supp}\left(\Delta^{*} \alpha\right)}\right)^{\oplus r}\right)$, it follows that

$$
\Delta^{!} c_{*}(\alpha)=(-1)^{n-1} c\left(\left(\left.T M\right|_{\operatorname{Supp}\left(\Delta^{*} \alpha\right)}\right)^{\oplus r}\right) \cap q_{*}\left(c\left(\mathcal{O}_{\mathbb{P}\left(T^{*} M\right)}(1)\right)^{-1} \cap\left[\mathbb{P}\left(\operatorname{Ch}\left(\Delta^{*} \alpha\right)\right)\right]\right)
$$

and then,

$$
\begin{aligned}
\Delta^{!} c^{F J}(Z(t) ; \alpha)= & (-1)^{n-1} c\left(\left.\Delta^{*} E\right|_{Z\left(\Delta^{*} t\right)}\right)^{-1} \cap \iota^{*}\left(c\left(\left(\left.T M\right|_{\operatorname{Supp}\left(\Delta^{*} \alpha\right)}\right)\right)^{\oplus r}\right) \cap \\
& \left.\cap q_{*}\left(c\left(\mathcal{O}_{\mathbb{P}\left(T^{*} M\right)}(1)\right)^{-1} \cap\left[\mathbb{P}\left(\operatorname{Ch}\left(\Delta^{*} \alpha\right)\right)\right]\right)\right) \\
= & (-1)^{n-1} c\left(\left.\Delta^{*} E\right|_{Z\left(\Delta^{*} t\right)}\right)^{-1} c\left(\iota^{*}\left(\left.T M\right|_{\operatorname{Supp}\left(\Delta^{*} \alpha\right)}\right)^{\oplus r}\right) \cap \\
& \cap \iota^{*} q_{*}\left(c\left(\mathcal{O}_{\mathbb{P}\left(T^{*} M\right)}(1)\right)^{-1} \cap\left[\mathbb{P}\left(\operatorname{Ch}\left(\Delta^{*} \alpha\right)\right)\right]\right) \\
= & (-1)^{n-1} c\left(\left.\Delta^{*} E\right|_{Z\left(\Delta^{*} t\right)}\right)^{-1} c\left(\iota^{*}\left(\left.T M\right|_{\operatorname{Supp}\left(\Delta^{*} \alpha\right)}\right)^{\oplus r-1}\right) . \\
& \cdot c\left(\iota^{*}\left(\left.T M\right|_{\operatorname{Supp}\left(\Delta^{*} \alpha\right)}\right)\right) \cap \iota^{*} q_{*}\left(c\left(\mathcal{O}_{\mathbb{P}\left(T^{*} M\right)}(1)\right)^{-1} \cap\left[\mathbb{P}\left(\operatorname{Ch}\left(\Delta^{*} \alpha\right)\right)\right]\right) \\
= & (-1)^{n-1} c\left(\left.\Delta^{*} E\right|_{Z\left(\Delta^{*} t\right)}\right)^{-1} c\left(\left(\left.T M\right|_{Z\left(\Delta^{*} t\right)}\right)^{\oplus r-1}\right) \cap \\
& \cap \iota^{*}\left(c\left(\left.T M\right|_{\operatorname{Supp}\left(\Delta^{*} \alpha\right)}\right) \cap q_{*}\left(c\left(\mathcal{O}_{\mathbb{P}\left(T^{*} M\right)}(1)\right)^{-1} \cap\left[\mathbb{P}\left(\operatorname{Ch}\left(\Delta^{*} \alpha\right)\right)\right]\right)\right) \\
= & c\left(\left.\Delta^{*} E\right|_{Z\left(\Delta^{*} t\right)}\right)^{-1} c\left(\left(\left.T M\right|_{Z\left(\Delta^{*} t\right)}\right)^{\oplus r-1}\right) \cap \iota^{*} c_{*}\left(\Delta^{*} \alpha\right) \\
= & c\left(\left(\left.T M\right|_{Z\left(\Delta^{*} t\right)}\right)^{\oplus r-1}\right) \cap c^{F J}\left(Z\left(\Delta^{*} t\right) ; \Delta^{*} \alpha\right) .
\end{aligned}
$$

As a consequence of the above propositions, we have the following result on the Milnor class.

Proposition 3.2.4 We have that,

$$
\Delta^{\prime}(\mathcal{M}(Z(t) ; \alpha))=(-1)^{n r-n} c\left(\left(\left.T M\right|_{Z\left(\Delta^{*} t\right)}\right)^{\oplus r-1}\right) \cap \mathcal{M}\left(Z\left(\Delta^{*} t\right) ; \Delta^{*} \alpha\right)
$$

Proof. Indeed, using the definition of the Milnor class and the above propositions, we have that

$$
\begin{aligned}
\Delta^{!}(\mathcal{M}(Z(t) ; \alpha))= & \Delta^{!}\left((-1)^{n r-n}\left(c^{F J}(Z(t) ; \alpha)-c_{*}(Z(t) ; \alpha)\right)\right) \\
= & (-1)^{n r-n} c\left(\left(\left.T M\right|_{Z\left(\Delta^{*} t\right)}\right)^{\oplus r-1}\right) \cap \\
& \cap\left(c^{F J}\left(Z\left(\Delta^{*} t\right) ; \Delta^{*} \alpha\right)-c_{*}\left(Z\left(\Delta^{*} t\right) ; \Delta^{*} \alpha\right)\right) \\
= & (-1)^{n r-n} c\left(\left(\left.T M\right|_{Z\left(\Delta^{*} t\right)}\right)^{\oplus r-1}\right) \cap \mathcal{M}\left(Z\left(\Delta^{*} t\right) ; \Delta^{*} \alpha\right) .
\end{aligned}
$$

### 3.3 Intersection product formulas relative to constructible functions

Consider $M$ a compact complex manifold with dimension equals to $n$. For each $i=1, \cdots, r$, let $E_{i}$ be a holomorphic vector bundle of rank $d_{i}$ over $M$ and let $X_{i}:=s_{i}^{-1}(0)$ be a $\left(n-d_{i}\right)$-dimensional local complete intersection, where $s_{i}$ is a regular holomorphic section on $E_{i}$. Here we are assuming that the product $X_{1} \times \cdots \times X_{r}$ is equiped with a Whitney stratification such that the diagonal embedding $\Delta$ is transversal to all strata. For each $i$, let $p_{i}: M^{(r)} \longrightarrow M$ be the $i^{t h}$-projection. Now, consider the holomorphic exterior product section

$$
s=s_{1} \oplus \cdots \oplus s_{r}: M^{(r)} \longrightarrow\left(p_{1}^{*} E_{1}\right) \oplus \cdots \oplus\left(p_{r}^{*} E_{r}\right),
$$

given by $s\left(x_{1}, \cdots, x_{r}\right)=\left(s_{1}\left(x_{1}\right), \cdots, s_{r}\left(x_{r}\right)\right)$. Therefore, $Z(s)=X_{1} \times \cdots \times X_{r}$ and $X:=Z\left(\Delta^{*}(s)\right)=X_{1} \cap \cdots \cap X_{r}$.

Let $\alpha_{1}, \cdots, \alpha_{r}$ be constructible functions on $M$ and let $\Delta: M \longrightarrow M^{(r)}$ be the diagonal morphism. Set $\alpha:=\alpha_{1} \otimes \cdots \otimes \alpha_{r}$ the constructible function on $M^{(r)}$ such that $\left(\alpha_{1} \otimes \cdots \otimes \alpha_{r}\right)\left(x_{1}, \cdots, x_{r}\right)=\alpha_{1}\left(x_{1}\right) \cdots \alpha_{r}\left(x_{r}\right)$, for all $\left(x_{1}, \cdots, x_{r}\right) \in M^{(r)}$. Note that, $\Delta^{*} \alpha(x)=\left(\alpha_{1} \otimes \cdots \otimes \alpha_{r}\right) \circ \Delta(x)=\alpha_{1}(x) \cdots \alpha_{r}(x)$ for all $x \in M$. Moreover, $\Delta^{*} \alpha(x)=0$ if $x \notin X$; and $\Delta^{*} \alpha(x)=\alpha_{1}(x) \cdots \alpha_{r}(x)$ if $x \in X$. In this way, $\Delta^{*}\left(\alpha_{1} \otimes \cdots \otimes \alpha_{r}\right)$ is also denoted by $\alpha_{1} \otimes \cdots \otimes \alpha_{r}$.

Lemma 3.3.1 With the same notation, we have
$\mathcal{M}(X ; \alpha)=(-1)^{n r-n} c\left(\left(\left.T M\right|_{X}\right)^{\oplus r-1}\right)^{-1} \cap \sum(-1)^{\left(n-d_{1}\right) \epsilon_{1}+\cdots+\left(n-d_{r}\right) \epsilon_{r}} P_{1} \cdots P_{r} \in H_{*}(X)$,
where

$$
\epsilon_{i}= \begin{cases}1, & \text { if } P_{i}=c_{*}\left(X_{i} ; \alpha_{i}\right) \\ 0, & \text { if } P_{i}=\mathcal{M}\left(X_{i} ; \alpha_{i}\right) .\end{cases}
$$

Proof. Using the Proposition 3.2.4,

$$
\Delta^{\prime}(\mathcal{M}(Z(s) ; \alpha))=(-1)^{n r-n} c\left(\left(\left.T M\right|_{Z\left(\Delta^{*} s\right)}\right)^{\oplus r-1}\right) \cap \mathcal{M}\left(Z\left(\Delta^{*} s\right) ; \Delta^{*} \alpha\right) .
$$

Then,

$$
\mathcal{M}(X ; \alpha)=(-1)^{n r-n} c\left(\left.T M^{\oplus r-1}\right|_{X}\right)^{-1} \cap \Delta^{!}\left(\mathcal{M}\left(X_{1} \times \cdots \times X_{r} ; \alpha\right)\right)
$$

By Theorem 3.1.4, it follows that

$$
\mathcal{M}(X ; \alpha)=(-1)^{n r-n} c\left(\left.T M^{\oplus r-1}\right|_{X}\right)^{-1} \cap \sum(-1)^{\left(n-d_{1}\right) \epsilon_{1}+\cdots+\left(n-d_{r}\right) \epsilon_{r}} \Delta^{!}\left(P_{1} \times \cdots \times P_{r}\right)
$$

where the sum runs over all choices of $P_{i} \in\left\{c_{*}\left(X_{i} ; \alpha_{i}\right), \mathcal{M}\left(X_{i} ; \alpha_{i}\right)\right\}, i=1, \cdots, r-1$, except $\left(P_{1}, \cdots, P_{r-1}\right)=\left(c_{*}\left(X_{1} ; \alpha_{1}\right), \cdots, c_{*}\left(X_{r-1} ; \alpha_{r-1}\right)\right)$ and

$$
\epsilon_{i}= \begin{cases}1, & \text { if } P_{i}=c_{*}\left(X_{i} ; \alpha_{i}\right) \\ 0, & \text { if } P_{i}=\mathcal{M}\left(X_{i} ; \alpha_{i}\right)\end{cases}
$$

Since $\Delta^{!}\left(P_{1} \times \cdots \times P_{r}\right)=P_{1} \cdots P_{r}$, the result follows.

This means that, in particular, for $r=2$ we have

$$
\begin{aligned}
\mathcal{M}\left(X ; \alpha_{1} \otimes \alpha_{2}\right)= & c\left(\left.T M\right|_{X}\right)^{-1} \cap\left((-1)^{n} \mathcal{M}\left(X_{1} ; \alpha_{1}\right) \mathcal{M}\left(X_{2} ; \alpha_{2}\right)+\right. \\
& \left.+(-1)^{d_{1}} c_{*}\left(X_{1} ; \alpha_{1}\right) \mathcal{M}\left(X_{2} ; \alpha_{2}\right)+(-1)^{d_{2}} \mathcal{M}\left(X_{1} ; \alpha_{1}\right) c_{*}\left(X_{2} ; \alpha_{2}\right)\right) .
\end{aligned}
$$

Moreover, for $r=3$, we have

$$
\begin{aligned}
& \mathcal{M}(X ; \alpha)=c\left(\left(\left.T M\right|_{X}\right)^{\oplus 2}\right)^{-1} \cap\left(\mathcal{M}\left(X_{1} ; \alpha_{1}\right) \mathcal{M}\left(X_{2} ; \alpha_{2}\right) \mathcal{M}\left(X_{3} ; \alpha_{3}\right)+\right. \\
& +(-1)^{d_{1}+d_{2}} c_{*}\left(X_{1} ; \alpha_{1}\right) c_{*}\left(X_{2} ; \alpha_{2}\right) \mathcal{M}\left(X_{3} ; \alpha_{3}\right)+(-1)^{d_{1}+d_{3}} c_{*}\left(X_{1} ; \alpha_{1}\right) \mathcal{M}\left(X_{2} ; \alpha_{2}\right) c_{*}\left(X_{3} ; \alpha_{3}\right)+ \\
& +(-1)^{d_{2}+d_{3}} \mathcal{M}\left(X_{1} ; \alpha_{1}\right) c_{*}\left(X_{2} ; \alpha_{2}\right) c_{*}\left(X_{3} ; \alpha_{3}\right)+(-1)^{n-d_{1}} c_{*}\left(X_{1} ; \alpha_{1}\right) \mathcal{M}\left(X_{2} ; \alpha_{2}\right) \mathcal{M}\left(X_{3} ; \alpha_{3}\right)+ \\
& \left.+(-1)^{n-d_{2}} \mathcal{M}\left(X_{1} ; \alpha_{1}\right) c_{*}\left(X_{2} ; \alpha_{2}\right) \mathcal{M}\left(X_{3} ; \alpha_{3}\right)+(-1)^{n-d_{3}} \mathcal{M}\left(X_{1} ; \alpha_{1}\right) \mathcal{M}\left(X_{2} ; \alpha_{2}\right) c_{*}\left(X_{3} ; \alpha_{3}\right)\right) .
\end{aligned}
$$

Now we will present the main result in this text.
Theorem 3.3.2 With the same notation, we have the following formulas:

$$
\begin{equation*}
c^{F J}(X ; \alpha)=c\left(\left(\left.T M\right|_{X}\right)^{\oplus r-1}\right)^{-1} \cap c^{F J}\left(X_{1} ; \alpha_{1}\right) \cdot \ldots \cdot c^{F J}\left(X_{r} ; \alpha_{r}\right), \tag{3.3}
\end{equation*}
$$

$$
\begin{equation*}
c^{S M}(X ; \alpha)=c\left(\left(\left.T M\right|_{X}\right)^{\oplus r-1}\right)^{-1} \cap c^{S M}\left(X_{1} ; \alpha_{1}\right) \cdot \ldots \cdot c^{S M}\left(X_{r} ; \alpha_{r}\right) \tag{3.4}
\end{equation*}
$$

and

$$
\begin{aligned}
\mathcal{M}(X ; \alpha)= & (-1)^{\operatorname{dim} X} c\left(\left(\left.T M\right|_{X}\right)^{\oplus r-1}\right)^{-1} \cap\left(c^{F J}\left(X_{1} ; \alpha_{1}\right) \cdots c^{F J}\left(X_{r} ; \alpha_{r}\right)-\right. \\
& \left.-c_{*}\left(X_{1} ; \alpha_{1}\right) \cdots c_{*}\left(X_{r} ; \alpha_{r}\right)\right)
\end{aligned}
$$

where $\alpha$ denotes the constructible function $\alpha_{1} \otimes \cdots \otimes \alpha_{r}$.
Proof. By Proposition 3.2.3, one has

$$
\Delta^{!}\left(c^{F J}(Z(s) ; \alpha)\right)=c\left(\left(\left.T M\right|_{Z\left(\Delta^{*} s\right)}\right)^{\oplus r-1}\right) \cap c^{F J}\left(Z\left(\Delta^{*} s\right) ; \Delta^{*} \alpha\right)
$$

Then, using the Proposition 3.1.3, it follows that

$$
\begin{aligned}
c^{F J}(X ; \alpha) & =c\left(\left(\left.T M\right|_{X}\right)^{\oplus r-1}\right)^{-1} \cap \Delta^{!}\left(c^{F J}\left(X_{1} \times \cdots \times X_{r} ; \alpha_{1} \otimes \cdots \otimes \alpha_{r}\right)\right) \\
& =c\left(\left(\left.T M\right|_{X}\right)^{\oplus r-1}\right)^{-1} \cap \Delta^{!}\left(c^{F J}\left(X_{1} ; \alpha_{1}\right) \times \cdots \times c^{F J}\left(X_{r} ; \alpha_{r}\right)\right)
\end{aligned}
$$

Since $\Delta^{!}\left(c^{F J}\left(X_{1} ; \alpha_{1}\right) \times \cdots \times c^{F J}\left(X_{r} ; \alpha_{r}\right)\right)=c^{F J}\left(X_{1} ; \alpha_{1}\right) \cdot \ldots \cdot c^{F J}\left(X_{r} ; \alpha_{r}\right)$, one has that

$$
c^{F J}(X ; \alpha)=c\left(\left(\left.T M\right|_{X}\right)^{\oplus r-1}\right)^{-1} \cap c^{F J}\left(X_{1} ; \alpha_{1}\right) \cdot \ldots \cdot c^{F J}\left(X_{r} ; \alpha_{r}\right) .
$$

Now, the Proposition 3.2.2 states that

$$
\Delta^{!}\left(c^{S M}(Z(s) ; \alpha)\right)=c\left(\left(\left.T M\right|_{Z\left(\Delta^{*} s\right)}\right)^{\oplus r-1}\right) \cap c^{S M}\left(Z\left(\Delta^{*} s\right) ; \Delta^{*} \alpha\right)
$$

In similar fashion, using the Proposition 3.1.2, we have

$$
c^{S M}(X ; \alpha)=c\left(\left(\left.T M\right|_{X}\right)^{\oplus r-1}\right)^{-1} \cap c^{S M}\left(X_{1} ; \alpha_{1}\right) \cdot \ldots \cdot c^{S M}\left(X_{r} ; \alpha_{r}\right) .
$$

Lastly, using the definition of Milnor class of $X$ relative to $\alpha$ and the formulas (3.3) (3.4), it follows that

$$
\begin{aligned}
\mathcal{M}(X ; \alpha)= & (-1)^{\operatorname{dim} X} c\left(\left(\left.T M\right|_{X}\right)^{\oplus r-1}\right)^{-1} \cap\left(c^{F J}\left(X_{1} ; \alpha_{1}\right) \cdots c^{F J}\left(X_{r} ; \alpha_{r}\right)-\right. \\
& \left.-c_{*}\left(X_{1} ; \alpha_{1}\right) \cdots c_{*}\left(X_{r} ; \alpha_{r}\right)\right) .
\end{aligned}
$$

The above result also follows from Lemma 3.3.1. To exemplify the approach of the computation, let us look at the case $\mathrm{r}=2$. By Lemma 3.3.1, we have

$$
\begin{aligned}
\mathcal{M}\left(X ; \alpha_{1} \otimes \alpha_{2}\right)= & (-1)^{2 n-n} c\left(\left.T M\right|_{X}\right)^{-1} \cap\left(\mathcal{M}\left(X_{1} ; \alpha_{1}\right) \mathcal{M}\left(X_{2} ; \alpha_{2}\right)+\right. \\
& \left.+(-1)^{n-d_{1}} c_{*}\left(X_{1} ; \alpha_{1}\right) \mathcal{M}\left(X_{2} ; \alpha_{2}\right)+(-1)^{n-d_{2}} \mathcal{M}\left(X_{1} ; \alpha_{1}\right) c_{*}\left(X_{2} ; \alpha_{2}\right)\right) .
\end{aligned}
$$

For each $i=1,2$, one has $\mathcal{M}\left(X_{i} ; \alpha_{i}\right)=(-1)^{n-d_{i}}\left(c^{F J}\left(X_{i} ; \alpha_{i}\right)-c_{*}\left(X_{i} ; \alpha_{i}\right)\right)$. Then, it follows that

$$
\begin{aligned}
\mathcal{M}\left(X ; \alpha_{1} \otimes \alpha_{2}\right)= & (-1)^{2 n-n} c\left(\left.T M\right|_{X}\right)^{-1} \cap\left((-1)^{2 n-d_{1}-d_{2}}\left(c^{F J}\left(X_{1} ; \alpha_{1}\right)-c_{*}\left(X_{1} ; \alpha_{1}\right)\right) \cdot\right. \\
& \cdot\left(c^{F J}\left(X_{2} ; \alpha_{2}\right)-c_{*}\left(X_{2} ; \alpha_{2}\right)\right)+(-1)^{2 n-d_{1}-d_{2}} c_{*}\left(X_{1} ; \alpha_{1}\right)\left(c^{F J}\left(X_{2} ; \alpha_{2}\right)-\right. \\
& \left.\left.-c_{*}\left(X_{2} ; \alpha_{2}\right)\right)+(-1)^{2 n-d_{1}-d_{2}}\left(c^{F J}\left(X_{1} ; \alpha_{1}\right)-c_{*}\left(X_{1} ; \alpha_{1}\right)\right) c_{*}\left(X_{2} ; \alpha_{2}\right)\right) \\
= & (-1)^{\operatorname{dim} X} c\left(\left.T M\right|_{X}\right)^{-1} \cap\left(c^{F J}\left(X_{1} ; \alpha_{1}\right) c^{F J}\left(X_{2} ; \alpha_{2}\right)-\right. \\
& \left.-c_{*}\left(X_{1} ; \alpha_{1}\right) c_{*}\left(X_{2} ; \alpha_{2}\right)\right)
\end{aligned}
$$

because $\operatorname{dim} X=n-d_{1}-d_{2}$.
Forthwith, we have the following consequence:
Corollary 3.3.3 The number $(-1)^{\operatorname{dim} X} \mu(X ; \alpha)$ is equals to the degree

$$
\int_{X} c\left(\left(\left.T M\right|_{X}\right)^{\oplus r-1}\right)^{-1} \cap\left(c^{F J}\left(X_{1} ; \alpha_{1}\right) \cdots c^{F J}\left(X_{r} ; \alpha_{r}\right)-c_{*}\left(X_{1} ; \alpha_{1}\right) \cdots c_{*}\left(X_{r} ; \alpha_{r}\right)\right) .
$$

Proof. The result follows immediately from Theorem 2.2.5 and Theorem 3.3.2.
In particular, when we have $\alpha_{i}=\mathbf{1}_{X_{i}}$ for all $i=1, \ldots, r$, we retrieve the original formulas of $[B-M-S]$.

Corollary 3.3.4 We have that

$$
\begin{aligned}
c^{F J}(X) & =c\left(\left(\left.T M\right|_{X}\right)^{\oplus r-1}\right)^{-1} \cap c^{F J}\left(X_{1}\right) \cdot \ldots \cdot c^{F J}\left(X_{r}\right), \\
c^{S M}(X) & =c\left(\left(\left.T M\right|_{X}\right)^{\oplus r-1}\right)^{-1} \cap c^{S M}\left(X_{1}\right) \cdot \ldots \cdot c^{S M}\left(X_{r}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{M}(X)= & (-1)^{\operatorname{dim} X} c\left(\left(\left.T M\right|_{X}\right)^{\oplus r-1}\right)^{-1} \cap\left(c^{F J}\left(X_{1}\right) \cdots c^{F J}\left(X_{r}\right)-\right. \\
& \left.-c_{*}\left(X_{1}\right) \cdots c_{*}\left(X_{r}\right)\right) .
\end{aligned}
$$

## Capítulo 4

## Segre classes relative to a constructible function

Let $X$ be a proper closed subscheme of a variety $Y$. Consider $\widetilde{Y}$ the blow-up of $Y$ along $X, \widetilde{X}=P\left(N_{X} Y\right)$ the exceptional divisor and $\eta: \widetilde{X} \longrightarrow X$ the projection, where $N_{X} Y$ is the normal bundle. It can be organized in the following diagram:


The Segre class of $X$ in $Y$ is characterized by

$$
s(X, Y)=\sum_{i \geq 0} \eta_{*}\left(c_{1}(\mathcal{O}(1))^{i} \cap\left[P\left(N_{X} Y\right)\right]\right)
$$

where $\mathcal{O}(1)$ is the canonical line bundle on $P\left(N_{X} Y\right)$. Now, assume that $Y=M$ is non-singular. We know that the Fulton class of $X$ is defined by $c^{F}(X)=c\left(\left.T M\right|_{X}\right) \cap$ $s(X, M) \in A_{*}(X)$.

We have already commented on the generalization of the Milnor class to an arbitrary constructible function due to Schürmann. Such a definition produces the Fulton class of $X$ relative to a constructible function $\alpha$ on $M$ which is given by $c\left(N_{X} M\right)^{-1} \cap \iota^{*}\left(c_{*}(\alpha)\right)$, where $\iota^{*}$ is the Gysin homomorphism. Thus, given a constructible function $\alpha$ on $M$, we define the Segre class of $X$ in $M$ relative to $\alpha$ as the
following element in $A_{*} X$

$$
s(X \subset M ; \alpha)=\eta_{*}\left(\sum_{i \geq 0} c_{1}\left(\mathcal{O}_{\mathbb{P}\left(N_{X} M\right)}(1)\right)^{i} \cap \eta^{*}\left(c\left(\left.T M\right|_{X}\right)^{-1} \cap \iota^{*} c_{*}(\alpha)\right)\right)
$$

where $c_{*}$ represents the MacPherson class. When there is no ambiguity in relation to the variety $M$, we shall denote $s(X \subset M ; \alpha)$ only by $s(X ; \alpha)$.

Remark 4.0.1 For any vector bundle $E$ of rank $e+1$ on $X$ and for any $\beta \in A_{*}(X)$, we have

$$
s_{i}(E) \cap \beta=b_{*}\left(c_{1}\left(\mathcal{O}_{P(E)}(1)\right)^{e+i} \cap b^{*}(\beta)\right),
$$

where $P(E)$ is the projective bundle of lines in $E, b$ is the projection from $P(E)$ to $X$ and $\mathcal{O}_{P(E)}(1)$ denotes the canonical line bundle on $P(E)$. Since $N_{X} M$ is a vector bundle on $X$, it follows that

$$
s(X \subset M ; \alpha)=c\left(N_{X} M\right)^{-1} \cdot c\left(\left.T M\right|_{X}\right)^{-1} \cap \iota^{*} c_{*}(\alpha),
$$

for all constructible function $\alpha$ on $M$. In this case, we are making $E=N_{X} M$ and $\beta=c\left(\left.T M\right|_{X}\right)^{-1} \cap \iota^{*} c_{*}(\alpha)$.

Motivating us to define a certain type of Fulton class with respect to a arbitrary constructible function.

Definition 4.0.2 Given a constructible function $\alpha$ on $M$, we define the Fulton class of $X$ relative to $\alpha$ as

$$
c^{F}(X \subset M ; \alpha)=c\left(\left.T M\right|_{X}\right) \cap s(X \subset M ; \alpha) .
$$

When there is no ambiguity in relation to the variety $M$, we shall denote $c^{F}(X \subset$ $M ; \alpha)$ only by $c^{F}(X ; \alpha)$. In this way, we have

$$
\begin{aligned}
c^{F}(X \subset M ; \alpha) & =c\left(\left.T M\right|_{X}\right) \cap s(X \subset M ; \alpha) \\
& =c\left(\left.T M\right|_{X}\right) \cap \eta_{*}\left(\sum_{i \geq 0} c_{1}\left(\mathcal{O}_{\mathbb{P}\left(N_{X} M\right)}(1)\right)^{i} \cap \eta^{*}\left(c\left(\left.T M\right|_{X}\right)^{-1} \cap \iota^{*} c_{*}(\alpha)\right)\right) \\
& =c\left(\left.T M\right|_{X}\right) c\left(N_{X} M\right)^{-1} c\left(\left.T M\right|_{X}\right)^{-1} \cap \iota^{*} c_{*}(\alpha) \\
& =c\left(N_{X} M\right)^{-1} \cap \iota^{*} c_{*}(\alpha) .
\end{aligned}
$$

This shows that Definition 4.0.2 coincides with the definition of Fulton class relative to a construtible function due to Schürmann.

The next result shows a relevant property of the Segre class presented above.

Proposition 4.0.3 Let $X_{1}$ and $X_{2}$ be schemes which can be imbedded as subshemes of nonsingular varieties $M_{1}$ and $M_{2}$, respectively. For each $i=1,2$, consider $\alpha_{i}$ a constructible function on $M_{i}$. Then, we get

$$
s\left(X_{1} \times X_{2} ; \alpha_{1} \otimes \alpha_{2}\right)=s\left(X_{1} ; \alpha_{1}\right) \times s\left(X_{2} ; \alpha_{2}\right) .
$$

Proof. For each $i=1,2$, let $\widetilde{X_{i}}$ be the exceptional divisor of the blow-up of $M_{i}$ along $X_{i}$, with projection $\eta_{i}: \widetilde{X_{i}} \longrightarrow X_{i}$. Set $\eta=\eta_{1} \times \eta_{2}$. By definition, $s\left(X_{1} \times X_{2} ; \alpha_{1} \otimes \alpha_{2}\right)$ is equals to

$$
\eta_{*}\left(\sum_{i \geq 0} c_{1}(\mathcal{O}(1))^{i} \cap \eta^{*}\left(c\left(\left.T M\right|_{X_{1} \times X_{2}}\right)^{-1} \cap\left(\iota_{1} \times \iota_{2}\right)^{*} c_{*}\left(\alpha_{1} \otimes \alpha_{2}\right)\right)\right)
$$

where $M$ denotes the product $M_{1} \times M_{2}$ and $\mathcal{O}(1)=\mathcal{O}_{\mathbb{P}\left(N_{\left.X_{1} \times X_{2} M\right)}\right)}$ (1). Using the Theorem 3.1.1 and Example 1.5.10, it follows that $s\left(X_{1} \times X_{2} ; \alpha_{1} \otimes \alpha_{2}\right)$ is equals to

$$
\eta_{*}\left(\sum_{i \geq 0} c_{1}(\mathcal{O}(1))^{i} \cap \eta^{*}\left(c\left(\left.T M\right|_{X_{1} \times X_{2}}\right)^{-1} \cap\left(\iota_{1}^{*} c_{*}\left(\alpha_{1}\right) \times \iota_{2}^{*} c_{*}\left(\alpha_{2}\right)\right)\right)\right) .
$$

Note that $\left.T M\right|_{X_{1} \times X_{2}}=\left.\left.T M_{1}\right|_{X_{1}} \oplus T M_{2}\right|_{X_{2}}$. Thus, $s\left(X_{1} \times X_{2} ; \alpha_{1} \otimes \alpha_{2}\right)$ is equals to

$$
\eta_{*}\left(\sum_{i \geq 0} c_{1}(\mathcal{O}(1))^{i} \cap \eta^{*}\left(c\left(\left.\left.T M_{1}\right|_{X_{1}} \oplus T M_{2}\right|_{X_{2}}\right)^{-1} \cap\left(\iota_{1}^{*} c_{*}\left(\alpha_{1}\right) \times \iota_{2}^{*} c_{*}\left(\alpha_{2}\right)\right)\right)\right) .
$$

With a similar argument that we use in the proof of Proposition 3.1.3, we get $c\left(\left.T M_{1}\right|_{X_{1}} \oplus\right.$ $\left.T M_{2} \mid X_{2}\right)^{-1} \cap\left(\iota_{1}^{*} c_{*}\left(\alpha_{1}\right) \times \iota_{2}^{*} c_{*}\left(\alpha_{2}\right)\right)=\left(c\left(\left.T M_{1}\right|_{X_{1}}\right)^{-1} \cap \iota_{1}^{*} c_{*}\left(\alpha_{1}\right)\right) \times\left(c\left(T M_{2} \mid X_{2}\right)^{-1} \cap \iota_{2}^{*} c_{*}\left(\alpha_{2}\right)\right)$. Using successively the Example 1.5.11, $s\left(X_{1} \times X_{2} ; \alpha_{1} \otimes \alpha_{2}\right)$ is equals to

$$
\begin{aligned}
& \eta_{*}\left(\sum_{i \geq 0}\left(c_{1}\left(\mathcal{O}_{\mathbb{P}\left(N_{X_{1}} M_{1}\right)}(1)\right)^{i} \cap \eta_{1}{ }^{*}\left(c\left(\left.T M_{1}\right|_{X_{1}}\right)^{-1} \cap \iota_{1}^{*} c_{*}\left(\alpha_{1}\right)\right)\right) \times\right. \\
&\left.\times\left(c_{1}\left(\mathcal{O}_{\mathbb{P}\left(N_{X_{2}} M_{2}\right)}(1)\right)^{i} \cap \eta_{2}^{*}\left(c\left(\left.T M_{2}\right|_{X_{2}}\right)^{-1} \cap \iota_{2}^{*} c_{*}\left(\alpha_{2}\right)\right)\right)\right) .
\end{aligned}
$$

Lastly, by Proposition 1.5.4(b), it follows that

$$
\begin{aligned}
& s\left(X_{1} \times X_{2} ; \alpha_{1} \otimes \alpha_{2}\right) \\
= & \left(\eta_{1}\right)_{*}\left(\sum_{i \geq 0}\left(c_{1}\left(\mathcal{O}_{\mathbb{P}\left(N_{X_{1}} M_{1}\right)}(1)\right)^{i} \cap \eta_{1}{ }^{*}\left(c\left(\left.T M_{1}\right|_{X_{1}}\right)^{-1} \cap \iota_{1}^{*} c_{*}\left(\alpha_{1}\right)\right)\right)\right) \times \\
& \times\left(\eta_{2}\right)_{*}\left(\sum_{i \geq 0}\left(c_{1}\left(\mathcal{O}_{\mathbb{P}\left(N_{X_{2}} M_{2}\right)}(1)\right)^{i} \cap \eta_{2}{ }^{*}\left(c\left(\left.T M_{2}\right|_{X_{2}}\right)^{-1} \cap \iota_{2}^{*} c_{*}\left(\alpha_{2}\right)\right)\right)\right) \\
= & s\left(X_{1} ; \alpha_{1}\right) \times s\left(X_{2} ; \alpha_{2}\right) .
\end{aligned}
$$

We can use the above proposition to inductively show the following result.

Corollary 4.0.4 Let $X_{1}, \cdots, X_{r}$ be schemes which can be imbedded as subshemes of nonsingular varieties $M_{1}, \cdots, M_{r}$, respectively. For each $i=1, \ldots, r$, consider $\alpha_{i} a$ constructible function on $M_{i}$. Then

$$
s\left(X_{1} \times \cdots \times X_{r} ; \alpha_{1} \otimes \cdots \otimes \alpha_{r}\right)=s\left(X_{1} ; \alpha_{1}\right) \times \cdots \times s\left(X_{r} ; \alpha_{r}\right) .
$$

Proposition 4.0.5 Consider $f: M^{\prime} \longrightarrow M$ a proper and flat morphism of nonsingular schemes, $X \subset M$ a closed subscheme, $X^{\prime}=f^{-1}(X)$ the inverse image scheme, $g: X^{\prime} \longrightarrow X$ the induced morphism. Given a constructible function $\alpha^{\prime}$ on $M^{\prime}$, set $\alpha=f_{*} \alpha^{\prime}$. Then

$$
g^{*}(s(X \subset M ; \alpha))=c\left(N_{X^{\prime}} M^{\prime}\right)^{-1} c\left(\left.f^{*}(T M)\right|_{X^{\prime}}\right)^{-1} \cap \iota^{\prime *} c_{*}\left(\alpha^{\prime}\right) .
$$

Proof. Let $B$ be the blow-up of $M$ along $X$ and let $B^{\prime}$ be the blow-up of $M^{\prime}$ along $X^{\prime}$. Denote by $\widetilde{X}$ the exceptional divisor in $B$ with projection $\eta: \widetilde{X} \longrightarrow X$, and denote by $\widetilde{X^{\prime}}$ the exceptional divisor in $B^{\prime}$ with projection $\eta^{\prime}: \widetilde{X^{\prime}} \longrightarrow X^{\prime}$. Consider $F: B^{\prime} \longrightarrow B$ the induced morphism such that $F^{*} \widetilde{X}=\widetilde{X^{\prime}}$ and consider $G$ the induced morphism from $\widetilde{X^{\prime}}$ to $\widetilde{X}$. Moreover, let $\mathcal{O}(1)$ be the canonical line bundle on $\widetilde{X}$. In this way, $G^{*} \mathcal{O}(1)$ is the canonical line bundle on $\widetilde{X^{\prime}}$. Let us look at the following commutative diagram:

that is, $g \circ \eta^{\prime}=\eta \circ G$ and $f \circ \iota^{\prime}=\iota \circ g$. Below we have a sequence of equalities in which we use several times the commutativity of the previous diagram and the projection
formula:

$$
\begin{aligned}
g^{*}(s(X \subset M ; \alpha)) & =g^{*} \eta_{*}\left(\sum_{i \geq 0} c_{1}(\mathcal{O}(1))^{i} \cap \eta^{*}\left(c\left(\left.T M\right|_{X}\right)^{-1} \cap \iota^{*} c_{*}(\alpha)\right)\right) \\
& =g^{*} \eta_{*}\left(\sum_{i \geq 0} c_{1}(\mathcal{O}(1))^{i} \cap \eta^{*}\left(c\left(\iota^{*}(T M)\right)^{-1} \cap \iota^{*} c_{*}\left(f_{*} \alpha^{\prime}\right)\right)\right) \\
& \left.=g^{*} \eta_{*}\left(\sum_{i \geq 0} c_{1}(\mathcal{O}(1))^{i} \cap \eta^{*}\left(\iota^{*}(T M)\right)^{-1} \cap \iota^{*} f_{*} c_{*}\left(\alpha^{\prime}\right)\right)\right) \\
& =g^{*} \eta_{*}\left(\sum_{i \geq 0} c_{1}(\mathcal{O}(1))^{i} \cap \eta^{*}\left(c\left(\iota^{*}(T M)\right)^{-1} \cap g_{*} \iota^{\prime *} c_{*}\left(\alpha^{\prime}\right)\right)\right) \\
& =g^{*} \eta_{*}\left(\sum_{i \geq 0} c_{1}(\mathcal{O}(1))^{i} \cap \eta^{*} g_{*}\left(c\left(g^{*} \iota^{*}(T M)\right)^{-1} \cap \iota^{\prime *} c_{*}\left(\alpha^{\prime}\right)\right)\right) \\
& =g^{*} \eta_{*}\left(\sum_{i \geq 0} c_{1}(\mathcal{O}(1))^{i} \cap G_{*} \eta^{\prime *}\left(c\left(\iota^{*} f^{*}(T M)\right)^{-1} \cap \iota^{\prime *} c_{*}\left(\alpha^{\prime}\right)\right)\right) \\
& =\eta_{*}^{\prime}\left(\sum_{i \geq 0} c_{1}\left(G^{*} \mathcal{O}(1)\right)^{i} \cap \eta^{\prime *}\left(c\left(\left.f^{*}(T M)\right|_{X^{\prime}}\right)^{-1} \cap \iota^{\prime *} c_{*}\left(\alpha^{\prime}\right)\right)\right) \\
& =c\left(N_{X^{\prime}} M^{\prime}\right)^{-1} c\left(\left.f^{*}(T M)\right|_{X^{\prime}}\right)^{-1} \cap \iota^{\prime *} c_{*}\left(\alpha^{\prime}\right) .
\end{aligned}
$$

## Capítulo 5

## Apêndice A

### 5.1 Algebraic sets

Let us denote by $k$ the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. As references we cite [Milnor] and [Suwa2].

A subset $V \subset k^{n}$ is called an algebraic set if $V$ is the locus of common seros of some collection of polynomial functions on $k^{n}$. Let $I(V) \subset k\left[x_{1}, \ldots, x_{n}\right]$ be the ideal consisting of those polynomials which vanish throughout $V$. By the Hilbert basis theorem, we know that $I(V)$ is finitely generated. A non-vacuous algebraic set $V$ is called variety or an irreducible algebraic set if it cannot be expressed as the union of two proper algebraic subset. One has that, $V$ is irreducible if and only if $I(V)$ is a prime ideal.

Given an irreducible algebraic set $V$, the integral domain $k\left[x_{1}, \ldots, x_{n}\right] / I(V)$ is called the field of rational functions on $V$. Consider $f_{1}, \ldots, f_{k} \in k\left[x_{1}, \ldots, x_{n}\right]$ which span the ideal $I(V)$ and, for each $x \in V$, consider the $k \times n$ matrix $\left(\partial f_{i} / \partial x_{j}\right)$ evaluated at $x$. Let $r$ be the largest rank which this matrix attains at any point of $V$. In this way, a point $x \in V$ is called non-singular if the matrix $\left(\partial f_{i} / \partial x_{j}\right)$ attains its maximal rank $r$ at $x$; and singular if $\operatorname{rank}\left(\partial f(x) / \partial x_{i}\right)<r$. Note that, the set of all singular points of $V$ forms a proper algebraic subset of $V$.

Now let us define an equivalence relation in the set of subsets of $k^{n}$. Let $p \in k^{n}$. Given $A$ and $B$ subsets of $k^{n}$, one defines $A \sim_{p} B$ if there is a neighborhood $U$ of $p$
such that $A \cap U=B \cap U$. The equivalence class represented by the set $A$ is denoted by $(A, p)$ or, simply, $A$. Moreover, consider two functions $f, g: k^{n} \longrightarrow k^{m}: f$ and $g$ are equivalent if there exists a neighborhood $U$ of $p$ such that $\left.f\right|_{U}=\left.g\right|_{U}$. This is an equivalence relation. The equivalence class represented by the function $f$ is denoted by $f:\left(k^{n}, p\right) \longrightarrow k^{m}$. In this case, one says that $f:\left(k^{n}, p\right) \longrightarrow k^{m}$ is a germ of function at $p$. When $f(p)=q$, it is denoted by $f:\left(k^{n}, p\right) \longrightarrow\left(k^{m}, q\right)$.

Analogously, we may define analytic sets, taking analytic functions instead polynomial functions.

### 5.2 Sheaves and schemes

The references are $[\mathrm{H}]$ and $[\mathrm{G}-\mathrm{W}]$.
Let $X$ be a topological space. A presheaf $\mathcal{F}$ on $X$ consist of the: for every open set $U$ of $X$ a set $\mathcal{F}(U)$; and for each pair of open set $U \subseteq V$ a map $\rho_{U}^{V}: \mathcal{F}(V) \longrightarrow \mathcal{F}(U)$, called restriction map, such that
(1) $\rho_{U}^{U}=i d_{\mathcal{F}(U)}$ for all open set $U \subseteq X$,
(2) for $U \subseteq V \subseteq W$ open sets of $X, \rho_{U}^{W}=\rho_{U}^{V} \circ \rho_{V}^{W}$.

Let $\mathcal{F}_{1}$ and $\mathcal{F}_{2}$ be presheaves on $X$. A morphism of presheaves $\varphi: \mathcal{F}_{1} \longrightarrow \mathcal{F}_{2}$ is a family of maps $\varphi(U): \mathcal{F}_{1}(U) \longrightarrow \mathcal{F}_{2}(U)$ for all $U \subseteq V$ open, such that for all pairs of open sets $U \subseteq V$ in $X$ the following diagram commutes


A presheaf $\mathcal{F}$ is called a sheaf if for all open sets $U$ in $X$ and every open covering $U=\cup_{i} U_{i}$ the following conditions hold
(a) Given $s, s^{\prime} \in \mathcal{F}(U)$ with $\rho_{U_{i}}^{U}(s)=\rho_{U_{i}}^{U}\left(s^{\prime}\right)$ for all $i$, one has $s=s^{\prime}$.
(b) Given $s_{i} \in \mathcal{F}\left(U_{i}\right)$ for all $i$ such that $\rho_{U_{i} \cap U_{j}}^{U}\left(s_{i}\right)=\rho_{U_{i} \cap U_{j}}^{U}\left(s_{j}\right)$ for all $i, j$, then there is an $s \in \mathcal{F}(U)$ such that $\rho_{U_{i}}^{U}(s)=s_{i}$.

A morphism of sheaves is a morphism of presheaves. In an analogous way, one defines the notion of a sheaf of abelian groups, a sheaf of rings, a sheaf of modules, or a sheaf of algebras. If $\mathcal{F}$ is a presheaf on $X$, and if $x$ is a point of $X$, we define the stalk $\mathcal{F}_{x}$ of $\mathcal{F}$ at $x$ to be the direct limit of the groups $\mathcal{F}(U)$ for all open sets $U$ containing $x$, via the restriction maps $\rho$.

Example 5.2.1 Let $X$ be a complex manifold. Denote by $\mathcal{O}_{X}$ the sheaf of holomorphic functions on $X$ defined as follows: Given an open set $U$ in $X$ one has

$$
\mathcal{O}_{X}(U)=\{f: U \longrightarrow \mathbb{C} \mid f \text { is holomorphic }\} .
$$

A ringed space is a pair $\left(X, \mathcal{O}_{X}\right)$, where $X$ is a topological space and $\mathcal{O}_{X}$ is a sheaf of (commutative) rings on $X$. Let $\left(X, \mathcal{O}_{X}\right)$ and $\left(Y, \mathcal{O}_{Y}\right)$ be ringed spaces. One defines a morphism of ringed spaces $\left(X, \mathcal{O}_{X}\right) \longrightarrow\left(Y, \mathcal{O}_{Y}\right)$ as a pair $\left(f, f^{\#}\right)$, where $f: X \longrightarrow Y$ is a continuous map and $f^{\#}: \mathcal{O}_{Y} \longrightarrow f_{*} \mathcal{O}_{X}$ is a morphism of sheaves of rings on $Y$. The sheaf $f_{*} \mathcal{O}_{X}$ is given by $f_{*} \mathcal{O}_{X}(U)=\mathcal{O}_{X}\left(f^{-1}(U)\right)$ for all open set $U$ in $Y$. Moreover, a locally ringed space is a ringed space $\left(X, \mathcal{O}_{X}\right)$ such that for all $x \in X$ the stalk $\mathcal{O}_{X, x}$ is a local ring. A morphism of locally ringed space $\left(X, \mathcal{O}_{X}\right) \longrightarrow\left(Y, \mathcal{O}_{Y}\right)$ is a morphism of ringed spaces $\left(f, f^{\#}\right)$ such that for all $x \in X$ the induced homomorphism on stalks $f_{x}^{\#}: \mathcal{O}_{Y, f(x)} \longrightarrow \mathcal{O}_{X, x}$ is a local ring homomorphism. An isomorphism of locally ringed spaces is a morphism with a two-sided inverse.

Let $\left(X, \mathcal{O}_{X}\right)$ be a ringed space. An $\mathcal{O}_{X}$-module is a sheaf $\mathcal{F}$ on $X$, such that for each open set $U \subseteq X$, the group $\mathcal{F}(U)$ is an $\mathcal{O}_{X}(U)$-module, and for each inclusion of open sets $U \subseteq V$, the restrition map $\mathcal{F}(V) \longrightarrow \mathcal{F}(U)$ is compatible with the module structures.

For any ring $A$, one can associate the topological space $\operatorname{Spec} A$, which is the set of all prime ideals of $A$ equipped with the so-called Zariski topology. One can also define $\mathcal{O}_{\text {Spec } A}$ on $\operatorname{Spec} A$. Given an open set $U \subseteq \operatorname{Spec} A$, consider $\mathcal{O}_{\operatorname{Spec} A}(U)$ the set of functions $s: U \longrightarrow \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}$, such that $s(\mathfrak{p}) \in A_{\mathfrak{p}}$ for each $\mathfrak{p}$, and such that $s$ is locally a quotient of elements of $A$. Note that, $\mathcal{O}_{\text {Spec } A}$ defines a sheaf of rings. Moreover, $\left(\operatorname{Spec} A, \mathcal{O}_{\mathrm{Spec} A}\right)$ is a locally ringed space, called the affine scheme.

A scheme is a locally ringed space $\left(X, \mathcal{O}_{X}\right)$ which admits an open covering $X=$ $\bigcup_{i \in I} U_{i}$ such that all locally ringed spaces $\left(U_{i},\left.\mathcal{O}_{X}\right|_{U_{i}}\right)$ which are isomorphic to affine schemes. A morphism of schemes is a morphism of locally ringed spaces.

### 5.3 Whitney stratification

Let $V$ be a complex analytic variety $V$ of dimension $n$ in a complex manifold $M$. An analytic stratification of $V$ is a locally finite family $\left(V_{\alpha}\right)_{\alpha}$ of non-singular analytic subspaces of $V$, which are called strata, such that:
(1) The family is a partition of $V$.
(2) For each $V_{\alpha}$, the closures in $V$ of both $\overline{V_{\alpha}}$ and $\overline{V_{\alpha}} \backslash V_{\alpha}$ are analytic in $V$.
(3) For each pair $\left(V_{\alpha}, V_{\beta}\right)$ such that $V_{\alpha} \cap \overline{V_{\beta}} \neq \emptyset$ one has $V_{\alpha} \subset \overline{V_{\beta}}$.

A Whitney stratification is a stratification $\left(V_{\alpha}\right)_{\alpha}$ that satisfies the following conditions, known as the Whitney conditions (a) and (b), for every pair ( $V_{\alpha}, V_{\beta}$ ) such that $V_{\alpha} \subset \overline{V_{\beta}}$. Given $y \in V_{\alpha}$, consider $x_{i} \in V_{\beta}$ a sequence converging to $y$, and $y_{i} \in V_{\alpha}$ another sequence that also converges to $y$. Suppose these sequences are such that the sequence of secant lines $l_{i}=\overline{x_{i} y_{i}}$ also converges to some limiting line $l$, and the tangent planes $T_{x_{i}} V_{\beta}$ converges to some limiting plane $\tau$. The Whitney conditions (a) and (b) are the following:
(a) The limit space $\tau$ contains the tangent space of the stratum $V_{\alpha}$ at $y$, that is, $T_{y} V_{\alpha} \subset \tau$.
(b) The limit space $\tau$ contains all the limits of secants, that is, $l \subset \tau$.

There are some interesting facts about a Whitney stratification. Among them, we have: Every closed analytic subset of an analytic manifold admits a Whitney stratification; Whitney stratified spaces can be triangulated compatibly with th stratifacation; and Whitney stratifications are locally topological trivial along the strata.

### 5.4 Chern-Weil Theory

The main references in this section are [Chern] and [Milnor-Stasheff].
Let $E$ be a complex $r$-vector bundle on a $n$-dimensional smooth manifold $M$. Denote by $T_{\mathbb{C}}^{*} M=T^{*} M \otimes \mathbb{C}$ the complexified dual tangent bundle of $M, \Omega^{1}(M)$ the module of smooth sections of $T_{\mathbb{C}}^{*} M$ and $\Gamma(E)$ the module of smooth sections of $E$.

Definition 5.4.1 $A$ connection $\pi$ on $E$ is a $\mathbb{C}$-linear map $\nabla: \Gamma(E) \longrightarrow \Omega^{1}(M) \otimes \Gamma(E)$ satisfying Leibnitz' rule

$$
\nabla(f s)=d f \otimes s+f \nabla(s)
$$

where $s \in \Gamma(E)$ and $f: M \longrightarrow \mathbb{C}$ is smooth.

Let $\underline{s}=s_{1}, \ldots, s_{r}: U \longrightarrow E$ be a frame, where $U$ is open subset in $M$. Given a connection $\nabla$ on $E$, we can decompose $\nabla\left(s_{i}\right)$ into its components, writing $\nabla\left(s_{i}\right)=$ $\sum_{j=1}^{r} \theta_{i j} s_{j}$. The matrix $\theta=\left(\theta_{i j}\right)$ of 1-forms is called the connection matrix of $\nabla$ with repect to $s$.

A connection $\nabla$ on $E$ induces a unique $\mathbb{C}$-linear function $\nabla: \Omega^{1}(M) \otimes \Gamma(E) \longrightarrow$ $\Omega^{2}(M) \otimes \Gamma(E)$ that satisfies $\nabla(\omega \otimes s)=d \omega \otimes s-\omega \wedge \nabla s$ for all $\omega \in \Omega^{1}(M)$ and $s \in \Gamma(E)$. The $\mathbb{C}$-linear function $K_{\nabla}:=\nabla^{2}$ is called the curvature tensor of the connection $\nabla$.

Denote by $M_{n}(\mathbb{C})$ the algebra consisting of all $n \times n$ complex matrices. An invariant polynomial on $M_{n}(\mathbb{C})$ is a function $P: M_{n}(\mathbb{C}) \longrightarrow \mathbb{C}$, which can be expressed as a complex polynomial in the entries of the matrix, and satisfies $P(X Y)=P(Y X)$, or equivalently, $P\left(T X T^{-1}\right)=P(X)$ for all non-singular matrix $T$. Note that, the trace and determinant functions are examples of invariant polynomials. For any invariant polynomial $P$, we have a well-defined global differential form, denoted by $P\left(K_{\nabla}\right)$.

Theorem 5.4.2 Given $P$ an invariant polynomial, we have
(a) The form $P\left(K_{\nabla}\right)$ is closed, that is $d P\left(K_{\nabla}\right)=0$.
(b) The cohomology class $[P(K)]=\left[P\left(K_{\nabla}\right)\right]$ is independent of the connection $\nabla$.

Given any $A \in M_{n}(\mathbb{C})$, let $\sigma_{k}(A)$ the $k$-th elementary symmetric function of the eigenvalues of $A$. One has $\operatorname{det}(I d+t A)=1+t \sigma_{1}(A)+\cdots+t^{n} \sigma_{n}(A)$.

Definition 5.4.3 The $k$-th Chern class of $E$ is defined by

$$
c_{k}(E):=\left[\sigma_{k}\left(\frac{\sqrt{-1}}{2 \pi} K_{\nabla}\right)\right] \in H_{D R}^{2 k}(M ; \mathbb{C})
$$

The total Chern class of $E$ is defined by

$$
c(E):=1+c_{1}(E)+\cdots+c_{r}(E) \in H_{D R}^{\text {even }}(M ; \mathbb{C}) .
$$

We have expected properties, such as $c\left(f^{*} E\right)=f^{*} c(E)$ for all smooth map $f: M^{\prime} \longrightarrow$ $M$; and $c(E \oplus F)=c(E) c(F)$, known as the Whitney sum formula.

### 5.5 Derived categories

Let $X$ be a complex analytic space. One denotes by $\mathcal{D}_{c}^{b}(X)$ the derived category of bounded, constructible complexes of sheaves of $\mathbb{C}$-vector spaces on $X$. Given $F^{\bullet}$, the shifted complex $F^{\bullet}[l]$ is defined by $\left(F^{\bullet}[l]\right)^{k} F^{k+l}$ with differential given by $d_{[l]}^{k}=(-1)^{l} d^{k+l}$. For any $F^{\bullet}=\mathcal{D}_{c}^{b}(X)$ and $p \in X$, one denotes by $\mathcal{H}^{k}\left(F^{\bullet}\right)_{p}$ the stalk cohomology of $F^{\bullet}$ at $p$, and thus, the Euler characteristic of $F^{\bullet}$ at $p$ is given by

$$
\chi\left(F^{\bullet}\right)_{p}=\sum_{k}(-1)^{k} \operatorname{dim}_{\mathbb{C}} \mathcal{H}^{k}\left(F^{\bullet}\right)_{p}
$$

Moreover, the Euler characteristic of $X$ with coefficients in $F^{\bullet}$, denoted by $\chi\left(X, F^{\bullet}\right)$, is given by

$$
\chi\left(X, F^{\bullet}\right)=\sum_{k}(-1)^{k} \operatorname{dim}_{\mathbb{C}} \mathbb{H}^{k}\left(X, F^{\bullet}\right)
$$

where $\mathbb{H}^{\bullet}\left(X, F^{\bullet}\right)$ denotes the hypercohomology groups of $X$ with coefficients in $F^{\bullet}$. Now, consider $\mathcal{S}$ a Whitney stratification of $X$. Given $p \in S$, set $\chi\left(F_{S}^{\bullet}\right):=\chi\left(F^{\bullet}\right)_{p}$. Then, we have

$$
\chi\left(X, F^{\bullet}\right)=\sum_{S \in \mathcal{S}} \chi\left(F_{S}^{\bullet}\right) \chi(S) .
$$

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